[1]S. F. Edwards and R. E. Peierls, Proc. Roy. Soc. (London) 224A, 24 (1954).

[2]I. M. Gol'fand and R. A. Minlos, Dokl. Akad. Nauk SSSR 97, 209 (1954).

[3]N. N. Bogoliubov, Dokl. Akad. Nauk SSSR 99, 225 (1954).

[4]P. Mathews and A. Salam, Nuovo Cimento 12, 563 (1954).

[5]K. Symanzik, Z. Naturforsch 9a, 809 (1954).

[6]E. S. Fradkin , Dokl. Akad. Nauk SSSR 98, 47 (1954).

[7]N. P. Klepikov, Dokl. Akad. Nauk SSSR 100, 1057 (1955).

[8]S. F. Edwards, Proc. Roy. Soc. (London) 228A, 411 (1955).

[9]F. Bloch and A. Nordsieck, Phys. Rev. 52, 54 (1937).

[10]V. A. Fock, Z. Physik Sowjetunion 12, 404 (1937).

[11]W. Pauli and F. Villars, Rev. Mod. Phys. 21, 434 (1949). Russian translation in the volume "Shifts in the Energy Levels of Atomic Electrons" IIL (1950) page 139.

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A Method of Obtaining Nonstationary Solutions of Boltzmann's Kinetic Equation

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A method is considered for obtaining nonstationary solutions of Boltzmann's kinetic equation. This method is free from the limitations of the Chapman-Enskog method. By way of an example, the dispersion of plane sound waves in a monatomic gas is considered.

1. IN recent times the most widely used method of solution of the Boltzmann equation has been the method of Chapman-Enskog[1]. However, this method is not applicable for a series of problems. Let us consider, for example, one such problem -- the problem of the dispersion of plane sound waves in a monatomic gas without consideration of the internal degrees of freedom of the atoms. Let the frequency of vibration of the external source be sufficiently low in comparison with the "characteristic frequency" of the gas, i.e., in comparison with the mean frequency of atomic collisions. Such a problem can be solved by making use of the equations of Navier-Stokes and Burnett[2], i.e., by making use of different approximations than those of the Chapman-Enskog method. If the frequency of vibration of the external source is comparable with or larger than the "characteristic frequency" of the gas, then these approximations lose their meaning. Actually, we use as the small parameter of the method of successive approximation employed in obtaining the equations of Navier-Stokes, Burnett, the ratio $\Delta t_p/\Delta t$, where Δt is a characteristic time interval for the process under consideration, for example, the period of vibration, Δt_p is the relaxation time of the gas. Thus we have as the condition of applicability of the Chapman-Enskog method the relation

$$\Delta t_p / \Delta t \ll 1. \quad (1)$$

Another method is necessary, consequently, to obtain a solution differing widely from the quasi-equilibrium solution of the Boltzmann equation. Such a possibility is given by the method of "moments" (see, for example, Ref. 3).

Let us formulate a modification of this method. We use as the zeroth approximation the stationary solution of the Boltzmann equation. Then the first approximation gives the deviation of the density, velocity and temperature from the stationary distribution, and also the corresponding viscous force and heat flow. As conditions for the application of this method of small perturbations we have the relations

$$\frac{\Delta\rho}{\rho_0}; \quad \frac{\Delta u_i}{c_0}; \quad \frac{\Delta\Theta}{\rho_0 c_0^2}; \quad \frac{p_{ij}}{\rho_0 c_0^2}; \quad \frac{S_i}{\rho_0 c_0^3} \ll 1, \quad (2)$$

where ρ_0, $\Theta_0 = kT_0 = mc_0^2$ are, respectively, the particle density and temperature in the stationary state; m is the mass of an atom of the gas; k is the Boltzmann constant; $\Delta\rho$, Δu_i, $\Delta\Theta$, p_{ij}, S_i are the deviations of the density, velocity and temperature from the stationary state and also the viscosity tensor and the heat flow (first approximation).

