UNIDIMENSIONAL MOTION IN MAGNETOHYDRODYNAMICS

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The motion of a perfectly conducting gas in a magnetic field is investigated. The Riemann invariants for a number of gases are computed and some nonstationary problems are solved. A new method is proposed for obtaining an approximate general solution of the magnetohydrodynamic equations. Some types of motion involving shock waves are considered.

THE qualitative picture of unidimensional motion in magnetohydrodynamics is fairly clear. It appears to be useful to develop methods for solving unidimensional problems in order to be able to evaluate quantitatively those new features that the introduction of a field adds to the dynamics of a conducting gas.

We shall restrict ourselves to the case of infinite conductivity, since only in this case, and even then not always, can we obtain solutions in analytic form. Moreover, this case is of interest as a limiting case in the sense that here the influence of the field is particularly pronounced. The effect of finite conductivity can be estimated qualitatively by treating classical gas dynamics as another limiting case.

1. NONSTATIONARY MOTION

The principal results in this field were obtained by Kaplan and Staniukovich,[1] who found the characteristics and a particular solution of the magnetohydrodynamic equations. However, their formulas are in a form that cannot yet be applied to the solution of specific problems.

For the case of isentropic motion the magnetohydrodynamic equations can be written as

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{1}{\rho}\frac{\partial p^*}{\partial x} = 0; \qquad (1.1)$$

$$\frac{\partial p^*}{\partial t} + v\frac{\partial p^*}{\partial x} + \rho c_m^2\frac{\partial v}{\partial x} = 0. \qquad (1.2)$$

Here $p^* = p + h^2/2$, $h\sqrt{4\pi} = H$ is the intensity of the magnetic field such that $\mathbf{H}\perp\mathbf{v}$; $c_m^2 = c^2 + V_a^2$, where $c^2 = (\partial p/\partial\rho)_S$ is the speed of ordinary sound, while $V^2 = H^2/4\pi\rho$ is the Alfven velocity. Equation (1.2) can be obtained from the equation of continuity if we take into account that

$$\partial\rho/\partial z = (\partial\rho/\partial p^*)_S\partial p^*/\partial z = c_m^{-2}\partial p^*/\partial z,$$

where $z = x$ or t.

After dividing (1.2) by ρc_m, adding and subtracting (1.1), we can write the result in the form

$$[\partial/\partial t + (v \pm c_m)\partial/\partial x]J_\pm = 0,$$

where we have denoted

$$J_\pm = v \pm \int dp^*/\rho c_m = v \pm \int c_m d\ln\rho. \qquad (1.3)$$

The quantities $J_\pm$ are conserved along the characteristics $dx/dt = v \pm c_m$ and go over into the classic Riemann invariants as $h \to 0$. The integral in (1.3) may be conveniently written in the form

$$\frac{2c_0}{\gamma-1}\int(1+\eta y^{2(2-\gamma)/(\gamma-1)})^{1/2}dy. \qquad (1.4)$$

Here $\gamma = c_p/c_v$, $\eta = V_{a0}^2/c_0^2$; $y = c/c_0$; the subscript 0 refers to quantities defined for $v = 0$. The integral (1.4) is a Chebyshev differential binomial and can be evaluated as a whole only in the cases $\gamma = (4n+1)/(2n+1)$ and $\gamma = 2 - 1/2n$, where $n = 1, 2, 3 \ldots$ In the first case when $n = 1$ we have the practically important case $\gamma = 5/3$. All other values of n can be used only for approximations. This does not include the case of isothermal motion[2] of interest for certain applications which is given by $\gamma = 1$. In this case we must start with (1.3). We state the results of integration for certain values of γ which are of practical interest:

$$\gamma = 1,\ J_\pm = v \pm \left[2(c_T^2 + b^2\rho)^{1/2} + b^2\ln\frac{(c_T^2+b^2\rho)^{1/2} - b}{(c_T^2+b^2\rho)^{1/2}+b}\right];$$

$$\gamma = \frac{3}{2},\ J_\pm = v \pm 2c_0[y(1+\eta y^2)^{1/2} + \eta^{-1/2}\ln(y + (y^2 + 1/\eta)^{1/2})]; \qquad (1.5)$$

$$\gamma = \frac{5}{3},\ J_\pm = v \pm (2c_0/\eta)(1+\eta y)^{3/2}.$$

Here C_T is the isothermal speed of sound, while

$b = h/\rho = \text{const}$.

