Existence of Quantum Stationary States of Point Nucleons Which Interact with the Meson Field

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Investigation is carried out of the problem of the existence of stationary quantum states of a system consisting of point nucleons which interact with a field of neutral and charged mesons. Various types of interaction operators between the nucleon and the meson-field are considered. With the motion of the nucleons treated non-relativistically, the absence of stationary states is proved for the case when the interaction is of a pseudo-vector character. The proof involves none of the approximations employed, for example, in the weak or strong coupling method.

1. INTRODUCTION

A variant of the meson theory of nuclear forces most popular at the present time is based on the assumption that π - mesons are describable in terms of pseudo-scalars and that their interaction with nucleons has a pseudo-vector character. It is known that this theory based on the weak coupling approximation for the nucleons in the meson field, leads to the conclusion that in the g^2 approximation the interaction potential of two nucleons separated by a small distance r is proportional to $1/r^3$; this is why no stationary states are obtained in the deuteron problem. Thus, even if we ignore the infinite self-energies of the interaction between the nucleons and the meson field, the system has no stationary quantum states. Tamm proved[1] that the same difficulty is encountered when the motion of the nucleons is treated relativistically, and also if several other approximations of the theory, for example, the quasi-static point of view, are not made.

When the next approximation of the weak coupling theory, of the order of g^4, was computed, new terms[2] of an order of magnitude that is higher than the previous terms of the g^2 approximation appeared in the interaction potential for two nucleons. In particular, if the distance of separation is small, the new terms are proportional to $1/r^6$, corresponding to a repulsion, and predominate over the terms of the earlier approximation; the $-1/r^3$ difficulty is therefore eliminated.

The fact that going to the next approximation alters even the qualitative nature of the theoretical result is very unfavorable to the weak coupling approximation. The question of the existence of stationary states of the system remains open, for it is not at all excluded that high approximations may show that at small values of r the interaction potential for two nucleons is an alternating series of increasing powers in $1/r$. This means formally that the existence of the stationary states depends on whether the last term in this series is positive or negative. In essence this means that for small values of r the potential cannot be represented by this series, and it may be that introducing a quasi-static potential is in general meaningless. This article therefore treats the question of the existence of stationary quantum states of nucleons in a meson field without resorting to the weak coupling approximation, and also without introducing quasi-static potentials for the nucleon interaction.

We shall express the Hamiltonian of the meson field without sources in the conventional form:

$$\hat{H}_0 = \frac{1}{2}\sum_{\alpha=1}^{3}\int\left[\frac{\pi_\alpha^2}{c^2} + (\nabla\varphi_\alpha)^2 + \mu^2\varphi_\alpha^2\right]d\tau. \quad (1)$$

Here all the φ_α and π_α terms are real: φ_1 and φ_2 are the fields of the charged mesons, and φ_3 is the field of the neutral mesons; π_α are fields that are canonically conjugate to φ_α. The masses of the positive, negative, and neutral mesons are assumed to be the same, and equal to m_π, with $\mu = m_\pi c/\hbar$.

Introducing a complete set of functions

$$\chi_{\vec{\varkappa}}(\mathbf{r}) = \sqrt{\frac{2}{L^3}}\begin{cases}\cos\vec{\varkappa}\mathbf{r} & \text{for } \varkappa_x \leqslant 0,\\ \sin\vec{\varkappa}\mathbf{r} & \text{for } \varkappa_x > 0,\end{cases} \quad (2)$$

which are orthonormalized in the volume L^3, it is possible to expand φ_α and π_α in Fourier series:

$$\varphi_\alpha = \sum_{\vec{\varkappa}}\varphi_{\alpha\vec{\varkappa}}\,\chi_{\vec{\varkappa}}(\mathbf{r}), \quad (3)$$

[1] I. E. Tamm, Journ. of Phys. USSR **9**, 449 (1945).

