QUADRUPOLE OSCILLATIONS OF DEFORMED NUCLEI

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Formulas are derived for quadrupole oscillations of deformed nuclei. The interaction between the rotation and the oscillations is considered, and a general formula is given for the corrections to the level energy of the ground rotational band. Presently-available experimental data for nuclei in the rare earth group are discussed and the difficulties encountered in the analysis are mentioned.

EQUATIONS FOR BETA AND GAMMA-OSCILLATIONS

A study of the oscillational levels yields new information on the properties of deformed nuclei. Two types of quadrupole oscillations are considered: (1) Beta oscillations about the equilibrium value β_0 along the deformation axis for an axially-symmetrical nucleus ($\gamma = 0$). (2) Gamma oscillations perpendicular to the deformation axis at equilibrium value β_0.

We start with the Bohr Hamiltonian[1] for a nucleus consisting of a core and n external nucleons.

$$\hat{H} = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\left(\beta^4\frac{\partial}{\partial\beta}\right)+\frac{1}{\beta^2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\left(\sin 3\gamma\frac{\partial}{\partial\gamma}\right)\right] + \frac{\hbar^2}{8B\beta^2}\sum_{\varkappa=1}^{3}\left(\hat{I}_\varkappa - \sum_{i=1}^{n}\hat{j}_\varkappa^{(i)}\right)^2 \Big/ \sin^2\left(\gamma-\frac{2\pi}{3}\varkappa\right) + \frac{1}{2}C_{\text{core}}\beta^2 + \sum_{i=1}^{n}\hat{H}_p(x_i;\beta,\gamma)+\frac{1}{2}\sum_{i,k}\hat{W}_{ik}. \qquad (1)$$

In the hydrodynamic approximation, in which the Hamiltonian (1) is written, there is a single inertia parameter B. However, such an approximation was found to be poor for real nuclei, and it became necessary to introduce three parameters: the inertia parameters of the β and γ oscillations, B_β and B_γ, and the moment of inertia J.* We assume the adiabatic approximation to hold, at least for the first oscillation levels of even-even nuclei, and average (1) over the internal states of the nucleons. Confining ourselves to small oscillations about to the equilibrium values $\beta = \beta_0$ and $\gamma = 0$, we can write the Hamiltonian (1) as

$$\hat{H} = E(\beta_0) + \hat{H}_\beta + \hat{H}_\gamma + \hat{H}_r + \hat{U}, \qquad (2)$$

*In the Bohr Hamiltonian, $B_\beta = B$, $B_\gamma = B\beta^2$, and $J = 3B\beta^2$.

where

$$\hat{H}_\beta = -\frac{\hbar^2}{2B_\beta}\left(\frac{\partial^2}{\partial\beta_1^2}+\frac{4}{\beta_0}\frac{\partial}{\partial\beta_1}\right)+\frac{1}{2}C_\beta\beta_1^2, \qquad (2a)$$

$$\hat{H}_\gamma = -\frac{\hbar^2}{2B_\gamma}\left[\frac{1}{\gamma}\frac{\partial}{\partial\gamma}\left(\gamma\frac{\partial}{\partial\gamma}\right)-\frac{\hat{I}_3^2}{4\gamma^2}\right]+\frac{1}{2}C_\gamma\gamma^2, \qquad (2b)$$

$$\hat{H}_r = (\hbar^2/2J)(\hat{I}^2-\hat{I}_3^2); \qquad (2c)$$

$$\hat{U} = (\gamma\hbar^2/\sqrt{3}J)(\hat{I}_1^2-\hat{I}_2^2). \qquad (2d)$$

Here $\beta_1 = \beta - \beta_0$, $\beta_1 \ll \beta_0$; C_β and C_γ are parameters of the stability of the nucleus relative to β and γ oscillations, which will be considered below. H_β is the Hamiltonian of the β oscillations, H_γ that of the γ oscillations, and H_r that of the rotation of the nucleus; finally, $\hat{U}$ takes into account the interaction between the rotation and the γ oscillations.* In the derivation of the Hamiltonian (2) we assumed that in even-even nuclei $\Omega = \Sigma\Omega_i = 0$. Considering the rotation and oscillations to be independent in the zero approximation, we obtain a nuclear wave function in the form

$$\Psi_{IKMn_\beta n_\gamma} = \sqrt{N^2/2(1+\delta_{K0})}\,\psi_{n_\beta}(\beta_1)\,\varphi_{n_\gamma}(\gamma)\,X(x_1\ldots x_n)\times[D^I_{MK}+(-1)^I D^I_{M-K}], \qquad (3)$$

where N is the normalizing factor.

