Motion of a Rarefied Plasma in a Variable Magnetic Field

IA. P. TERLETSKII
Moscow State University
(Submitted to JETP editor December 20, 1956)
J. Exptl. Theoret. Phys. (U.S.S.R.) 32, 927-928 (April, 1957)

CONSIDER A RAREFIED PLASMA in a quasistationary magnetic field which varies in time. Let us limit ourselves to the case where the free path length of the electrons and ions may be considered as great as we like and the magnetic field **H** is intense enough to fulfill always the condition

$$\rho\,|\nabla H|\,/\,H \ll 1, \quad (1)$$

where $\rho = p_\perp c/eH$ is the radius of curvature of the trajectory of a free charged particle in the magnetic field, $p_\perp$ is the component of momentum of the particle perpendicular to the direction of the magnetic field, e is the charge of an electron, and c is the velocity of light. It is assumed that condition (1) is true for all electrons and all ions.

In the case under consideration, in accordance with Ref. 1–3, the motion of the electrons and ions of the plasma may be represented as circular motion of radius ρ around the direction of the magnetic field with cyclical frequency $\Omega = eH/mc$, while the center of rotation (we will call the point representing this center of rotation the drifting center) drifts in a direction perpendicular to the magnetic field with velocity

$$\mathbf{v}_\perp = c[\mathbf{E} \times \mathbf{H}]\,/H^2 - (c\mu/eH^2)[\nabla_\perp H \times \mathbf{H}] \quad (2)$$

and in a direction parallel to the magnetic field with a velocity determined by the equation

$$d\,(m v_\parallel)\,/\,dt = e\,E_\parallel = (c\mu\,/\,|e|)\,\nabla_\parallel H, \quad (3)$$

where $\mu = |e|\,\Omega\rho^2/2c$ is the magnetic moment produced by the particle rotation with frequency Ω, $\nabla_\perp$ and $\nabla_\parallel$ are the components of the gradient perpendicular and parallel to the magnetic field, and m is the mass of the particle. As a consequence of the conservation of the adiabatic invariant $\oint(m\mathbf{v} + e\mathbf{A}/c)d\mathbf{l}$, the magnitude of $\Omega\rho^2$ remains unchanged[2] in the case of a sufficiently slowly changing magnetic field ($H^{-1}\,|\,dH/dt\,| \ll \Omega/2\pi$). Hence

$$\mu = e\Omega\rho^2\,/\,2c = e^2H\rho^2\,/\,2mc^2 = \text{const}, \quad (4)$$

whence the energy of the rotational motion of the particle* is

$$W = p_\perp^2\,/\,2\,m = m\Omega^2\rho^2\,/\,2 = \mu H. \quad (5)$$

The drift in a direction perpendicular to H with velocity

$$\mathbf{u} = c[\mathbf{E} \times \mathbf{H}]/H^2, \quad (6)$$

may be considered as a motion "together with the lines of force," just as in the case of magnetic hydrodynamics[4]. Actually, if we determine the "velocity of motion of the lines of force," starting from the equation

$$\frac{d}{dt}\int \mathbf{H}\,d\sigma = \oint [\mathbf{u} \times \mathbf{H}]\cdot d\mathbf{l}, \quad (7)$$

then, in virtue of the law of electromagnetic induction, the velocity (6) coincides with the value of **u** occuring in Eq. (7) in the case where $\mathbf{E} \perp \mathbf{H}$. Thus even in the case of an extremely rarefied gas the picture of a substance "fastened" to the lines of force, or "frozen" into the lines of force, may be used as a descriptive representation of the motion of the plasma.

Let us consider the motion of the plasma in two simple cases with axial symmetry.

1) Let **H** be directed along the axis of symmetry z and be generated by sources situated on the outside of a cylinder of radius R. In virtue of the stipulation of a quasistationary state within this cylinder, we may consider $H_z = H(t)$. If r, z and θ are the cylindrical coordinates of the drifting center of the particle, then in virtue of Eq. (6) and the law of electromagnetic induction [or in view of Eq. (7)], we have

$$u = dr\,/\,dt = -\,(r\,/\,2\,H)\,dH\,/\,dt, \quad (8)$$

whence, as a result of integration, we obtain r^2H = const. By virtue of Eq. (4) ρ^2H = const. also, and thus we have