[2] K. Nakabayasi and I. Sato, Phys. Rev. **88**, 144 (1952).

$$\pi_\alpha = \sum_{\vec{\varkappa}} \pi_{\alpha\vec{\varkappa}} \chi_{\vec{\varkappa}}(\mathbf{r}). \tag{4}$$

Then, assuming

$$\omega_{\vec{\varkappa}} = c\sqrt{\mu^2 + \vec{\varkappa}^2}\,(\omega_{\vec{\varkappa}} > 0) \tag{5}$$

and introducing the dimensionless canonically conjugate quantities

$$q_{\alpha\vec{\varkappa}} = \varphi_{\alpha\vec{\varkappa}} \frac{1}{c}\sqrt{\omega_{\vec{\varkappa}}/\hbar}, \tag{6}$$

$$g_{\alpha\vec{\varkappa}} = \pi_{\alpha\vec{\varkappa}}(1/c\sqrt{\hbar\omega_{\vec{\varkappa}}}),$$

the Hamiltonian of the meson field can be rewritten as:

$$\hat{H}_0 = \frac{1}{2}\sum_{\alpha\vec{\varkappa}} \hbar\omega_{\vec{\varkappa}} [q^2_{\alpha\vec{\varkappa}} + g^2_{\alpha\vec{\varkappa}}]. \tag{7}$$

Instead of considering $q_{\alpha\vec{\varkappa}}$ and $g_{\alpha\vec{\varkappa}}$ as operators that obey the known commutation rules, we shall take $q_{\alpha\vec{\varkappa}}$ to be the independent variable, and represent the operators $g_{\alpha\vec{\varkappa}}$ in the following form:

$$g_{\alpha\vec{\varkappa}} = -i\,\partial/\partial q_{\alpha\vec{\varkappa}}. \tag{8}$$

As a result we obtain

$$\hat{H}_0 = \frac{1}{2}\sum_{\alpha\vec{\varkappa}} \hbar\omega_{\vec{\varkappa}} [q^2_{\alpha\vec{\varkappa}} - (\partial^2/\partial q^2_{\alpha\vec{\varkappa}})]. \tag{9}$$

The non-relativistic Hamiltonian of the ith nucleon in the absence of a meson field will have the following form:

$$K_i = -\frac{\hbar^2}{2m}\Delta_i. \tag{10}$$

The masses of the proton and of the neutron are assumed to be the same and equal to m.

The interaction between the ith nucleon and the meson field is chosen to be of the following form:

$$\hat{H}'_i = -\sqrt{4\pi}\sum_{\alpha=1} g_\alpha \tau_{\alpha i} \{\hat{A}_i\varphi_\alpha(\mathbf{r}_i), \hat{L}_i\}, \tag{11}$$

$$\{\hat{A}_i\varphi_\alpha, \hat{L}_i\} \equiv \sum_\beta (\hat{A}_{\beta i}\varphi_\alpha)\hat{L}_{\beta i}, \tag{12}$$

where g_α are constants, $\tau_{\alpha i}$ are the usual "isotopic spin" operators, $\mathbf{r}_i$ are the nucleon coordinates, and $\hat{A}_{\beta i}$ is an operator acting on $\varphi_\alpha(\mathbf{r}_i)$. It is assumed that operators $B_{\beta i}$ conjugate to $A_{\beta i}$ exist and satisfy the following relationship

$$\int f(\mathbf{r}_i)\hat{A}_{\beta i}\chi(\mathbf{r}_i)\,d\tau_i = \int \chi(\mathbf{r}_i)\hat{B}_{\beta i} f(\mathbf{r}_i)\,d\tau_i \tag{13}$$

for arbitrary quadratically-integrable f and χ. The operator $L_{\beta i}$ acts only on the spatial and spin coordinates of the nucleon, but not on its charge degree of freedom. Thus $\hat{L}_{\beta i}$ commutes with $\tau_{\alpha i}$.