The function ψ_{n_β} satisfies the following equation

$$\hat{H}_\beta\psi_{n_\beta} = \varepsilon_\beta\psi_{n_\beta}; \qquad (4)$$

Putting $\psi = \exp(-2\beta_1/\beta_0)$, we bet

$$-\frac{\hbar^2}{2B_\beta}\frac{d^2f}{d\beta_1^2}+\frac{1}{2}C_\beta\beta_1^2 f = \varepsilon_\beta f, \qquad (4a)$$

*It must be noted that $\hat{U}$ does not take the interaction into account too accurately. In fact, the formula $\hat{U} = \frac{\gamma}{\sqrt{3}}\frac{\hbar^2}{J}(\hat{I}_1^2-\hat{I}_2^2)$ contains still another factor on the order of unity, depending on the model. In the hydrodynamic approximation this factor equals unity exactly.

from which we get

$$\varepsilon_\beta = \hbar\omega_\beta (n_\beta + {}^1/_2),\ \omega_\beta = \sqrt{C_\beta / B_\beta},\ n_\beta = 0,\ 1,\ 2, \ldots, \quad (5a)$$

$$f_{n_\beta} = \exp(-B_\beta\omega_\beta\beta_1^2 / 2\hbar)\, H_{n_\beta}(\beta_1 \sqrt{B_\beta\omega_\beta / \hbar}); \quad (5b)$$

H_{n_β} is the Hermite polynomial.

The equation for the γ oscillations is

$$-\frac{\hbar^2}{2B_\gamma}\frac{1}{\gamma}\frac{d}{d\gamma}\left(\gamma\frac{d\varphi}{d\gamma}\right) + \frac{\hbar^2}{8B_\gamma}\frac{K^2}{\gamma^2}\varphi + \frac{1}{2}C_\gamma\gamma^2\varphi = \varepsilon_\gamma\varphi, \quad (6)$$

where K is the eigenvalue of the operator $\hat{I}_3$. Solution of (6) is

$$\varepsilon_\gamma = \hbar\omega_\gamma (n_\gamma + 1), \quad \omega_\gamma = \sqrt{C_\gamma / B_\gamma}, \quad (6a)$$

$$n_\gamma = K/2,\ K/2+2,\ K/2+4 \ldots = \begin{cases} 0,\ 2,\ 4, \ldots & \text{for } K=0, \\ 1,\ 3,\ 5, \ldots & \text{for } K=2, \end{cases} \quad (6b)$$

$$\varphi_{n_\gamma} = \exp\left(-\frac{B_\gamma\omega_\gamma}{2\hbar}\gamma^2\right) \times \gamma^{K/2} F\left(-\frac{2n_\gamma - K}{4};\ \frac{K}{2}+1;\ \frac{B_\gamma\omega_\gamma}{\hbar}\gamma^2\right), \quad (6c)$$

F is the hypergeometrical polynomial.

The rotational energy ϵ_r, according to (2c), is

$$\varepsilon_r = (\hbar / 2J)\,[I(I+1) - K^2]. \quad (7)$$

For the total energy of the nucleus we obtain

$$E_{IKn_\beta n_\gamma} = E(\beta_0) + \hbar\omega_\beta(n_\beta + {}^1/_2) + \hbar\omega_\gamma(n_\gamma + 1) + (\hbar^2/2J)\,[I(I+1) - K^2]. \quad (8)$$

In the ground state

$$E_0 = E(\beta_0) + {}^1/_2\,\hbar\omega_\beta + \hbar\omega_\gamma. \quad (8a)$$

the energy of the first β-oscillation level is

$$\varepsilon(I = K = 0,\quad n_\beta = 1,\quad n_\gamma = 0) = E_{0010} - E_0 = \hbar\omega_\beta, \quad (8b)$$

and the energy of the first γ-oscillation level is

$$\varepsilon(I = K = 2,\quad n_\beta = 0,\quad n_\gamma = 1) = \hbar\omega_\gamma + \hbar^2 / J. \quad (8c)$$

The operator of the E2 transition is

$$\mathfrak{M}_\mu(E2) = \frac{3}{4\pi} ZeR_0^2\alpha_\mu^* = \frac{3}{4\pi} ZeR_0^2\left[\beta\cos\gamma D_{\mu 0}^2 + \frac{\beta}{\sqrt{2}}\sin\gamma\,(D_{\mu 2}^2 + D_{\mu -2}^2)\right]. \quad (9)$$

