# *SURFACE OSCILLATIONS OF A CHARGED COLUMN IN A LONGITUDINAL MAGNETIC FIELD*

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An analysis is given of the surface oscillations and stability of a hydrodynamic charged column in an external magnetic field. This analysis represents an extension of the well-known work of Kruskal and Schwarzchild and Tayler on the stability of an uncharged hydrodynamic column. The surface-wave dispersion equation is derived and the features of the oscillation spectrum are investigated. From the experimental point of view these results are of value for investigating the nature of the radiation from a plasma in a magnetic field, an effect which is as yet not fully understood.

## 1. INTRODUCTION

THE stability of a column (pinch) has been studied by Kruskal and Schwarzchild,[1] who considered a plasma cylinder in which the current flows along the surface. It was shown that a system of this kind is unstable against lateral distortions. A further step in this problem was made by Shafranov[2] and Tayler[3,4] who considered perturbations of other kinds. The analysis was extended to include a more general current distribution over the column cross section and to the case in which there is an arbitrary magnetic field and a conducting wall. It has been shown that there is some improvement in stability in the latter case.

In the work cited above the electric charge of the plasma was not taken into account. However, it is known experimentally, and from the theory of ambipolar diffusion, that a plasma column exhibits a total resultant charge, even though this charge is weak. It is of interest to extend the results of the above work to the case of a charged column since the presence of an external magnetic field leads to peculiar effects: these result from the interaction of the self electric field with the external magnetic field.

There are extensive experimental data on the oscillatory properties of a plasma in an external magnetic field. The frequency of these oscillations covers a wide range (from tens of cycles per second to ultra high frequencies) (cf. for example, reference 5). In certain cases the frequency increases as the external magnetic field is increased. There is a minimum value of the magnetic field at which the field begins to have an effect on the oscillation frequency. Under certain conditions there is a uniformly spaced line spectrum. The number of lines which are observed varies from two to ten. It is difficult to interpret these in terms of "overtones" of a single spectral line. Up to this time this effect has not been explained theoretically.

Without undertaking a complete analysis of these effects, in the present paper we shall examine magnetohydrodynamic surface waves in connection with the above problems.

## 2. BASIC EQUATIONS

Following Kruskal and Schwarzchild we consider the problem from a magnetohydrodynamic standpoint, assuming infinite conductivity.

The internal region of the plasma is described by the usual system, comprising Maxwell's equations and the hydrodynamic equations:

$$\begin{gathered}\operatorname{div}\mathbf{E}=0,\ \operatorname{div}\mathbf{H}=0,\ \operatorname{curl}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t},\\ \operatorname{curl}\mathbf{H}=\frac{4\pi}{c}\mathbf{j}+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t},\quad \mathbf{E}+\frac{1}{c}[\mathbf{v}\times\mathbf{H}]=\frac{1}{\sigma}\mathbf{j}=0,\\ \frac{\partial\rho}{\partial t}+\operatorname{div}(\rho\mathbf{v})=0,\quad \rho\frac{dv}{dt}=-\operatorname{grad}p+\frac{1}{c}[\mathbf{j}\times\mathbf{H}],\\ \frac{1}{p}\frac{dp}{dt}=\frac{\gamma}{\rho}\frac{d\rho}{dt}\end{gathered}\tag{1}$$

($\gamma$ is the ratio of the specific heats of the plasma).

It is assumed that the electrical conductivity of the external medium is zero and that the pressure is constant. Hence we deal with Maxwell's equations for source-free conditions.

The boundary conditions at the interface are of the form:

$$(\mathbf{n}\cdot\{\mathbf{E}\}) = 4\pi\varepsilon^*,\ (\mathbf{n}\cdot\{\mathbf{H}\}) = 0,$$
$$[\mathbf{n}\times\{\mathbf{E}\}] = \frac{u}{c}\{\mathbf{H}\},\ [\mathbf{n}\times\{\mathbf{H}\}] = \frac{4\pi}{c}\mathbf{j}^* - \frac{u}{c}\{\mathbf{E}\}, \tag{2}$$

where $\mathbf{n}$ is a unit vector in the direction of the outward normal and $u = \mathbf{n}\cdot\mathbf{v}$; $\{\ldots\}$ denotes the difference in the values of a quantity at the interface. Here and below in all cases we shall include the convection current $\epsilon^*\mathbf{v}$ in the expression for the total current $\mathbf{j}^*$. Also

$$d\mathbf{n}/dt = \mathbf{n}\times[\mathbf{n}\times\operatorname{grad} u], \tag{3}$$

$$\varepsilon^*\overline{\mathbf{E}} + \frac{1}{c}[\mathbf{j}^*\times\overline{\mathbf{H}}] - \mathbf{n}\{p\} = 0, \tag{4}$$

where the bar denotes the average value of a quantity; as in reference 1 this average can be taken simply as half the sum of the values on the two sides of the interface.