Thus we have the kinetic equation of Boltzmann:

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} = F_B + \frac{1}{m} \frac{\partial f}{\partial v_i} \frac{\partial V}{\partial x_i}, \quad (3)$$

where F_B is the Boltzmann collision integral, V is the potential of the external field. We set

$$f(\mathbf{r}, \mathbf{v}, t) = f_0(\mathbf{r}, \mathbf{v}) + f_1(\mathbf{r}, \mathbf{v}, t) + \ldots,$$

$$V(\mathbf{r}, t) = V_0(\mathbf{r}) + V_1(\mathbf{r}, t) + \ldots. \quad (4)$$

Then, substituting (4) in (3), we get a system of equations of successive approximations of the described method:

$$v_i \frac{\partial f_0}{\partial x_i} = F_B(f_0) + \frac{1}{m} \frac{\partial f_0}{\partial v_i} \frac{\partial V_0}{\partial x_i}; \quad (5)$$

$$\frac{\partial f_1}{\partial t} + v_i \frac{\partial f_1}{\partial x_i} = F_B(f_0, f_1) + \frac{1}{m} \frac{\partial f_1}{\partial v_i} \frac{\partial V_0}{\partial x_i} + \frac{1}{m} \frac{\partial f_0}{\partial v_i} \frac{\partial V_1}{\partial x_i}.$$

The normalization conditions for the function f are written in the following way:

$$\rho = \int f d\mathbf{v}; \quad (6)$$

$$u_i = \frac{1}{\rho} \int v_i f d\mathbf{v}; \quad \Theta + \frac{mu^2}{2} = \frac{1}{\rho} \int \frac{mv^2}{2} f d\mathbf{v};$$

$$\Pi_{ij} = \frac{1}{\rho} \int m v_i v_j f d\mathbf{v}; \qquad S_i = \frac{1}{\rho} \int v_i \frac{mv^2}{2} f d\mathbf{v}.$$

From Eqs. (4) and (6) there stems the possibility of the following normalization for the functions of the different approximations:

$$\rho_0(\mathbf{r}) = \int f_0 d\mathbf{v}; \quad \Delta\rho(\mathbf{r}, t) = \int f_1 d\mathbf{v}; \ldots \quad (7)$$

$$\rho_0 u_{0i} = \int v_i f_0 d\mathbf{v}; \quad u_{0i} \Delta\rho + \rho_0 \Delta u_i = \int v_i f_1 d\mathbf{v}, \ldots$$

with similar relations for Θ, Π_{ij}, S_i.

In many problems (in particular, in the problem noted above on the propagation of sound waves) we can set $V_0 \equiv 0$. Then the equation of zeroth approximation (5), with account of (7), has the well-known solutions of Maxwell:

$$f_0 = \rho_0 c_0^{-3} (2\pi)^{-3/2} \exp\{-v^2/2c_0^2\}, \quad (8)$$

where ρ_0, c_0 are constants, $u_{0i} \equiv 0$.

We now write the equation of the first approximation of (5), introducing dimensionless functions and variables:

$$g_0(\xi) = (2\pi)^{-3/2} e^{-\xi^2/2}; \quad f_0(\mathbf{v}) = \rho_0 c_0^{-3} g_0(\xi);$$
$$f_1(\mathbf{r}, \mathbf{v}, t) = \rho_0 c_0^{-3} g_1(\mathbf{r}, \xi, t); \quad \xi = \mathbf{v}/c_0. \quad (9)$$

We get*

$$\frac{\partial g_1}{\partial t} + c_0 \xi_i \frac{\partial g_1}{\partial x_i} = F_B(g_0, g_1) + \frac{1}{mc_0} \frac{\partial g_0}{\partial \xi_i} \frac{\partial V_1}{\partial x_i}. \quad (10)$$

We shall seek the solution of (10) in the form of an expansion in Hermite polynomials:

$$g_1 = g_0 \sum_{n=0}^{\infty} \sum_{i, j, \ldots} \frac{1}{n!} \alpha_{i, j, \ldots}^{(n)} H_{i, j, \ldots}^{(n)} \quad (11)$$

where the coefficients α are unknown functions, $\mathbf{r}$, t, $H_{i,j,\ldots}^{(n)}$ are the orthonormal Hermite polynomials, for example,

$$H^{(0)} = 1; \quad H_{i,j}^{(2)} = \xi_i \xi_j - \delta_{ij};$$
$$H_i^{(1)} = \xi_i; \quad H_{i,j,k}^{(3)} = \xi_i \xi_j \xi_k - \xi_i \delta_{jk} - \xi_j \delta_{ik} - \xi_k \delta_{ij}.$$

Because of the orthonormality of these polynomials, we have the relation

$$\frac{1}{n!} \int g_0 H^{(m)} H^{(n)} d\xi = \delta_{mn}; \quad \alpha^{(m)} = \int H^{(m)} g_1 d\xi. \quad (12)$$

Here the indices $i, j, \ldots$ are omitted.