$$\rho\,/\,\rho_0 = r\,/\,r_0 = \sqrt{H_0\,/\,H}. \quad (9)$$

*In the relativistic case, obviously,

$$W = mc^2\sqrt{1 + 2\mu H\,/\,mc^2}.$$

An increasing magnetic field of the type considered may be created by the rapid compression of a conducting cylinder by means of a convergent explosive wave[5]. In this case, by virtue of the conservation of the current of the vector **H**, $H(t)R^2(t) = H(0)R^2(0)$, where $R(t)$ is the inside radius of a compressed cylinder. Thus, analogously to Eq. (9), $R/R_0 = \sqrt{H_0/H}$, that is, all the dimensions and distances are decreased in proportion to R. At the same time the energy of the rotational motion of the particle according to Eq. (5) increases as*:

$$W / W_0 = H / H_0 = [R_0 / R]^2. \tag{10}$$

If the magnetic field is not strictly parallel to the z axis, but decreases with distance from the axis, that is, has a barrel-shaped form, then as a result of Eq. (3), forces appear which draw the particles of the plasma together to the surface $H_z(r, z, t) = 0$ and guarantee the stability of the motion of the particles in the z direction.

2) Let the magnetic field be generated by an axially symmetrical current directed parallel to the z axis. In order to simplify the calculations we shall assume that the current density

$$j(r, t) = \begin{cases} j(t) & \text{for } r \leqslant R(t), \\ 0 & \text{for } r > R(t). \end{cases} \tag{11}$$

In this case, in accordance with Eq. (2) and Maxwell's equations, with cylindrical coordinates r and z for the drifting center, we have for $r > R$:

$$u_r = \frac{dr}{dt} = -\left\{\frac{1}{J}\frac{dJ}{dt}\left[\frac{1}{2} + \ln\left(\frac{r}{R}\right)\right] - \frac{1}{R}\frac{dR}{dt} + \frac{c^2}{2J}E_1\right\} r, \tag{12}$$

$$u_z = dz / dt \, (c\mu / e) / r, \tag{13}$$

where $J(t) = \pi R^2(t) j(t)$ is the total current strength and $E_1 = E_z$ for $r = 0$. On integrating Eq. (12) we obtain

$$\frac{rR_0}{r_0 R} = \left[\frac{r_0}{R_0} e^{1/2}\right]^{(J_0/J)-1} \exp\left\{-\frac{c^2}{J}\int_0^t E_1\, dt\right\}, \tag{14}$$

whence, in accordance with Eq. (5)

$$\frac{W}{W_0} = \frac{H}{H_0} = \frac{JR_0}{J_0 R}\left[\frac{r_0}{R_0} e^{1/2}\right]^{1-J_0/J} \exp\left\{\frac{c^2}{J}\int_0^t E_1\, dt\right\}. \tag{15}$$

The velocity in the z direction is obtained by substituting Eq. (14) into Eq. (13).

Formula (15) shows that even in the constant-current case the energy of rotational motion of the particles of a rarefied plasma may be increased many fold as a result of a decrease in R. In the light of this result, the manifold increase in the kinetic energy of electrons and ions which was discovered by Artsimovich and his coworkers[6,7] during the passage of powerful discharges through rarefied gases is not unexpected. (One can assume, for example, that the electrons and ions formed after the first compression in the region where $E_1 > H$ are first accelerated by the field E_1 to energies comparable to the potential difference between the discharge electrodes and then, as a result of the increase in density of the discharge current, the particles far from the axis enter the region $H > E$, where a supplementary manifold increase in the energy occurs as a result of the induction mechanism considered above.) The fields obtained in this case guarantee the fulfillment of condition (1) and, consequently, the applicability of formula (15).

[1] H. Alfven, *Cosmic Electrodynamics*, (Russian Translation), 1952.

[2] Ia. P. Terletskii, J. Exptl. Theoret. Phys. (U.S.S.R.) 19, 1059 (1949).

[3] Ia. P. Terletskii, J. Exptl. Theoret. Phys. (U.S.S.R.) 16, 403 (1946).

[4] P. E. Kolpakov and Ia. P. Terletskii, Dok. Akad. Nauk SSSR 76, 185 (1951).

[5] Ia. P. Terletskii, J. Exptl. Theoret. Phys. (U.S.S.R.) 32, 387 (1957), Soviet Physics 5, 301 (1957).

[6] Artsimovich, Andrianov, Bazilevskaia, Prokhorov and Filippov, Atomnaia Energiia 3, 76 (1956).

[7] I. V. Kruchatov, Atomnaia Energiia 3, 65 (1956).

Translated by
M. G. Gibbons
193

*In the ultrarelativistic case, obviously,

$$W / W_0 = \sqrt{H / H_0} = R_0 / R.$$