The reduced probability of transition to the β-oscillation level with spin I = 2 (we are speaking, naturally, of a suitable level of the rotational band) is

$$B_{0\to 2}^{(\beta)}(E2) = \left(\frac{3}{4\pi} ZeR_0^2\right)^2 \sum_\mu |\langle 20\mu\, 10 \,|\, \beta_1 D_{\mu 0}^2 \,|\, 00000\rangle|^2 = \left(\frac{3}{4\pi} ZeR_0^2\right)^2 \frac{\hbar}{2\sqrt{B_\beta C_\beta}} \quad (10)$$

In (10) we used the following matrix element

$$\langle 1 \,|\, \beta_1 | 0\rangle = \sqrt{\hbar / 2B_\beta\omega_\beta} = (\hbar^2 / 4B_\beta C_\beta)^{1/4}.$$

The reduced probability of a transition from the ground state to the γ-oscillation level is

$$B_{0\to 2}^{(\gamma)}(E2) = \left(\frac{3ZeR_0^2}{4\pi}\right)^2 \beta_0^2 \sum_\mu |\langle 22\mu 01 \,|\, \frac{\gamma}{\sqrt{2}}(D_{\mu 2}^2 + D_{\mu -2}^2) \,|\, 00000\rangle|^2 = \left(\frac{3}{4\pi} ZeR_0^2\right)^2 \beta_0^2 \frac{\hbar}{\sqrt{B_\gamma C_\gamma}}. \quad (11)$$

Here we use the matrix element

$$\langle 1 \,|\, \gamma \,|\, 0\rangle = \sqrt{\hbar / B_\gamma\omega_\gamma} = (\hbar^2 / B_\gamma C_\gamma)^{1/4}.$$

It is interesting to compare (11) with the reduced probability of transition to the first rotational level of the ground band

$$B_{0\to 2}^{(\gamma)}(E2) = B_{0\to 2}^{\mathbf{gr}}(E2)\,\hbar\omega_\gamma / C_\gamma. \quad (12)$$

For γ-oscillation levels at the borders of the rare-earth group (isotopes of Sm, W, Os), $\hbar\omega_\gamma$ is on the order of 1 Mev, and C_γ on the order of 20 Mev, hence

$$B^{(\gamma)}(E2) / B^{\mathbf{gr}}(E2) \approx {}^1/_{20}. \quad (12a)$$

2. CONNECTION BETWEEN C_β AND C_γ

Let us consider in greater detail the problem of the averaging of the Hamiltonian (1) over the entire states of the nucleons. The Hamiltonian of the nucleon in a deformed nucleus is of the form

$$\hat{H}_p = -\frac{\hbar^2}{2m}\Delta + V(r) + \frac{\alpha}{r}\frac{dV}{dr}\mathbf{l}\cdot\mathbf{s} + \frac{\beta^2}{8\pi}\left(r^2\frac{d^2V}{dr^2} + 4r\frac{dV}{dr}\right) - \beta\cos\gamma\, r\frac{dV}{dr}Y_{20} - \frac{\beta}{\sqrt{2}}\sin\gamma\, r\frac{dV}{dr}(Y_{22} + Y_{2,-2}). \quad (13)$$

The Hamiltonian of n nucleons contains, in addition to h_p, also the energy of the residual interaction and therefore

$$\hat{H}_{\mathbf{nucl}} = \sum_{i=1}^{n} \hat{H}_p(i) + \sum_{i<k} \hat{W}_{ik}. \quad (14)$$

It can be shown (see, for example, reference 3) that in the observed nuclear deformations the energy of the residual interaction W is no longer dependent on β and is constant. Let us average (13), choosing the solution for $\gamma = 0$ as the zeroth approximation. The β-dependent portion of the nuclear energy will then be

$$E(\beta) = W + \frac{\beta^2}{2}\Big[C_c + \frac{1}{4\pi}\sum_{i=1}^{n}\Big\langle \Omega_i \Big| r^2\frac{d^2V}{dr^2} + 4r\frac{dV}{dr}\Big|\Omega_i\Big\rangle\Big] - \beta\sum_{i=1}^{n}\Big\langle\Omega_i\Big| r\frac{dV}{dr}Y_{20}\Big|\Omega_i\Big\rangle$$

$$= W + \frac{1}{2}C_\beta\beta^2 - \beta\sum_{i=1}^{n}\Big\langle\Omega_i\Big| r\frac{dV}{dr}Y_{20}\Big|\Omega_i\Big\rangle. \quad (15)$$