Substituting (11) in (10), we obtain a system of linear differential equations for the unknown functions $\alpha^{(m)}$. This same system can be obtained by multiplying Eq. (10) by $H^{(m)}$ and integrating over ξ. We then get

$$\int H^{(m)} \frac{\partial g_1}{\partial t} d\xi + c_0 \int \xi_i H^{(m)} \frac{\partial g_1}{\partial x_i} d\xi \quad (13)$$
$$= \int H^{(m)} F_B(g_0, g_1) d\xi.$$

We get for the first term in Eq. (13) [keeping Eq. (12) in mind]

$$\int H^{(m)} \frac{\partial g_1}{\partial t} d\xi = \frac{\partial \alpha^{(m)}}{\partial t}. \quad (14)$$

To transform the second term in Eq. (13), we make use of relations of the form

* In view of the fact that the term which takes into account the external field is not essential in what follows, it is omitted.

$$H_i^{(1)} H^{(m)} = H_i^{(m+1)} + \delta_i H^{(m-1)}. \quad (15)$$

Using Eqs. (11), (12) and (15), we get

$$c_0 \frac{\partial}{\partial x_i} \int H_i^{(1)} H^{(m)} g_1 \, d\xi = c_0 \frac{\partial}{\partial x_i} \{\alpha_i^{(m+1)} + \delta_i \alpha^{(m-1)}\}. \quad (16)$$

We can write the right hand side of Eq. (13) in the form

$$\int H_{i,j,\ldots}^{(m)} F_B(g_0 g_1) \, d\xi = c_0 \rho_0 \sum_{n=2}^{\infty} \sum_{i,j,\ldots} \frac{1}{n!} \alpha_{i,j,\ldots}^{(n)} \int d\xi g_0(\xi) H_{i,j,\ldots}^{(m)}(\xi) \int d\xi_1 g_0(\xi_1) Y_{i,j,\ldots}^{(n)}$$

$$Y_{i,j,\ldots}^{(n)} = |\xi_1 - \xi| \int_0^\infty \varepsilon d\varepsilon \int_0^{2\pi} d\varphi \{H_{i,j,\ldots}^{(n)}(\xi_1')$$

$$+ H_{i,j,\ldots}^{(n)}(\xi') - H_{i,j,\ldots}^{(n)}(\xi_1) - H_{i,j,\ldots}^{(n)}(\xi)\} \, d\varphi \quad (17)$$

if we apply Eq. (11) and the laws of conservation of energy and momentum. The integrals $Y_{i,j,\ldots}^{(n)}$ were computed (see, for example, Ref. 3), whereupon we obtained

$$Y_{ij}^{(2)} = \frac{C}{c_0} \{w^2 \delta_{ij} - 3 w_i w_j\},$$

$$Y_{ijk}^{(3)} = \frac{C}{2c_0} \{w^2 [W_k \delta_{ij} + W_j \delta_{ik} + W_i \delta_{jk}]$$

$$- 3 [w_i w_k W_j + w_i w_j W_k + w_j w_k W_i]\}, \quad (18)$$

where

$$C = c_0 2\pi w \int_0^\infty \varepsilon d\varepsilon \sin^2 \theta \cos^2 \theta,$$

$$w = |\xi_1 - \xi|; \quad W = |\xi_1 + \xi|.$$

Further calculations are carried out only for Maxwell molecules with interaction action $\Phi(r) = br^{-4}$. Then

$$C = 0.6841 \sqrt{8b/m}. \quad (19)$$

Further simple calculations lead to the following values of the integrals in Eq. (17):

$$\int d\xi_1 g_0(\xi_1) Y_{ij}^{(2)} = \frac{C}{c_0} \{\delta_{ij} \operatorname{Sp} H_{ii}^{(2)} - 3 H_{ij}^{(2)}\}; \quad (20)$$

$$\int d\xi_1 g_0(\xi_1) Y_{ijk}^{(3)} = \frac{C}{2c_0} \{9 H_{ijk}^{(3)} - H_i^{(3)} \delta_{ik} - H_j^{(3)} \delta_{ik} - H_k^{(3)} \delta_{ij}\},$$

where, for example, $H_1^{(3)} = H_{111}^{(3)} + H_{122}^{(3)} + H_{133}^{(3)}$.