We shall deal henceforth only with a monatomic gas. We obtain the constant $J_\pm$ from the condition that $y = 1$ when $v = 0$. Then

$$v = \mp (2c_0/\eta)\,[(1 + \eta y)^{3/2} - (1 + \eta)^{3/2}], \qquad (1.6)$$

from which it follows that

$$c = c_0 y = (c_0/\eta)\,\{[(1 + \eta)^{3/2} \pm \eta v/2c_0]^{2/3} - 1\}. \qquad (1.7)$$

By using the Poisson adiabatic we can determine the dependence on v and on the other parameters of the gas. Formulas (1.6) and (1.7) may be used for the description of self-similar motion and for unidimensional running waves given by the special solution[1] $x = (v \pm c_m)\,t + f(v)$ which can now be written in the following form convenient for practical purposes:

$$x = t\left\{\frac{3}{2} v \pm \frac{c_0}{\eta}\left[(1 + \eta)^{3/2} - \left((1 + \eta)^{3/2} \pm \frac{\eta v}{2c_0}\right)^{1/3}\right]\right\} + f(v). \qquad (1.8)$$

By using this formula we can solve, for example, the problem of a piston entering with a speed $U = at$ into a tube which contains a perfectly conducting gas in a magnetic field ($a = \text{const}$).

This solution is completely analogous to the case $h = 0$, although it is more awkward. The shock wave is formed at the instant

$$t = \frac{c_0}{a}\,\frac{6\,(1 + \eta)^{3/2}}{8 + \eta}, \qquad (1.9)$$

which in the case $\eta = 0$ reduces to the well known result[3a] for $\gamma = 5/3$: $t = 3c_0/4a$. As the field increases, the shock wave is formed at progressively later times and progressively further away from the piston, since the field decreases the compressibility of the medium, the perturbations are propagated with greater speed, and it is more difficult for a compression wave to develop in this case than in a medium in the absence of the field and for the same piston speed. One can arrive at the same conclusion by investigating graphically the system of characteristics for this problem.

The condition $c > 0$ enables us to determine from (1.7) the maximum rate of nonstationary efflux of monatomic gas into a vacuum where there is also no field:

$$v \leqslant 2c_0\,[(1 + \eta)^{3/2} - 1]\,/\,\eta. \qquad (1.10)$$

From the Bernoulli equation of magnetohydrodynamics we can obtain the rate of stationary efflux:

$$v = [2\,(i_0 + b^2\rho_0)]^{1/2} = \sqrt{2}\,c_0\left(\frac{1}{\gamma - 1} + \eta\right)^{1/2}. \qquad (1.11)$$

Both rates given by (1.10) and (1.11) increase as the field increases. The limit of their ratio approaches $\sqrt{2}$ as $\eta \to \infty$, and is equal to $\sqrt{3}$ for $\eta = 0$ in the case $\gamma = 5/3$.

The magnetohydrodynamic equations for isentropic flows can be reduced in the usual way[3b] to a single second order linear equation with variable coefficients, and in doing so we must take t and x as the dependent variables, while v and ρ or c should be taken as the independent variables. However, the equation which is obtained as a result of this does not have an analytic solution for any value of γ. In order to obtain an approximate general solution we can approximate the equation of state as follows[1]

$$p = A\rho^{(2n+3)/(2n+1)} + B - {}^1/_2\, b^2\rho^2,$$

where A, n, B are arbitrary constants. However, such an approximation is not convenient in the case when the magnetic pressure is greater than the thermodynamic pressure. We shall give the equation of state in a different form which is convenient in the majority of cases for obtaining the general solution:

$$p^* = A\rho^\gamma + {}^1/_2\, b^2\rho^2 = (A + {}^1/_2 b^2\rho_1^{2-\gamma})\,\rho^\gamma = C\rho^\gamma, \qquad (1.12)$$

where ρ_1 is the constant value of the density defined in a suitable manner. In the case $\gamma = (2n + 3)/(2n + 1)$, where n is an integer, we can obtain the exact solution of the corresponding Darboux equation. If we interpret p^* as the ordinary pressure, then all the equations in terms of whatever variables are employed, and all the thermodynamic relations will be of exactly the same form as in ordinary gas dynamics, only the velocity of sound will be greater in the present case. Therefore all the results of Landau and Lifshitz[3b] and of Staniukovich[4] may be carried over to this case without any changes.

2. MOTION INVOLVING SHOCK WAVES

We shall consider in detail a number of problems where the presence of a shock discontinuity plays an essential role. The front of the discontinuity is parallel to the magnetic field and is perpendicular to the velocity of the flow. The relations between the parameters of different types of discontinuity have been investigated in detail by Helfer[5] and by Lust.[6] We shall discuss a number of topics ab initio, having in mind the solution of certain problems.

In the system of coordinates associated with the surface of discontinuity the following conditions must be fulfilled:

$$\rho_1 u_1 = \rho_2 u_2; \qquad (2.1)$$

$$\rho_1 u_1^2 + p_1 + h_1^2/2 = \rho_2 u_2^2 + p_2 + h_2^2/2; \qquad (2.2)$$

$$i_1 + u_1^2/2 + h_1^2/\rho_1 = i_2 + u_2^2/2 + h_2^2/\rho_2; \qquad (2.3)$$

$$h_1 u_1 = h_2 u_2. \qquad (2.4)$$

The subscripts 1 and 2 refer to the two sides of the surface of discontinuity.

