Some Relativistic Properties of the Behavior of Spin-½ Particles

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It is shown that in the case of a scalar field the Klein paradox does not arise for spin-½ particles tunneling through a potential barrier. The particle does not fall toward the center of a centrally symmetric field even when a high-order pole exists in the center. This result also holds in classical relativistic theory.

1\. FOR a scalar interaction, the interaction potential between a particle and the field is an invariant. This is not the case for an electrostatic (vector) interaction, when the potential is the fourth component of a four-vector. Therefore, the Dirac equation for a spin-½ particle in a scalar field can be written

$$\left\{E - i\left(\alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y} + \alpha_3 \frac{\partial}{\partial z}\right) + \rho_3 (E_0 + U)\right\} \psi = 0. \quad (1)$$

Here E is the total energy of the particle, E_0 is its rest energy U is the potential energy of the particles in the scalar field, and α_1, α_2, α_3, and ρ_3 are the Dirac matrices; Planck's constant and the velocity of light are set equal to unity.

We shall consider one-dimensional motion of a particle along the x axis in a field which can be represented by a square potential barrier of the form

$$U = 0\,(x < 0), \quad U = U_0\,(x > 0). \quad (2)$$

In this case, as usual, we look for a solution of (1) in the form of plane waves:

$$\begin{aligned} \psi_i &= a_i e^{ip_1 x} + b_i e^{-ip_1 x} \quad (x < 0), \\ \psi_i &= c_i e^{ip_2 x} \quad (x > 0). \end{aligned} \quad (3)$$

Inserting these solutions into the equation gives the dependence of the particle momentum on its energy, namely,

$$p_1^2 = E^2 - E_0^2, \quad p_2^2 = E^2 - (E_0 + U_0)^2. \quad (4)$$

It is clear that the reflection coefficient is equal to unity when p_2 is pure imaginary, or when $E < E_0 + U_0$. In the case of a vector (electrostatic) interaction, p_2 is of the form

$$p_2^{(\text{vect})} = \sqrt{(E - U_0)^2 - E_0^2} \quad (5)$$

and is imaginary when $E - E_0 < U_0 < E + E_0$. It is seen from this that for a scalar interaction, the particles will not tunnel through a sufficiently high barrier (the Klein paradox will not occur).

A no less "paradoxical" phenomenon, however, does take place for the scalar interaction. In this case p_2 becomes imaginary and the reflection coefficient becomes unity for a sufficiently high negative value of U_0, when $|U_0| < E + E_0$. In other words, not only is it impossible for the particle to pass through a sufficiently high barrier, but it is also impossible for it to penetrate into a sufficiently deep well.

These results remain valid not only in the case of a square barrier, but also for a "smoothed out" barrier. Let us consider, for instance, the behavior of a particle in a field whose potential for instance, the behavior of a particle in a field whose potential is of the form

$$U(x) = V/(1 + e^{-ax}), \; V = \text{const}. \quad (6)$$

In this case it is possible to find an exact solution for the Dirac equation (1) in terms of hypergeometric functions. This solution is of the form

$$\psi_1 = \psi_2 = cy^{\mu}\left[\frac{(1-y)^{\lambda}}{\sqrt{\alpha_1\beta_1}} F(\alpha_1, \beta_1, \gamma_1; y) + \frac{(1-y)^{-\lambda}}{\sqrt{\alpha_2\beta_2}} F(\alpha_2, \beta_2, \gamma_2; y)\right],$$
$$\psi_3 = \psi_4 = -icy^{\mu}\left[\frac{(1-y)^{\lambda}}{\sqrt{\alpha_1\beta_1}} F(\alpha_1, \beta_1, \gamma_1; y) + \frac{(1-y)^{-\lambda}}{\sqrt{\alpha_2\beta_2}} F(\alpha_2, \beta_2, \gamma_2; y)\right], \quad (7)$$

where $F(\alpha, \beta, \gamma; y)$ are hypergeometric functions, c is a constant,

$$y = -e^{-ax}; \quad \mu^2 = a^{-2}[(E_0+V)^2 - E^2];$$
$$\nu^2 = a^{-2}[E_0^2 - E^2]; \quad \lambda = -\nu/a;$$
$$\alpha_1 = \mu+\nu+\lambda; \quad \alpha_2 = \mu+\nu-\lambda; \quad (8)$$
$$\beta_1 = \mu-\nu+\lambda;$$
$$\beta_2 = \mu-\nu-\lambda; \quad \gamma_1 = \gamma_2 = 2\mu+1.$$

Making use of the asymptotic expressions for the the hypergeometric functions, as well as several of their properties, we can calculate the reflection coefficient, which is found to be

$$R = \left|\frac{\Gamma(2\nu)\Gamma(\mu-\nu+\lambda)\Gamma(\mu-\nu-\lambda)}{\Gamma(-2\nu)\Gamma(\mu+\nu+\lambda)\Gamma(\mu+\nu-\lambda)}\right|^2 \times \frac{|(\mu-\nu)^2-\lambda^2|^2}{|(\mu+\nu)^2-\lambda^2|^2}. \quad (9)$$

Since λ is real and ν is pure imaginary, it follows from (9) that the reflection coefficient $R=1$ when μ is real. It follows from Eq. (8) that, independent of the magnitude of a, which determines the "steepness" of the potential energy curve, the particle can neither tunnel through a barrier of height $V < E - E_0$ nor penetrate into a well of depth $|V| < E + E_0$.

