tillation counters,[16] although a detailed comparison of the present results with those would be difficult.

In the above we have proceeded from the entirely reasonable requirement that close-pair stars should be created simultaneously. To be perfectly logical, however, we have not ruled out the possibility that the close pairs are formed with a delay time which is beyond the resolution capabilities of this method. This question can be resolved when a considerably greater amount of experimental data is accumulated. In the event of a negative result, there would seem to be a fundamental contradiction between the statistical analysis and the results of the present work; this would seem to indicate the presence of some methodological error in the statistical analysis. In this connection factors such as nonuniformity of exposure, inhomogeneities in the amount of AgBr, differences in the thickness of the emulsion in different parts of the plates, and the higher counting efficiency in the vicinity of neighboring stars as compared with single stars should be considered. A detailed analysis of these and similar effects should be the subject of a special report.

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[16] J. B. Harding. Nature 169, 747 (1952).

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On the Theory of Stochastic Processes in Cosmic Radiation

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We study a generalization of the kinetic equation of L. Janossy for the case where the physical quantity $\xi(t)$ does not remain constant between two consecutive jumps, but changes in accordance with some causal law. The equation that is introduced can be successfully applied to various problems concerning stochastic processes in cosmic radiation.

IN the present paper we consider some problems of the theory of stochastic processes, which play a very important role in nuclear physics and in the theory of cosmic radiation[1]. We shall not touch upon concrete problems, since there are a large number of papers in this direction in the literature[2]; however, a unified treatment of these problems is lacking.

For the stochastic description of an arbitrary physical process (energy loss, scattering, etc.) we must first of all determine the distribution function of the physical quantities which play a decisive role in the given process. If it is known that the value of some physical quantity $\xi(t)$ at the instant of time t is equal to x, then in certain kinds of stochastic processes (processes without after-effect*) it is easy to determine the probability that the value of the random variable $\xi(\tau)$ is greater than y at any instant of time $\tau \geq t$. Let us designate this probability [the distribution function of the random process $\xi(t)$] by $F(t, x; \tau, y)$. As is well known, the function $F(t, x; \tau, y)$ is determined by two integro-differential equations, introduced by Kolmogoroff[3] and Feller[4].

Let us denote by $P(t, x; y)$ the probability that the random process $\xi(t)$ discontinuously changes its value to $\xi(t+0) \leqslant y$ under the condition that a jump occurred at the instant of time t, and that immediately before the jump $\xi(t-0)$ was

[1] L. Janossy, J. Exptl. Theoret. Phys. (U.S.S.R.) 26, 386, 518 (1954).

[2] A. Bekessy, L. Janossy and L. Pal, Magyar. Fizikai Folyoirat (in press).

* Processes without after-effect are often called Markoff processes.

[3] A. Kolmogoroff, Math. Ann. 104, 415 (1931).

[4] W. Feller, Math. Ann. 113, 113 (1936).

equal to x. Let $Q(t, x)\Delta t + o(\Delta t)$ be the probability that in the time interval $(t, t + \Delta t)$ there occurs at least one jump of the random process $\xi(t)$ [$o(\Delta t)$ is the probability that in this interval there occurs more than jump of the quantity $\xi(t)$] under the condition that immediately before the jump $\xi(t-0)$ was equal to x. In physical applications $Q(t, x)$ is called the "total cross section" and the expression $w(t, x; y)\,dy = Q(t, x)\,dP_y(t, x; y)$ is called the "differential cross section".

The Kolmogoroff-Feller equations, which determine the distribution function $F(t,x; \tau, y)$, have the following form:

$$\frac{\partial F(t, x;\tau, y)}{\partial t} = Q(t, x)\{F(t, x; \tau, y) - \int F(t, z; \tau, y)\,dP_z(t, x; z)\}; \tag{1}$$

$$\frac{\partial F(t, x; \tau, y)}{\partial \tau} = \int\limits_{(y)} Q(\tau, z)\,dF_z(t, x; \tau, z) - \int Q(\tau, z)\,P(\tau, z; y)\,dF_z(t, x; \tau, z). \tag{2}$$

The equation (1) is especially suitable for solving a whole series of problems of cascade theory[1]. Janossy[1], using the method of the "first collision", worked out a very graphic method of obtaining a kinetic equation of the type (1) in the case of processes that are homogeneous in time.