The taking into account of finite conductivity alters only the last relation[7] by smearing out the front of the discontinuity. However, formulas (2.1) to (2.4) will hold at large distances from the front. Therefore, the formulas which we obtain below should also hold in the case of finite conductivity.

By introducing the specific volume $V = 1/\rho$ and the energy per unit volume of the field $\epsilon = H^2/8\pi$ we can obtain the equation of the shock adiabatic:

$$i_1 - i_2 + \frac{1}{2}(V_1 + V_2)(p_2 + \varepsilon_2 - p_1 - \varepsilon_1) + 2(\varepsilon_1 V_1 - \varepsilon_2 V_2) = 0. \qquad (2.5)$$

We now introduce $\rho_2/\rho_1 = x = V_1/V_2$ representing the discontinuity in the density of the medium. As a result of the field being frozen-in $\epsilon_2 = \epsilon_1 x^2$. After taking into account the fact that $i = \gamma pV/(\gamma - 1)$ we shall obtain from (2.5) after a number of transformations:

$$\alpha x - 1 - (\alpha - x)\, p_2/p_1 + \eta_1 (x - 1)^3 = 0,$$
$$\alpha = (\gamma + 1)/(\gamma - 1), \quad \eta_1 = \varepsilon_1/p_1 = H_1^2/8\pi p_1. \qquad (2.6)$$

In the case $\eta_1 = 0$ we obtain the usual Hugoniot adiabatic. The parameter η_1 differs by the factor $\gamma/2$ from the parameter η of the preceding section.

We note that although Eq. (2.6) is cubic in x, it is linear in p_2/p_1, and η_1. Therefore, by fixing one of the parameters we can easily calculate the dependence of the other parameter on x. Figure 1 shows the Hugoniot adiabatics in the presence of the field in the case $\gamma = 5/3$ in the system of coordinates p_2/p_1 and $V_2/V_1 = 1/x$. They lie progressively higher than the usual adiabatic as the initial field H_1 increases, i.e., the compressibility of the medium decreases. The adiabatics in the presence of the field are fourth order curves with two asymptotes:

$$V_2/V_1 = 1/\alpha = (\gamma - 1)/(\gamma + 1),$$
$$p_2/p_1 = -(1 + \eta_1)/\alpha. \qquad (2.7)$$

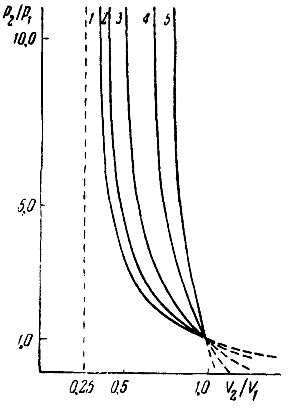

FIG. 1. Curve $1 - \eta_1 = 0$; $2 - \eta_1 = 1$; $3 - \eta_1 = 10$; $4 - \eta_1 = 10^2$; $5 - \eta_1 = 10^3$.

A rarefaction shock wave cannot exist also in the case of magnetohydrodynamics in consequence of the principle of increase of entropy at a discontinuity of the type under investigation,[8] therefore states represented by points of the adiabatics below the point (1, 1) are not realized.

Due to the occurrence of "freezing-in" in order to specify completely the state of the medium beyond the discontinuity we must know in addition to the initial parameters only any one of the three quantities p_2, ρ_2, h_2.

The discontinuity in the temperature is determined by the equation of state, from which it follows that $T_2/T_1 = p_2/xp_1$. Since $1/x$ increases with η_1 for a given value of the ratio p_2/p_1, the temperature discontinuity must also increase. Although in actual fact the problem cannot be stated so simply, nevertheless we must not forget the possibility of such an increase in temperature, since it could occur in the case of some nonstationary introduction of a field into a medium which already contains a discontinuity and when p_2/p_1 remains constant or decreases insignificantly.