## 3. LINEARIZATION AND SOLUTION OF THE EQUATIONS

We take all quantities in the form

$$q = q_0 + \delta q(r)\exp\{i(m\theta + kz + \omega t)\}, \tag{5}$$

where $q_0$ is the equilibrium value of a quantity $q$.

We shall specify the equilibrium state as follows: at the surface of the column there is a charge of uniform density $\epsilon_0^*$ which produces a radial electric field $E_0 = 4\pi\epsilon_0^*$. In the equilibrium state there is no current flow. The external magnetic field $H_0$ is along the $z$ axis. The electric forces are balanced by the pressure difference $\{p\}_0 = E_0^2/8\pi$.

Substituting Eq. (5) in the original equations we obtain a system of linearized equation from which we can obtain an expression for $\delta p$ in the plasma:

$$\left(1 - \frac{k^2c_H^2}{\omega^2} + \frac{c_H^2}{c_s^2}\right)\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{m^2}{r^2}\right\}\delta p - \left(1 + \frac{c_H^2}{c^2} - \frac{k^2c_H^2}{\omega^2}\right)\left(k^2 - \frac{\omega^2}{c_s^2}\right)\delta p = 0, \tag{6}$$

where $c_s = (\gamma p_0/\rho_0)^{1/2}$ is the velocity of sound in the plasma, and $c_H = H_0/\sqrt{4\ \rho_0}$ is the velocity of the Alfvén waves. The solution of Eq. (6) is

$$\delta p = \alpha p_0 I_m(\eta r), \tag{7}$$

$$\eta^2 = \frac{(1 + c_H^2/c^2 - k^2c_H^2/\omega^2)\zeta^2}{(1 + c_H^2/c_s^2 - k^2c_H^2/\omega^2)} = \frac{(k^2 - \omega^2/c_H^2 - \omega^2/c^2)(k^2 - \omega^2/c_s^2)}{k^2 - \omega^2/c_H^2 - \omega^2/c_s^2}, \qquad \zeta^2 = k^2 - \omega^2/c_s^2, \tag{8}$$

$I_m$ is a Bessel function of imaginary argument, and $\alpha$ is the dimensionless amplitude.

From the foregoing it is easy to obtain all the remaining quantities of interest. All of these are expressed in terms of the same amplitude $\alpha$. The expressions which obtained for the pressure, density, velocity components and field are of the same form as those obtained by Tayler,[4] who did not take displacement current into account; there is one exception: the expression given by Tayler for $\alpha^2$, which in the present notation is written

$$(k^2 - \omega^2/c_H^2)(k^2 - \omega^2/c_s^2)/(k^2 - \omega^2/c_H^2 - \omega^2/c_s^2),$$

must be replaced by $\eta^2$ from (8).

The solutions for the original equations for the region outside the column are given in reference 1. These are characterized by two amplitudes and are proportional to $K'_m(\eta' r)$ or the derivative $K'_m(\eta' r)$, where $\eta'^2 = k^2 - \omega^2/c^2$ ($K_m$ is a Hankel function of imaginary argument).

The boundary conditions yield three independent linear homogeneous relations between the three amplitudes; the existence condition for a non-trivial solution yields the dispersion equation

$$(\omega^2 - c_H^2\zeta^2)\frac{I_m(\eta R)}{\eta R I'_m(\eta R)} = -\left\{\frac{m^2\omega^2\omega_E^2\zeta^2}{c^2\eta^2\eta'^2} + 2\omega\omega_m\frac{\zeta^2}{\eta^2} + \frac{c_H^2\zeta^2\eta'^2}{\eta^2}\right\}\frac{K_m(\eta' R)}{\eta' R K'_m(\eta' R)} + \frac{\omega_E^2\zeta^2}{\eta^2}\left(1 + \frac{k^2}{\eta^2}\frac{\eta' R K'_m(\eta' R)}{K_m(\eta' R)}\right), \tag{9}$$

where

$$\omega_E^2 = E_0^2/4\pi R\rho_0, \quad \omega_m = E_0H_0m/4\pi cR\rho_0, \tag{10}$$

where $R$ is the radius of the pinch.

Northrup has obtained similar results for plane boundaries.[6]

## 4. DISCUSSION OF THE DISPERSION EQUATION

We now consider several cases.