We assume that operators B_β and L_β are homogeneous, that is, they satisfy the following relationships

$$\hat{B}_\beta(k\mathbf{r}) = k^n\hat{B}_\beta(\mathbf{r}), \tag{14}$$

$$\hat{L}_\beta(k\mathbf{r}) = k^m\hat{L}_\beta(\mathbf{r}), \tag{15}$$

where k, n, and m are constants. The generally known forms of interaction, namely scalar, pseudoscalar, and pseudo-vector, are particular cases of (11) and possess propoerties (14) and (15). In case of a mixture of meson fields having different types of interaction $\hat{H}'_1$, the values of n and m in relationships (14) and (15) can be different for each type of interaction.

2. INVESTIGATION OF THE EXISTENCE OF STATIONARY STATES IN THE NON-RELATIVISTIC TREATMENT OF NUCLEON MOTION

The energy operator of the system has the following form:

$$\hat{H}(r,q) \equiv \hat{H}(\ldots r_i \ldots, \ldots q_{\alpha\vec{\varkappa}} \ldots)$$

$$= \hat{H}_0(q) + \hat{K}(r) + \hat{H}'(q,r),$$

$$\hat{K} = \sum_{i=1}^{A}\hat{K}_i(\mathbf{r}_i);\ \hat{H}' = \sum_{i=1}^{A}\hat{H}'_i(q,\mathbf{r}_i). \tag{16}$$

If the system has a stationary state, the lowest eigenvalue of operator (16) (the ground energy level) should have a finite value. According to the general assumptions of the quantum mechanics, the average energy value of any arbitrary state of the system $\Psi(\ldots\mathbf{r}_i\ldots q_{\alpha\vec{\varkappa}}\ldots)$ is not less than the lowest energy level W_0:

$$\overline{H} \equiv \int \Psi^*\hat{H}\Psi\,dr\,dq \geqslant W_0, \tag{17}$$

$$dr = \prod_{i=1}^{A} dx_i\,dy_i\,dz_i,\ dq = \prod_{\alpha\vec{\varkappa}} dq_{\alpha\vec{\varkappa}},$$

W_0 is the absolute minimum of the integral in the left half of (17), provided that $\Psi(\ldots\mathbf{r}_i \ldots q_{\alpha\vec{\varkappa}}\ldots)$ is chosen from among the admissible functions, i.e., normalized, continuous, with continuous first derivatives, finite, and single-valued. Were the system to contain only a single nucleon, Ψ would be a four-component function, for two values assume the spin index and two values assume the nuclear-charge index. If the system contains A nucleons, Ψ has 4^A components. Treating Ψ as a multi-dimensional vector, we can regard expressions such as $\Psi^*\Psi$ and $\Psi^*\hat{H}\Psi$ as being scalar products (sum of the products of the components that have equal indices).

Let us assume the following wave function for the system

$$\Psi(r, q) = \psi(\mathbf{r}_1 \ldots \mathbf{r}_A) \tag{18}$$

$$\times \exp\left\{-\frac{1}{2}\sum_{\alpha\vec{x}}\left[(q_{\alpha\vec{x}} - q^0_{\alpha\vec{x}})^2 + \frac{1}{2}\ln\pi\right]\right\},$$

where $\psi(\mathbf{r}_1 \ldots \mathbf{r}_A)$ is a certain quadratically-integrable function in the coordinate space $x_1, y_1, z_1 \ldots x_A, y_A, z_A$ of the nucleons.