In order to obtain an evaluation as to which wave is more advantageous from the point of view of obtaining a higher temperature — the wave in a medium with a field, or without a field — we shall solve the problem of the piston moving with a constant velocity U into an unbounded tube. A shock wave is formed in the gas which at the initial moment coincides with the position of the piston and then moves away from it. In the region between the wave and the piston the gas moves with velocity U. From (2.1) and (2.2) we can obtain the following formula:

$$u_1 - u_2 = [(p_2^* - p_1^*)(V_1 - V_2)]^{1/2}. \qquad (2.8)$$

Since the gas is at rest ahead of the discontinuity we obtain from (2.8):

$$U = [(p_2^* - p_1^*)(V_1 - V_2)]^{1/2}. \qquad (2.9)$$

This equation can be conveniently rewritten in the form

$$U = c_1 \left\{\frac{1}{\gamma}\left(1 - \frac{1}{x}\right)\left[\frac{p_2}{p_1} - 1 + \eta_1 (x^2 - 1)\right]\right\}^{1/2}. \qquad (2.10)$$

We can easily calculate the speed of the discontinuity front by expressing it in terms of the density

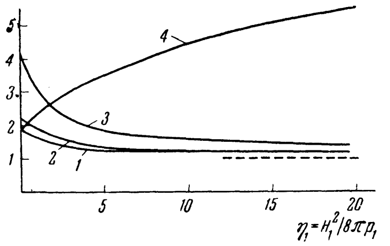

FIG. 2. Curve $1 - T_2/T_1$; $2 - x = \rho_2/\rho_1$; $3 - p_2/p$; $4 - v/c_1$.

discontinuity:

$$\frac{v}{c_1} = \left\{\frac{x\eta\,[(\alpha-3)x^2+4x-(\alpha+1)]+(\alpha+1)(x-1)}{\gamma\,(x-1)}\right\}^{1/2}. \quad (2.11)$$

With the aid of (2.10), (2.11), and (2.8) we can find for any specific value of U/c, the discontinuities in the pressure, the temperature, the density, and the value of v/c_1. Figure 2 shows the variation with η_1 of these parameters in the case $\gamma = 5/3$ and $U/c_1 = 1$. As the field increases all the discontinuities tend to 0, while v/c_1 increases, however the value of $v/c_{m_1} = v/c_1(1+\eta)^{1/2}$ approaches 1, i.e., the shock wave degenerates into a weak discontinuity. This result is physically clear: as the field increases the compressibility of the medium decreases, while progressively larger masses of gas are involved in the motion and progressively larger fractions of the work done must be expended in increasing the energy of the field. Therefore the discontinuities in p, ρ, T must decrease and tend to 0 as $\eta_1 \to \infty$. In a medium in which $p \ll H^2/8\pi$ with all the other conditions remaining the same, the shock wave has a low efficiency from the point of view of heating the gas. This should be taken into account in many astrophysical problems.

Let us consider the reflection of a shock wave from an absolutely rigid wall. Since the gas is at rest in the space between the wall and the shock wave, the relative velocities of the gas on the two sides of the discontinuity are the same for the incident and for the reflected wave. Therefore, by making use of (2.8), we obtain

$$(p_2^* - p_1^*)(V_1 - V_2) = (p_3^* - p_2^*)(V_2 - V_3),$$

where the subscript 3 denotes the region between the wall and the shock wave. This condition may be brought into the form:

$$(x_1 - 1)\left[\frac{p_2}{p_1} - 1 + \eta_1(x_1^2 - 1)\right] = \frac{p_2}{p_1}\left(1 - \frac{1}{x_2}\right)\left[\frac{p_3}{p_2} - 1 + \eta_1\frac{x_1^2}{p_2/p_1}(x_2^2 - 1)\right] \quad (2.12)$$

Here $x_1 = \rho_2/\rho_1$; $x_2 = \rho_3/\rho_2$. The equations of the adiabatics for the incident and the reflected shock waves are given by:

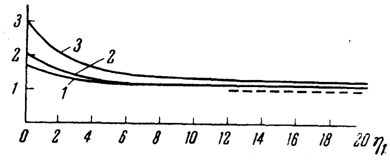

FIG. 3. Curve $1 - T_3/T_2$; $2 - x_2$; $3 - p_3/p_2$.

$$\alpha x_1 - 1 - (\alpha - x_1)\,p_2/p_1 + \eta_1(x_1 - 1)^3 = 0; \quad (2.6')$$

$$\alpha x_2 - 1 - (\alpha - x_2)\,p_3/p_2 + \eta_1 x_1^2(p_1/p_2)(x_2 - 1)^3 = 0. \quad (2.6'')$$

Here, just as in the case of the problem of the piston, we have not succeeded in finding an analytic solution, but the numerical solution of equations (2.12), (2.6′), and (2.6″) does not present any difficulties.

The results of the calculation are given in Fig. 3. For the sake of simplicity we have taken the wave of the preceding problem as the incident wave.

The use of the above results to investigage the decay of an arbitrary discontinuity in the medium considered by us, containing a field perpendicular to the velocity, also presents no difficulties in principle. Such an investigation can be carried out in exactly the same way as in the case of ordinary gas dynamics,[3c] and we can obtain the same qualitative results with somewhat altered estimates for the velocities.

In conclusion, I express my deep gratitude to K. P. Staniukovich for a discussion of the above results.

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