Now we can write down for the unknown functions $\alpha^{(m)}$ a set of differential equations which are equivalent to the Boltzmann equation of the first approximation (10). Actually, taking into account Eqs. (13), (14), (16), (17) and (20), we get, with accuracy to terms of fourth order:

$$\frac{\partial \alpha^{(m)}}{\partial t} + c_0 \frac{\partial}{\partial x_i} \{\alpha_i^{m+1} + \delta_i \alpha^{(m-1)}\} = C\rho_0 \sum_{i,j,k,\ldots} \Big\{ \alpha_{ij}^{(2)} \frac{1}{2!} \int d\xi g_0 H^{(m)} [\delta_{ij} \operatorname{Sp} H_{ii}^{(3)} - 3 H_{ij}] - \frac{\alpha_{ijk}^{(3)}}{2} \frac{1}{3!} \quad (21)$$

$$\times \int d\xi g_0 H^{(m)} [9 H_{ijk}^{(3)} - H_i^{(3)} \delta_{jk} - H_j^{(3)} \delta_{ik} - H_k^{(3)} \delta_{ij}] \Big\}.$$

We describe these equations in more detail for the very simple case of one dimensional flow. Here, we choose as unknown functions five coefficients $\alpha^{(n)}$ from (11):

$$\alpha^{(0)}, \; \alpha_1^{(1)}, \; \alpha_{11}^{(2)}, \; \alpha_{22}^{(2)} = \alpha_{33}^{(2)}, \; \alpha_{111}^{(3)} = \alpha_{122}^{(3)} = \alpha_{133}^{(3)}. \quad (22)$$

The physical meaning of these functions is made clear from conditions (6) and (7), which, along with Eq. (12), give

$$\alpha^{(0)} = \Delta\rho / \rho_0, \quad \alpha_i^{(1)} = \Delta u_i / c_0,$$

$$\operatorname{Sp} \alpha_{ii}^{(2)} = \Delta\Theta / \rho_0 c_0^2;$$

$$\tfrac{1}{2} (\alpha_{ij} + \alpha_{ji}) = p_{ij} / \rho_0 c_0^2;$$

$$\alpha_{iii}^{(3)} = 3 S_i / 5 \rho_0 c_0^3 - 3 \Delta\rho / \rho_0;$$

$$(\Pi_{ij} = p + p_{ij} + \ldots;$$

$$S_i = S_i^0 + S_i + \ldots; \; S_i^{(0)} = 0).$$

Keeping in mind (22) and (12), we get from (21) the following system:

$$\frac{\partial \alpha^{(0)}}{\partial t} + c_0 \frac{\partial \alpha^{(1)}}{\partial x} = 0; \quad \frac{\partial \alpha^{(1)}}{\partial t} + c_0 \frac{\partial}{\partial x} (\alpha_{11} + \alpha^{(0)}) = 0;$$

$$\frac{\partial \alpha_{11}}{\partial t} + c_0 \frac{\partial}{\partial x} (2\alpha^{(1)} + \alpha^{(3)}) = 2C\rho_0 (\alpha_{22} - \alpha_{11});$$

$$\frac{\partial \alpha_{22}}{\partial t} + c_0 \frac{\partial}{\partial x} \frac{1}{3} \alpha^{(3)} = C\rho_0 (\alpha_{11} - \alpha_{22}); \quad (23)$$

$$\frac{\partial \alpha^{(3)}}{\partial t} + c_0 \frac{\partial}{\partial x} \frac{3}{5} (3\alpha_{11} + 2\alpha_{22}) = -2C\rho_0 \alpha^{(3)}$$

by setting $H^{(m)}$ successively equal to

$$H^{(m)} = H^{(0)};\; H_1^{(1)};\; H_{11}^{(2)};\; H_{22}^{(2)} + H_{33}^{(2)};\; H_{111}^{(3)} + H_{122}^{(3)} + H_{133}^{(3)}.$$

(The indices which are superfluous for the one dimensional case have been omitted.)

The system (23) determines the functions $\alpha^{(n)}$ if the corresponding initial and boundary conditions are given. A solution of the Boltzmann equation in the form (11) is found by the same approximation. This solution will no longer be limited by the condition (1). Thus the method considered here of small perturbations is free from the limitation of the Chapman and Enskog method.