(1) The short-wave case:

$$k^2 \gg \omega^2/c^2,\ k^2 \gg \omega^2/c_s^2 \quad (c_s^2 \ll c^2). \tag{11}$$

As will be shown by calculation, the last inequality holds if the following condition is satisfied:

$$c_H^2 \ll c_s^2. \tag{12}$$

Under these conditions we can replace $\eta$, $\eta'$ and $\zeta$ by $k$ and Eq. (9) assumes the simpler form:

$$\omega^2 + 2\omega\omega_m I'_m K_m/I_m K'_m + c_H^2k^2(I'_m K_m/I_m K'_m - 1) - \omega_E^2(1 + kRK'_m/K_m)kRI'_m/I_m = 0. \tag{13}$$

In the last equation the argument of the cylindrical functions is $kR$ in all cases.

The stability condition is

$$\omega_m^2\left(\frac{I'_mK_m}{I_mK'_m}\right)^2 - c_H^2k^2\left(\frac{I'_mK_m}{I_mK'_m}-1\right) + \omega_E^2\left(1+\frac{kRK'_m}{K_m}\right)\frac{kRI'_m}{I_m} \geqslant 0. \tag{14}$$

Except when m = 0, the third term is negative while the first two are positive. We neglect the first term in (14) since it is small compared with the third term if $c_H^2 \ll c^2$, as follows directly from (13).

The absolute value of the second term is greater than that of the third only if

$$k \geqslant k_c, \tag{15}$$

where $k_c$ is some critical wave number, which can be determined by setting the left hand part of the inequality in (14) equal to zero; we have

$$F_m(k_cR) = E_0^2/H_0^2$$

$$F_m(x) = \frac{x^2(I'_m(x)K_m(x)/I_m(x)K'_m(x)-1)}{(1+xK'_m(x)/K_m(x))xI'_m(x)/I_m(x)}. \tag{16}$$

With the exception of the case m = 1, $F_m(x)$ increases monotonically from 0 to 2 so that stability is possible only when

$$2H_0^2 > E_0^2. \tag{17}$$

When $E_0 = 0$ (uncharged column) the result agrees with Eq. (3.12) given by Tayler[4] for the appropriate conditions (no surface current and no conducting walls). This result also agrees with that obtained by Kislovskii[7] for the case m = 1 if we take $\Sigma = 0$, $\mu_1 = \mu_2 = 1$, $H_0 = H_\infty$ in Eq. (15.45) of that reference. These oscillations are nothing more than magnetohydrodynamic waves which propagate in the cylinder.

If there is no external magnetic field ($H_0 = 0$) only the first and last terms remain in Eq. (13). The equation which results coincides with the dispersion relation obtained by A. A. Vlasov for a charged jet. Oscillations of this type are purely electrostatic in origin.

Vlasov has also obtained Eq. (13) without the third term and the factor $2I'_mK_m/I_mK'_m$ in the second term, starting with a completely different model — the inertia motion of a charged jet with an ellipsoidal velocity distribution.* The interesting term in Eq. (13) is the second term; this term, which arises because of the simultaneous presence of $E_0$ and $H_0$ can be considered the result of the interaction between the charged column and the external magnetic field.

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In the present case the frequency $\omega_m$ is the modulation frequency. Actually, the solution of Eq. (13) can be written in the explicit form

$$\omega = \omega_m\Phi_m \pm \{\omega_m^2\Phi_m^2 + \Omega_m^2\}^{1/2}, \tag{18}$$

where we have used the notation

$$\Phi_m = -I'_mK_m/I_mK'_m,$$

$$\Omega_m^2 = -c_H^2k^2(I'_mK_m/I_mK'_m - 1) - \omega_E^2(1+kRK'_m/K_m)kRI'_m/I_m,$$

where $k > k_c$, $\Omega^2 m > 0$, $\Phi_m \approx 1$.

Since $\Omega^2 m \gg \omega_m^2$, we can write

$$\omega \approx \pm\Omega_m + \omega_m\Phi_m. \tag{19}$$

We may recall that when the substitution $m \to -m$ is made $\omega_m$ changes sign, whereas $\Phi_m$ does not.

A linear combination of the four possible types results in oscillations of the form

$$\cos\Omega_m t\cos(\omega_m\Phi_m t + m\theta). \tag{20}$$

There are modulated oscillations with a modulation frequency $\omega_m\Phi_m$. In the first approximation the factor $\Phi_m$ can be taken as unity and we obtain an equally spaced line spectrum, $\omega_m$, with frequencies proportional to $E_0$ and $H_0$. $\Phi_m$ has some distorting effect on the equal spacing, especially for the first lines in the spectrum.