Let $q^0_{\alpha\vec{x}}$ be, in accordance with equations (6) and (3), the dimensionless Fourier coefficients of a certain constant prescribed meson field

$$\varphi^0_\alpha(\mathbf{r}) = \int \frac{\eta_\alpha[\psi, \mathbf{r}'] \exp\{-\mu|\mathbf{r}-\mathbf{r}'|\}}{|\mathbf{r}-\mathbf{r}'|}\, d\tau', \tag{19}$$

where

$$\eta_\alpha[\psi, \mathbf{r}] = \sum_{i=1}^{A} \eta_{i\alpha}[\psi, \mathbf{r}], \tag{20}$$

$$\eta_{\alpha i}[\psi, \mathbf{r}_i] \tag{21}$$

$$= \frac{g_\alpha}{\sqrt{4\pi}} \int \{\hat{B}_i, \psi\tau^*_{\alpha i}\hat{L}_i\psi\}\, d\tau_1 \ldots d\tau_{i-1}\, d\tau_{i+1} \ldots d\tau_A,$$

$$\{\hat{B}_i, \psi^*\tau_{\alpha i}\hat{L}_i\psi\} \equiv \sum_\beta \hat{B}_{\beta i}\psi^*\tau_{\alpha i}\hat{L}_{\beta i}\psi. \tag{22}$$

Elementary calculation, using Eqs. (9), (10), (12), (13), (16), (22), yield

$$\bar{H} = \frac{1}{2}\sum_{\alpha\vec{x}} \hbar\omega_{\vec{x}} - \frac{\hbar^2}{2m}\sum_{i=1}^{A}\int \psi^*\Delta_i\psi\, d\mathbf{r} \tag{23}$$

$$- 2\pi \sum_{\alpha=1}^{3} \iint \eta_\alpha[\psi, \mathbf{r}]\, \eta_\alpha[\psi, \mathbf{r}']$$

$$\times \frac{\exp\{-\mu|\mathbf{r}-\mathbf{r}'|\}}{|\mathbf{r}-\mathbf{r}'|}\, d\tau\, d\tau':$$

Furthermore, let

$$\psi(\mathbf{r}_1 \ldots \mathbf{r}_A) = k^{3A/2} f(k\mathbf{r}_1, \ldots, k\mathbf{r}_A), \tag{24}$$

where $f(\mathbf{r}_1 \ldots \mathbf{r}_A)$ is a certain fixed normalized admissible function. It is easy to see that if f is normalized, ψ and $\Psi(r,q)$ are also normalized for any positive constant k. Simple interchange of the variables under the integral sign results in

$$\sum_{i=1}^{A}\int \psi^*\Delta_i\psi\, dr = k^2 \sum_{i=1}^{A}\int f^*\Delta_i f\, dr, \tag{25}$$

$$\eta_\alpha[\psi, \mathbf{r}] = k^{3-n-m}\eta_\alpha[f, k\mathbf{r}]. \tag{26}$$

The last equation results from Eqs. (14) and (15). In (25) and (26) f is a function of $\mathbf{r}_i$ rather than of $k\mathbf{r}_i$.

Let us expand the last term of Eq. (23) in positive and negative powers of k; this is done most simply by expanding the exponential function under the integral sign in a series and taking the quantities kr and kr' as new integration variables. As a result, the expansion term having the highest power of k will be

$$- k^{1-2n-2m} 2\pi \sum_\alpha \iint \frac{\eta_\alpha[f, \mathbf{r}]\, \eta_\alpha[f, \mathbf{r}']}{|\mathbf{r}-\mathbf{r}'|}\, d\tau\, d\tau'. \tag{27}$$

Here the function under the integral sign no longer contains k. It is easy to see that the term (27) is always negative. At sufficiently large values of k this is the governing term of the expansion.

The next to the last term of (23) is always positive and is proportional to k^2, as can be seen from (25). Therefore if

$$1 + 2n + 2m < 0, \tag{28}$$

and k increases in (23), the (27) becomes the dominant term, and $\bar{H} \to -\infty$. As k increases, $\Psi(r,q)$ continues to remain an admissible function, and inequality (17) should be observed. Consequently, the system has no ground state with a finite energy W_0.

If the inequality sign is reversed in (28), then as k increases, $\bar{H} \to +\infty$, W_0 can be finite, and consequently the existence of stationary state is not excluded. However, there is no reason in this case for insisting on a continuous existence of stationary states, for it is not known whether the absolute minimum of $\bar{H}$ will be finite after we forego the particular form of the wave function (18) and turn to its more general expressions.