2. As an example, let us consider the problem of the dispersion of plane sound waves in a monatomic gas without regard to internal degrees of freedom of the gas atoms. This problem has been considered by a number of authors (see Refs. 4-7). The starting point for these researches is the Burnett equation[2]. Thus in these researches there were analyzed only the quasi-equilibrium solutions, the limitation of which follows from the requirement (1) of the method of Chapman-Enskog.

We consider this problem, not limiting ourselves to quasi-equilibrium solutions. For this purpose, we employ the system of equations (23), which is suitable for the one dimensional flow of gas. We write the corresponding system for the Fourier components of the unknown functions $\alpha^{(n)}$ from (23) as

$$\begin{aligned} &U\alpha^{(0)} + \alpha^{(1)} = 0,\; U\alpha^{(1)} + \alpha_{11} + \alpha^{(0)} = 0; \qquad (24)\\ &U\alpha_{11} + 2\alpha^{(1)} + \alpha^{(3)} = -\, i2U_0(\alpha_{22} - \alpha_{11});\\ &U\alpha_{22} + {}^1/_3\,\alpha^{(3)} = -\, iU_0(\alpha_{11} - \alpha_{22});\\ &U\alpha^{(3)} + {}^9/_5\,\alpha_{11} + {}^6/_5\,\alpha_{22} = i2U_0\alpha^{(3)}. \end{aligned}$$

Here $U = \omega/kc_0$, $U_0 = \omega_0/kc_0$, $\omega_0 = c\rho_0$; $\alpha_{ij}^{(n)}$ denotes the Fourier components of the corresponding functions from (23), ω is the vibration frequency, k the wave vector.

Equating the determinant of this system to zero, we obtain the necessary dispersion equation

$$(U^5 - {}^{26}/_5\,U^3 + 3U) - iU_0(5U^4 - 16U^2 + 5) - 2U_0^2U(3U^2 - 5) = 0. \qquad (25)$$

We now assume that ω is a given quantity and introduce the variable $\mu = \omega/\omega_0$.

The nontrivial solutions of this equation are

$$U = \pm\left[\frac{1}{2A}\left(-B \pm \sqrt{B^2 - 4AD}\right)\right]^{1/2}.$$

$$A = \mu^2 - 6 - i5\mu,$$

$$B = 10 - {}^{26}/_5\,\mu^2 + i16\mu,\quad D = 3\mu^2 - i5\mu. \qquad (26)$$

We write them in the form*

$$U_+ = \alpha_+ + i\beta_+,\quad U_- = \alpha_- + i\beta_-.$$

Introducing the phase velocity c and the absorption coefficient $\varkappa$ with the aid of the formulas

$$c = \omega/\mathrm{Re}\,k,\quad \varkappa = \mathrm{Im}\,k,$$

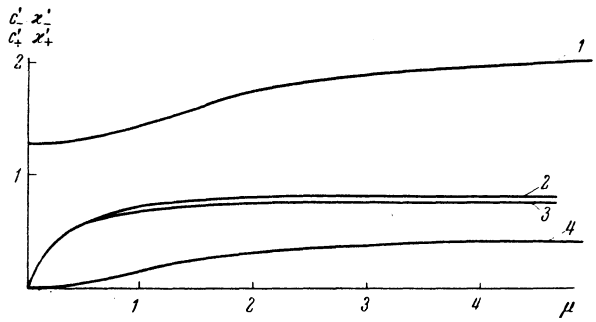
we finally obtain the solution of the dispersion equation for dimensionless velocity and absorption:

$$c'_\pm = c_\pm/c_0 = \varphi_1(\mu);\quad \varkappa'_\pm = \varkappa_\pm c_0/\omega_0 = \varphi_2(\mu),$$

$$\varphi_1(\mu) = [(\alpha_\pm)^2 + (\beta_\pm)^2]/\alpha_\pm;$$

$$\varphi_2(\mu) = \mu\beta_\pm/[(\alpha_\pm)^2 + (\beta_\pm)^2]. \qquad (27)$$

Numerical evaluation leads to the results plotted in the Figure.