The equilibrium deformation β_0 is determined, as usual, from the condition

$$\frac{dE}{d\beta} = C_\beta\beta_0 - \sum_{i=1}^{n}\Big\langle\Omega_i\Big| r\frac{dV}{dr}Y_{20}\Big|\Omega_i\Big\rangle = 0 \quad (16)$$

and, through the use of (16), $E(\beta)$ is represented by

$$E(\beta) = E(\beta_0) + {}^1/_2 C_\beta(\beta - \beta_0)^2. \quad (15a)$$

The portion of the energy that depends on γ (for fixed $\beta = \beta_0$), is, within accuracy to second-order perturbation-theory terms,

$$E(\gamma) \equiv -\beta_0\cos\gamma\sum_{i=1}^{n}\Big\langle\Omega_i\Big| r\frac{dV}{dr}Y_{20}\Big|\Omega_i\Big\rangle - \frac{1}{2}\beta_0^2\sin^2\gamma \times\sum_{i=1}^{n}\sum_{k}\Big|\Big\langle\Omega_k\Big| r\frac{dV}{dr}(Y_{22}+Y_{2,-2})\Big|\Omega_i\Big\rangle\Big|^2 \Big/ (E_{\Omega_k} - E_{\Omega_i}). \quad (17)$$

The summation over k is carried out over the unfilled states. It is seen from (17) that the equilibrium value of γ, determined from the condition $dE/d\gamma = 0$, is zero, i.e., deformed even-even nuclei have an axially-symmetrical equilibrium form.

The parameter of the stability of the nucleus relative to γ oscillations is

$$C_\gamma = \Big(\frac{d^2E}{d\gamma^2}\Big)_{\gamma=0} = \beta_0\sum_{i=1}^{n}\Big\langle\Omega_i\Big| r\frac{dV}{dr}Y_{20}\Big|\Omega_i\Big\rangle - \beta_0^2\sum_{i=1}^{n}\sum_{k}\Big|\Big\langle\Omega_k\Big| r\frac{dV}{dr}(Y_{22} + Y_{2,-2})\Big|\Omega_i\Big\rangle\Big|^2 \Big/ (E_{\Omega_k} - E_{\Omega_i}), \quad (18)$$

and, considering (16), we get

$$C_\gamma = \beta_0^2\Big[C_\beta - \sum_{i=1}^{n}\sum_{k} \times\Big|\Big\langle\Omega_k\Big| r\frac{dV}{dr}(Y_{22} + Y_{2,-2})\Big|\Omega_i\Big\rangle^2 \Big/ (E_{\Omega_k} - E_{\Omega_i})\Big] \leqslant C_\beta\beta_0^2. \quad (19)$$

Since the denominator in the sum contains the difference of energy levels for which we have no accurate values, it is impossible in general to calculate C_γ and C_β. However, if it is known that for a given nucleus there are no nearby levels, it is possible to use the calculations of Nilsson[2] to obtain a rough estimate of C_γ. We thus obtain C_γ = 24 Mev for W^{182} and C_γ = 27 Mev for W^{186}.

3. INTERACTION BETWEEN THE ROTATION AND THE OSCILLATIONS

The correction to the energy of the rotational levels in the ground band, proportional to $I^2(I+1)^2$, is due to two causes: (1) γ oscillations cause the shape of the nucleus to deviate from that of an axially-symmetrical top. (2) The moment of inertia changes during β oscillations.

The interaction between the rotation and the γ oscillations is taken into account by operator $\hat{U}$ [Eq. (2d)], which is not diagonal in the I, j_3 representation. We then obtain in the second perturbation-theory approximation

$$\Delta E_\gamma = -\frac{|\langle I201|\hat{U}|I000\rangle|^2}{\hbar\omega_\gamma} = -\frac{\hbar^2}{6B_\gamma}\Big(\frac{\hbar^2}{J}\Big)^2\frac{I^2(I+1)^2 - 2I(I+1)}{(\hbar\omega_\gamma)^2}. \quad (20)$$

The portion of ΔE_γ that is proportional to $I(I+1)$ can be included in the expression for the rotational energy in the zeroth zpproximation, since it leads to an insignificant change in the moment of inertia. The remaining portion will yield deviations from the $E \sim I(I+1)$ law, namely

$$\Delta E_\gamma = -\frac{\hbar^2}{6B_\gamma(\hbar\omega_\gamma)^2}\Big(\frac{\hbar^2}{J}\Big)^2 I^2(I+1)^2 = -\frac{1}{6C_\gamma}\Big(\frac{\hbar^2}{J}\Big)^2 I^2(I+1)^2. \quad (21)$$