For comparison, we remark that in the case of a vector field, the Klein paradox arises for a "smoothed-out" barrier only if the potential energy increases to $2E_0$ in a distance which is smaller than the Comp-wavelength of the particle.

2. It is known that in the nonrelativistic quantum mechanical treatment of a spin-½ particle in a centrally symmetric field, no stable states can take place if the potential in the neighborhood of the attracting center increases as r^{-n}, where $n<2$ (here r is the distance from the center). This result remains valid also in the relativistic treatment if the interaction is electromagnetic (of the vetor type). A different result is obtained for a scalar interaction.

Consider a particle in a scalar centrally symmetric field whose potential energy is given by

$$U = a/r^n. \quad (10)$$

Inserting (10) into the Dirac equation (1) and separating the radius and angle variables, we obtain the following radial equations:

$$(E+E_0-U)f + (dg/dr) - lg/r = 0,$$
$$-(E-E_0+U)g + (df/dr) + (l+2)f/r = 0, \quad (11)$$

where f and g are functions which depend only on r, and l is the azimuth quantum number. The double degeneracy of the Dirac equation lies in the fact that there exists another pair of radial equations which can be obtained from the first by replacing l by $-(l+1)$. The investigation of this second set of equations is unnecessary, since it is exactly analogous to the treatment presented below.

By eliminating one of the functions from (11), we obtain the second order equation

$$\frac{d^2f}{dr^2} + \left(\frac{2}{r} - \frac{dU/dr}{E-E_0+U}\right)\frac{df}{dr} + \left[E^2 - (E_0-U)^2 - \frac{l+2}{r}\frac{dU/dr}{E-E_0+U} + \frac{(l+1)(l+2)}{r^2}\right]f = 0. \quad (12)$$

The asymptotic behavior of the solutions of this equation for small r is given by the equation

$$\frac{d^2f}{dr^2} + \frac{n+2}{r}\frac{df}{dr} - \frac{a^2}{r^{2n}}f = 0. \quad (13)$$

The solution of Eq. (13) can be written in terms of Bessel functions

$$f = r^{-(n+1)/2} I_{(n+1)/2(n-1)}\left(\pm\frac{ia}{1-n}r^{1-n}\right), \quad (14)$$

and by making use of the asymptotic behavior of the Bessel functions for small r, we obtain

$$f \sim \exp\left\{\pm\frac{a}{1-n}\frac{1}{r^{n-1}}\right\}. \quad (15)$$

which is accurate up to a factor multiplying the exponential.

Thus when $r = 0$, one of the solutions becomes infinite and the other vanishes. The same result is easily obtained in solving the Dirac equation (in spherical coordinates) in the absence of a field. The fact that the number of finite solutions for $r=0$ does not change when a field is introduced indicates that in this field the particle does not fall into the attractive center.

3. It can be shown that the result of the previous section is not a quantum effect, but holds also in the classical (relativistic) theory.

In the quantum mechanical treatment, the energy of the scalar interaction between the particle and the field is given by a term of the form $\rho_3 U(r)$, where ρ_3 is a Dirac matrix. It is known that in the classical treatment this matrix corresponds to the factor $\sqrt{1-\beta^2}$, where β is the velocity of the particle (in the units being used).

Choosing the interaction energy between the particle and the field in the form

$$U = -ar^{-n}\sqrt{1-\beta^2}, \qquad (16)$$

when n is arbitrary, we obtain the following expression for the classical (relativistic) Lagrangian:

$$L = -E_0\sqrt{1-\beta^2} + ar^{-n}\sqrt{1-\beta^2}. \qquad (17)$$

Here E_0 is the rest energy of the particle, and a is the coupling constant between the field and the particle. Since this is a spherically symmetrical problem, it is clear that the total angular momentum is conserved. If the z axis is directed along the total angular momentum, the trajectory of the particle is in the xy plane. In cylindrical coordinates with the origin at the force center of the field, $\beta^2 = \dot{r}^2 + r^2\dot{\varphi}^2$, and since φ is a cyclic coordinate, the generalized momentum p_φ conjugate to it is conserved:

$$p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = \left(E_0 - \frac{a}{r^n}\right)\frac{r^2\dot{\varphi}}{\sqrt{1-\beta^2}} = \text{const}. \qquad (18)$$

Using the expression for p_φ and

$$p_r = \frac{\partial L}{\partial\dot{r}} = \left(E_0 - \frac{a}{r^n}\right)\dot{r}/\sqrt{1-\beta^2},$$

we find the second integral of the motion, namely the total energy of the particle in the field:

$$H = p_r\dot{r} + p_\varphi\dot{\varphi} - L \qquad (19)$$
$$= (E_0 - ar^{-n})/\sqrt{1-\beta^2} = \text{const}.$$

Further, eliminating $\dot{\varphi}$ from Eqs. (18) and 19 and making use of the expression for β^2, it is easy to obtain

$$\dot{r}^2 = 1 - H^{-2}[(E_0 - ar^{-n})^2 + p_\varphi^2/r^2]. \qquad (20)$$

It is immediatedly seen from this equation that the particle can not approach very close to the attactive center, since for sufficiently small r the right side of expression (20) becomes negative, whereas the left side is an essentially positive quantity. This result remains true for any sign of a, so that the particle can not fall into the center whether it be repulsive or attactive. From expression (20) it is also seen that in the attractive case the particle in the scalar field will move in closed trajectories when $H = E_0$. It is thus possible, in spite of a high order pole at the center, for the particle to have a stable state in the field.

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118