Curve $1 - c'_-$, $2 - c'_+$, $3 - \varkappa'_+$, $4 - \varkappa'_-$

To evaluate the results we need approximate formulas for the case $\mu \ll 1$:

$$c' = \sqrt{5/3}\,(1 + 0.1194\mu^2);\quad c'_+ = \sqrt{\mu}\,(1 - 0.3500\mu),$$
$$\varkappa'_- = 0.1807\,\mu^2;\quad \varkappa'_+ = \sqrt{\mu}\,(1 - 0.3500\,\mu). \qquad (28)$$

Thus the dispersion equation (25) has two solutions for the velocity and attenuation of small perturbations of the neutral gas.

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* The sign ± in front of the general square root has been omitted, since the choice of this sign is connected with the choice of the direction of propagation of the wave.

Let us consider the solution c'_-, $\varkappa'_-$ (see the Figure). As $\omega \to 0$, it undergoes transition into expressions known from the theory of continuous media: $c'_- = \sqrt{5/3}$, $\varkappa'_- = 0$. Hence we obtain the usual expression $c = (5RT/3M)^{1/2}$ for the sound velocity in a monatomic gas.

The solution just obtained contains in it the higher approximations of the Chapman-Enskog method (for example, the second approximation of Stokes-Navier, the third approximation of Burnett). In order to be convinced of this, we compare our results with the corresponding theoretical[4-7] and experimental results[7] (see the Table). Thus the solution actually contains in it, as a special case, the various hydrodynamic approximations. The connection assumed in hydrodynamics between the pressure and the velocity gradient is not assumed in our method but is obtained in a special case ($\omega \to 0$).

Comparison of Theoretical values of sound velocity with Experimental
(for argon at 0° and sound velocity $\omega/2\pi$ = 970.68 Kc;
a = 308.23 m/sec.

pressure p in cm Hg	Values of c/a (m/sec). experimental	according to Nanier-Stokes	according to Burnett	according to Eq. (27).
78.40	1	1	1	1
47.10	1	1	1	1
22.30	1	1	1	1
13.50	1	1	1	1
9.10	1.002	1	1	1
4.30	1.006	1	1	1.001
3.30	1.008	1.001	1.001	1.002
2.45	1.008	1.001	1.002	1.003
1.80	1.016	1.002	1.002	1.005
1.50	1.028	1.003	1.005	1.007
0.65	1.058	1.019	1.028	1.043
0.34	1.139	1.054	1.103	1.144
0.22	1.259	1.139	1.246	1.281

Increase in the vibration frequency leads to a comparatively large increase in the sound propagation velocity and to a simultaneous increase in the absorption. For sufficiently high frequencies ($\omega \gtrsim 5\omega_0$), $1/\varkappa_- \sim 2\lambda$ (λ = mean free path length). Consequently, this solution gives an analytic description of the Lebedev effect[8] which consists in the presence of a frequency limit for the propagation of ultrasound in neutral gases. This limit is a function of the gas density: the lower the gas density, the lower the frequencies of vibration that cannot be propagated in the gas.

Let us consider the second solution c. The fact of two solutions has already been pointed out in the literature (see, for example, Ref. 3). However, in rarefied gases, only a single wave is observed for small disturbances. This corresponds to the form of the solution c_+', $\varkappa'_+$. Actually, the solution c_+', $\varkappa'_+$ does not give propagating waves in practice: $c'_+ \approx \varkappa'_+$ and the damping $\varkappa'_+$ for $\omega \to 0$ tends to zero as $(\omega/\omega_0)^{1/2}$, while $\varkappa'_-$ tends to zero as $(\omega/\omega_0)^2$.

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[1] S. Chapman and T. Cowling, *The mathematical theory of nonuniform gases*, Cambridge, 1939.

[2] D. Burnett, Proc. Lond. Mathem. Soc. **39**, 385 (1935); **40**, 382 (1935).

[3] H. Grad, Comm. Pure Appl. Math. **2**, 407 (1949).

[4] H. Primakoff, J. Acoust. Soc. Am. **14**, 14 (1942).

[5] H. S. Tsein and R. Schamberg, J. Acoust. Soc. Am. **18**, 334 (1946).

[6] R. B. Lindsay, Am. J. Phys. **16**, 371 (1948).

[7] R. Boyer, J. Acoust. Soc. Am. **23**, 176 (1951).

[8] P. N. Lebedev, Ann. Physik **35**, 171 (1911).

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45