1. In connection with certain problems in the statistical theory of the penetration of elementary particles through matter, one needs a generalization of the Kolmogoroff-Feller equation for the case where the physical quantity $\xi(t)$ does not remain constant between two consecutive jumps, but changes in accordance with some causal law, i.e., if it is known that $\xi(t) = x$ at the instant of time t, then after a time u the random variable $\xi(t + u)$ takes the value $f(x, u)$, under the condition that the interval $(t, t + u)$ lies between two consecutive jumps of the random process. It is evident that without the introduction of new random quantities the Markoff character of the process can be preserved if and only if the function which describes the causal character of the random process $\xi(t)$ satisfies the functional equation

$$f(x, u_1 + u_2) = f\{f(x, u_1), u_2\}. \tag{3}$$

The equation (3) was studied in detail by Aczel[5]. As the basis for his study it is easily shown that a function which satisfies equation (3) has the following form:

$$f(x, u) = H\{H^{-1}(x) + u\}. \tag{4}$$

Here it is assumed that the function $H(x)$ has an inverse function $H^{-1}(x)$. Indeed, one can easily convince oneself that

$$f\{f(x, u_1) + u_2\} = H\{H^{-1}[f(x, u_1)] + u_2\}$$

$$= H\{H^{-1}(x) + u_1 + u_2\},$$

i.e., that

$$H\{H^{-1}(x) + u_1 + u_2\} = f(x, u_1 + u_2). \tag{5}$$

It is evident that the functions $xe^{\pm\lambda u}$, $x/(1 \pm \lambda xu)$, $x \pm \lambda u$ etc. satisfy Eq. (3).

In connection with statistical problems concerning recording of nuclear fission using an ionization chamber[6] we studied the solution of the kinetic equation for the case where the physical quantity $\xi(t)$ changes in accordance with an exponential law between consecutive jumps, i.e, $f(x, u) = xe^{-\lambda u}$. The generalized equation is also easily written for the case where all that is required of the function $f(x, u)$ is that it satisfy the condition (3).

First of all we determine the distribution function $F(t, x; t + \Delta t, z)$. It is easily expressed in terms of the functions $Q(t, x)$, $P(t, x; y)$ and $f(x, u)$; i.e.,

$$F(t, x; t + \Delta t, z) = \{1 - Q(t, x)\Delta t\}\,E\{z - f(x, \Delta t)\} + Q(t, x)\,P(t, x; z)\,\Delta t + o(\Delta t), \tag{6}$$

where

$$E(z - z') = \begin{cases} 0, & \text{if } z \leqslant z', \\ 1, & \text{if } z > z'. \end{cases}$$

From the Markoff equations (see Ref. 7, p. 245), we obtain the following equations, using (6):

[5] J Aczel, Publ. Math. 1, 243 (1950).

[6] L. Pal, Magyar. Fizikai Folyoirat 3, 31 (1955).

[7] V. V. Gnedenko, *A Course in Probability Theory*, Moscow-Leningrad, 1951.

$$\left\{\frac{\partial}{\partial t} - Q(t, x) - g(x)\frac{\partial}{\partial x}\right\} F(t, x; \tau, y) + Q(t, x)\int F(t, z; \tau, y)\, dP_z(t, x; z) = 0, \tag{7}$$

$$\left\{\frac{\partial}{\partial \tau} + g(y)\frac{\partial}{\partial y}\right\} F(t, x; \tau, y) - \int Q(\tau, z) P(\tau, z; y)\, dF_z(t, x; \tau, z) = \int_{(y)} Q(\tau, z)\, dF_z(t, x; \tau, z), \tag{8}$$

where $g(x) = (\partial f(x, u)/\partial u)_{u=0}$.

If $Q(t, x) = N$, $P(t, x; y) = H(y - x)$ and $F(t, x; \tau, y) = R(\tau - t, y)$, then, introducing the notation $\tau - t = v$, we obtain from (7) for the case $f(x, u) = xe^{-\alpha u}$ the equation

$$\frac{\partial R}{\partial v} - \alpha y \frac{\partial R}{\partial y} = -N\{R(v, y) - \int R(v, z)\, dH(y - z)\}, \tag{9}$$

which coincides with Eq. (2), studied in Ref. 6. An analogous equation is obtained from (8) in the homogeneous case.

2. In the preceding section we showed that for the class of random processes without after-effect the causal changes of the quantity $\xi(t)$ between consecutive jumps (discontinuities) cannot be arbitrary, but that the functions which describe these changes must satisfy Eq. (3). In the theory of stochastic processes in cosmic radiation, one often encounters such random processes which cannot be described within the framework of Markoff processes by using the distribution function of one random quantity. In these cases a description within the class of processes without after-effect can be achieved in principle with the aid of the distribution function of several mutually dependent random variables.

Thus for the stochastic description of the given physical system, we introduce the random quantities $\xi_1(t), \xi_2(t), \ldots, \xi_n(t)$. The study of such processes is of special interest in the case where, in addition to purely discontinuous changes, the random quantities $\xi_1(t), \ldots, \xi_n(t)$ can change in accordance with certain causal laws, i.e., if it is known that $\xi_k(t) = x_k$ $(k = 1, 2, \ldots, n)$, at the time t, then after the time u the random quantity takes the value $\xi_k(t + u)$ under the condition that the interval $(t, t + u)$ lies between two consecutive jumps of the random vector $\{\xi_1(t), \xi_2(t), \ldots, \xi_n(t)\}$.