The first term in equation (23) is the zero order energy of the meson field and can be disregarded in the discussion.

A) Pseudo-Vector Interaction between Nucleons and Meson Field

In this case the operator $\hat{H}'_i$ has the following form

$$\hat{H}_i' = -\sqrt{4\pi}\sum_{\alpha=1}^{3} g_\alpha \tau_{\alpha i}(\sigma_i, \nabla_i \varphi_\alpha), \tag{29}$$

where σ_i is the nucleon spin operator. Comparing this expression with (11) we see that

$$\hat{A}_{\beta i} = \partial/\partial x_{\beta i},\ \hat{L}_{\beta i} = \sigma_{\beta i},\ \beta = 1, 2, 3. \tag{30}$$

Consequently, according to definition (13), we get

$$\hat{B}_{\beta i} = -\partial/\partial x_{\beta i}. \tag{31}$$

The operators $\hat{B}_{\beta i}$ and $\hat{L}_{\beta i}$ actually have the homogeneity properties (14) and (15), whereby $n = -1$ and $m = 0$. Thus, condition (28) is satisfied, and consequently the system has no stationary states.

B) Scalar Interaction of Nucleons with Meson Field

In this case the operator $\hat{H}_i'$ has the following form

$$\hat{H}_i' = -\sqrt{4\pi}\sum_{\alpha=1}^{3} g_\alpha \tau_{\alpha i}\varphi_\alpha(\mathbf{r}_i). \tag{32}$$

Comparing this expression with (11) we see that

$$\hat{A}_i = 1,\ \hat{L}_i = 1. \tag{33}$$

Hence $\hat{B}_i = 1$, $n = 0$, $m = 0$, and consequently the inequality in (28) is reversed. Thus the existence of stationary states of the system is not excluded.

The two examples considered above show how simple it is to apply the general criterion (28) to specific variants of meson dynamics. In case of a mixture of meson fields with different forms of the operator $\hat{H}_i'$ and unequal μ, the results obtained above are generalized in a trivial manner: Instead of equation (23) we get

$$\overline{H} = \frac{1}{2}\sum_{\alpha \vec{x} p} \hbar\omega_{\vec{x}p} - \frac{\hbar^2}{2m}\sum_{l=1}^{A}\int \psi^* \Delta_l \psi dr - 2\pi \sum_{\alpha, p}\iint \eta_{\alpha p}[\psi, \mathbf{r}]\,\eta_{\alpha p}[\psi, \mathbf{r}'] \times \frac{\exp\{-\mu_p|\mathbf{r}-\mathbf{r}'|\}}{|\mathbf{r}-\mathbf{r}'|} d\tau d\tau'. \tag{34}$$

Here the index p runs over the different types of meson fields. All the integrals in the last term of equation (34) are positive, and it is therefore impossible for components with different values of p to cancel each other. Consequently if inequality (28) is valid for at least one of the meson fields, the system has no stationary states; no addition of any other meson fields can remedy the situation.

In connection with this it is appropriate to mention the work by Moller and Rosenfeld[3] and by Schwinger[4*], in which a mixture of pseudo-scalar and vector meson fields is considered. When each field is taken individually, no stationary states are obtained for the deuteron problem; but stationary states are obtained when mixtures of these fields are taken in a definite ratio. Actually this does not contradict the results of our work, for these authors obtained a finite energy for the deuteron only after using a quasi-static potential for the nucleon interaction and discarding known infinite self-energies of the nucleons in the meson field. By neglecting these terms, these authors obtained system energies that are totally unsuitable as eigenvalues of operator (16).

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[3] C. Moller and L. Rosenfeld, Kgl. Danske Vil. Sels. Math. fys. Medd. 17, 8 (1940).

[4] J. Schwinger, Phys. Rev. 67, 339 (1945)

* *Translator's note.*- This reference is actually to a paper by Ning Hu, not by Schwinger.

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