It is apparent from (19) that

$$\omega^2 \approx \Omega_m^2 \leqslant c_H^2k^2.$$

In order to satisfy the condition $k^2 \gg \omega^2/c_s^2$, on which the present calculation is based, the condition given in (12) must be fulfilled.

(2) Long-wave case. For simplicity we assume $k = 0$; thus

$$\eta'^2 = -\frac{\omega^2}{c^2},\quad \zeta^2 = -\frac{\omega^2}{c_s^2},$$

$$\eta^2 = -\left(\frac{c^2+c_H^2}{c_s^2+c_H^2}\right)\frac{\omega^2}{c^2}.$$

The arguments of $K_m$ and $K'_m$ are imaginary.

We now consider Eq. (9). $\eta$ and $\eta'$ appear in (9) only in quadratic terms or in the combinations $\eta RI'_m(\eta R)/I_m(\eta R)$, $\eta RK'_m(\eta' R)/K_m(\eta' R)$; it can be shown that $xI'_m(x)/I_m(x)$ is always real for real and imaginary x while $xK'_m(x)/K_m(x)$ is always complex for imaginary x. Whence it is clear that the left side of (9) is always real for real x but the right side is complex. Hence, in general there is no real solution for $\omega$. However, if the following condition is satisfied:

$$|\omega R/c| \gg 1 \tag{21}$$

it is possible to have a real frequency with a very small imaginary part. When $|x| \ll 1$ we have

$$ixK_m'(ix)/K_m(ix) \approx -m - i\pi x^{2m}/2^{2m-1}[(m-1)!]^2. \quad (22)$$

We assume that $\omega$ is so small that $|\eta R| \ll 1$. When $|x| \ll 1$ we have

$$ixI_m'(ix)/I_m(ix) \approx m.$$

Introducing these assumptions it is possible to make a considerable simplification in the dispersion equation (9). If we take

$$\omega = \omega_1 + i\gamma, \ (\gamma \ll \omega_1), \quad (23)$$

the equation for $\omega_1$ is

$$\beta\omega_1^2 - 2\omega_1\omega_m + m(m-1)\omega_E^2 = 0, \quad (24)$$

where

$$\beta = \frac{c_s^2(c^2 + c_H^2) + c_H^2(c^2 + 2c_H^2 + c_s^2)}{c^2(c_s^2 + c_H^2)}$$

$$\gamma = \frac{m^2\omega_E^2}{\omega_m - \beta\omega_1}\,\frac{\pi(\omega_1 R/2c)^{2m}}{[(m-1)!]^2} \quad (25)$$

We consider Eq. (24) for two cases.

(a) If $c^2 \gg c_H^2$, $\beta \approx 1$,

$$\omega_1 = \omega_m \pm \{\omega_m^2 - m(m-1)\omega_E^2\}^{1/2}$$
$$\approx \omega_m\{1 \pm [(1-m)/m]^{1/2}(c/c_H)\}, \quad (26)$$

and $\omega_1$ is real only when $m = 1$.

(b) If $c^2 \ll c_H^2$, $\beta^2 \approx 2c_H^2/c^2$,

$$\omega_1 = \frac{1}{2}(1 \pm \sqrt{(2-m)/m})\,\frac{E_0 cm}{H_0 R}, \quad (27)$$

and $\omega_1$ is real for $m = 1$ and $m = 2$.

(3) We now estimate the corrections for the quantities computed above. The following numerical values may be taken as typical:

$H_0 \approx 1500$ gausses, pressure $p = 10^{-1}$ to $10^{-5}$ mm Hg, i.e., the density of neutral particles $n \approx 10^{11}$ to $10^{15}$ cm$^{-3}$. Take the mass of the particles as $\mu \approx 2 \times 10^{-24}$ g; then $\rho_0 \approx 10^{-13}$ to $10^{-9}$ g/cm$^3$ and $c_H^2 \approx 10^7$ to $10^9$ cm/sec, $c_H^2/c^2 = 10^{-6}$ to $10^{-2}$.

If we assume that the radius of the column $R \approx 1$ cm, $E_0 \approx 2 \times 10^{-4}$ cgs electrostatic units, $E_0/H_0 \approx 10^{-7}$ and the modulation frequency for case (1) is $\omega_m = 10^2$ to $10^{-2}$ cps for $m = 1$.

The frequency $\omega_1$ in (26) is $\omega_m$ for $m = 1$ while in (27) $\omega_1 \approx 3 \times 10^{-3}$ cps for $m = 1$, $m = 2$.

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310