Let $Q(t, x_1; \ldots, x_n)\Delta t + o(\Delta t)$ be the probability that in the interval $(t, t + t)$ the state of the system, determined by the equations $\xi_k(t) = x_k$, changes discontinuously, and $P(t, x_1, \ldots, x_n; y_1, \ldots, y_n)$ the probability that $\xi_k(t + 0) = y_k$, $(k = 1, 2, \ldots, n)$ under the condition that a jump occurred at the instant of time t and that immediately before the jump the equations $\xi_k(t - 0) = x_k$ were valid. Let us designate by $F(t, x_1, \ldots, x_n; \tau, y_1, \ldots, y_n)$ the probability that $\xi_k(\tau) \leqslant y_k$, if it is known that at the instant of time t the equalities $\xi_k(t) = x_k$ $(k = 1, 2, \ldots, n)$ were valid.

In an analogous way it is easy to determine the generalized Kolmogoroff-Feller equations

$$\left\{\frac{\partial}{\partial t} - Q(t, x_1, \ldots, x_n) - \sum_k g_k(x_1, \ldots, x_n)\frac{\partial}{\partial x_k}\right\} \times F(t, x_1, \ldots, x_n; \tau, y_1, \ldots, y_n) + Q(t, x_1, \ldots, x_n) \times \int \cdots \int F(t, z_1, \ldots, z_n; \tau, y_1, \ldots, y_n)\, dP(t, x_1, \ldots, x_n; z_1, \ldots, z_n), \tag{10}$$

$$\left\{\frac{\partial}{\partial \tau} + \sum_k g_k(y_1, \ldots, y_n)\frac{\partial}{\partial y_k}\right\} F(t, x_1, \ldots, x_n; \tau, y_1, \ldots, y_n) - \int \cdots \int_{(y_1, \ldots, y_n)} Q(\tau, z_1, \ldots, z_n)\, dF(t, x_1, \ldots, x_n; \tau, z_1, \ldots, z_n) = \int \cdots \int Q(\tau, z_1, \ldots, z_n) P(\tau, z_1, \ldots, z_n; y_1, \ldots, y_n) \times dF(t, x_1, \ldots, x_n; \tau, z_1, \ldots, z_n). \tag{11}$$

In what follows we shall be concerned only with the first equation. If the process is homogeneous in time, i.e.,

$$F(t, x_1, \ldots, x_n; \tau, y_1, \ldots, y_n) = \Phi(v, x_1, \ldots, x_n; y_1, \ldots y_n),$$

then we obtain from Eq. (10)

$$\left\{\frac{\partial}{\partial v} + Q(x_1, \ldots, x_n) - \sum_k g_k(x_1, \ldots, x_n)\frac{\partial}{\partial x_k}\right\} \tag{12}$$

$$\times \Phi(v, x_1, \ldots, x_n; y_1, \ldots, y_n) = \int\cdots\int w(x_1, \ldots, x_n; z_1, \ldots, z_n)$$

$$\times \Phi(v, z_1, \ldots, z_n; y_1, \ldots, y_n)\, dz_1, \ldots, dz_n,$$

$$g_k(x_1, \ldots, x_n) = \left(\partial f_k(x_1, \ldots, x_n; u)/\partial u\right)_{u=0},$$

where

$$w(x_1, \ldots, x_n; z_1, \ldots, z_n)\, dz_1, \ldots, dz_n$$

$$= Q(t_1, x_1, \ldots, x_n)\, dP(t, x_1, \ldots, x_n; z_1, \ldots, z_n),$$

$$Q(x_1, \ldots, x_n) = \int\cdots\int w(x_1, \ldots, x_n; z_1, \ldots, z_n)\, dz_1 \ldots dz_n.$$

Equation (10) or (12) is a generalization of Janossy's equation[8] and has the advantage that it avoids difficulties connected with the appearance of one and the same quantity several times in the equation. In the next section, we shall show that the kinetic equation, studied in detail by Janossy[8], is easily obtained from our equation.

3. Elementary particles in traversing matter lose their energy and deviate from their original direction. For the stochastic description of the motion of a particle we introduce the following random quantities: $\xi_1(t)$, the energy of the particle, $\xi_2(t)$, the angle between the tangent to the projection of the trajectory of the particle on the (x, z) plane and the x axis, $\xi_3(t)$, the distance of the particle from the x axis in the (x, z) plane, and $\xi_4(t)$, the area determined by the projection of the trajectory of the particle on the (x, z) plane and the x axis. The quantities $\xi_3(t)$ and $\xi_4(t)$ change continuously. From elementary considerations it follows that

$$f_3(x_1, \ldots, x_4; u) \tag{13}$$

$$= x_3 + u \operatorname{tg} x_2 \sim x_3 + u x_2;$$

$$f_4(x_1, \ldots, x_4; u) = x_4 + x_3 u$$

$$+ {}^1\!/_2 u^2 \operatorname{tg} x_2 \sim x_4 + x_3 u + {}^1\!/_2 u^2 x_2.$$

Using Eqs. (12) and (13), we can write

$$\left\{\frac{\partial}{\partial v} + Q(x_1) - x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_4}\right\}\Phi(v, \mathrm{x}, \mathrm{y}) = \int w(\mathrm{x}, \mathrm{z})\,\Phi(v, \mathrm{z}, \mathrm{y})\, d\mathrm{z}, \tag{14}$$

where $\mathrm{x} = \{x_1, \ldots, x_4\}$ is a four-dimensional vector. The cross section $w(x, z)$ has the form

$$w(\mathrm{x}, \mathrm{z}) = q(x_1, x_2; z_1, z_2) \times \delta(z_3 - x_3)\,\delta(z_4 - x_4). \tag{15}$$

Since $q(x_1, x_2; z_1, z_2)$ has a sharp maximum at the points $x_1 = z_1$ and $x_2 = z_2$ and its values are very small at other points, we can expand $\Phi(v, \mathrm{x}, \mathrm{y})$ in powers of $(z_1 - x_1)$ and $(z_2 - x_2)$. In this way we get the following equation:

$$\left\{\frac{\partial}{\partial v} - x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_4} + a(x_1)\frac{\partial}{\partial x_1} - {}^1\!/_2\, b(x_1)\frac{\partial^2}{\partial x_1^2} - {}^1\!/_2\,\sigma(x_1)\frac{\partial^2}{\partial x_2^2}\right\}\Phi(v, \mathrm{x}, \mathrm{y}) = 0, \tag{16}$$

where $a(x_1)$, $b(x_1)$ and $\sigma(x_1)$ are determined by the following relations:

$$a(x_1) = \iint (x_1 - z_1)\, q(x_1, x_2, z_1, z_2)\, dz_1\, dz_2,$$

$$b(x_1) = \iint (x_1 - z_1)^2\, q(x_1, x_2; z_1, z_2)\, dz_1\, dz_2,$$

$$\sigma(x_1) = \iint (x_2 - z_2)^2\, q(x_1, x_2; z_1, z_2)\, dz_1\, dz_2.$$

Equation (16) was studied in detail in the work of Janossy for the case $b(x_1) = 0$. If we take into consideration the variance of the energy loss, then it is more difficult to obtain a solution of (16). However, in the neighborhood of the mean value of the random quantity $\xi_1(t)$, the solution of equation (16) can be easily found. Indeed, if in the expressions $a(x_1)$, $b(x_1)$, $\sigma(x_1)$ we replace x_1 by the mean value of the random quantity $\xi_1(t)$, then going from the distribution function $\Phi(v, \mathrm{x}, \mathrm{y})$ to

[8] L. Janossy, J. Exptl. Theoret. Phys. (U.S.S.R.) **30**, 351 (1956).

the probability density $\varphi(v, \mathrm{x}, \mathrm{y})$, and assuming that $\varphi(v, \mathrm{x}, \mathrm{y}) = \psi(v, x_2, x_3, x_4; y_2, y_3, \nu_4) \times \chi(v, x_1; y_1)$, we obtain the following equation

$$\frac{\partial \chi(v, x_1; y_1)}{\partial v} = \frac{1}{2} b[\bar{x}_1(v)] \frac{\partial^2 \chi(v, x_1; y_1)}{\partial x_1^2} \quad (17)$$

$$- a[\bar{x}_1(v)] \frac{\partial \chi(v, x_1; y_1)}{\partial x_1}$$

Equation (17) can be easily solved[9]. With the initial condition $\chi(0, x_1; y_1) = \delta(y_1 - x_1)$ we find that

$$\chi(v, x_1; y_1) = B^{-1}(2\pi)^{-1/2} \quad (18)$$

$$\exp\{-(y_1 - x_1 - A)^2 / 2B^2\},$$

$$A = \int_0^v a[\bar{x}_1(s)]\, ds, \quad (19)$$

$$B^2 = \int_0^v b[x_1(s)]\, ds.$$

In the expressions for A and B^2, $x_1(s)$ designates the mean value of $\xi_1(s)$.

The determination of $\eta\ \nu_4)$ can be found in the work of Janossy[8].

For a practical application of the approximation in question we refer to one of some forthcoming papers.

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[9] L. Pal, Vestn. Moscow State University 6, 111 (1953)

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