In the model of a rotating liquid drop, $J = 3B_\gamma$, and this leads to Bohr and Mottelson's Eq. (6.35) of reference 4.

$$\Delta E_\gamma = -\frac{1}{2(\hbar\omega_\gamma)^2}\Big(\frac{\hbar^2}{J}\Big)^3 I^3(I+1)^2. \quad (21a)$$

The correction due to the interaction with the β oscillations can be determined by taking it into account that the moment of inertia is a function of β. Expanding $\hbar^2/2J$ in powers of β_1, we obtain in the second approximation

$$\Delta E_\beta = -\frac{1}{2C_\beta}\Big(\frac{\partial}{\partial\beta}\frac{\hbar^2}{2J}\Big)^2 I^2(I+1)^2. \quad (22)$$

The total correction to the energy of the levels of the principal rotational band will be

$$\Delta E = \Delta E_\gamma + \Delta E_\beta = -\Big[\frac{1}{6C_\gamma}\Big(\frac{\hbar^2}{J}\Big)^2 + \frac{1}{2C_\beta}\Big(\frac{\partial}{\partial\beta}\frac{\hbar^2}{2J}\Big)^2\Big] I^2(I+1)^2 = -FI^2(I+1)^2. \quad (23)$$

F depends on the three parameters J, C_β, and C_γ as well as on the derivative $(\partial/\partial\beta)\hbar^2/2J$. These parameters can be determined from Eqs. (7), (10), and (11). The derivative can then be determined from (23).

4. ANALYSIS OF AVAILABLE EXPERIMENTAL DATA

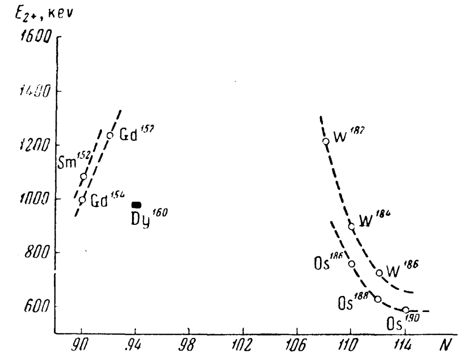
Tables I and II list the experimental data on the oscillational and rotational levels of nuclei near the rare-earth group. What is most striking in an examination of the β-oscillation levels is the strong dependence of the energies ϵ_γ of the γ levels on the parameter β_0 of the equilibrium deformation. At the borders deformed-nuclei regions $\epsilon_\gamma \approx$ 600 kev (Os^{190}); when the deformation increases, ϵ_γ rises rapidly, reaching 1220 kev for W^{182} and on the order of 2 Mev and above for Dy, Er, and Yd. It is quite possible that the ϵ_γ vs. β_0 curve has a fine structure, so that all the isotopes of the given element fit a separate curve (see diagram).

Let us consider now the behavior of the derivative $(\partial/\partial\beta)(\hbar^2/2J)$, making use of the data of Tables I and II.

We begin with the isotopes of tungsten, for

TABLE I. Oscillational levels in deformed even-even nuclei

γ-oscillation levels, I = K = 2+		γ-oscillation levels, I = K = 2+		β-oscillation levels, I = K = 0+	
Isotope,	Energy, kev	Isotope	Energy, kev	Isotope	Energy, kev
Sm^{152}	1086	Os^{188}	633	Sm^{152}	685
Gd^{154}	996.6	Os^{190}	586	Gd^{154}	679
Gd^{156}	1160	Th^{228}	964	Er^{166}	1460?
Dy^{160}	996	Th^{230}	1060?	Th^{232}	720
Er^{166}	786	U^{232}	868	U^{234}	805
Er^{168}	822	Pu^{238}	1030	Pu^{238}	935
W^{182}	1222	Pu^{240}	1020	Pu^{240}	940
W^{184}	890	Cm^{246}	1300		
W^{186}	730	Cf^{250}	1200		
Os^{186}	764	Fm^{254}	700		



which C_γ have been measured by the Coulomb-excitation method. Inserting in

$$\Delta_\beta^2 = \frac{1}{C_\beta}\left(\frac{\partial}{\partial\beta}\frac{\hbar^2}{2J}\right)^2 = 2F - \frac{1}{3C_\gamma}\left(\frac{\hbar^2}{J}\right)^2 \qquad (24)$$

the values $C_\gamma = 21$ Mev, $\hbar^2/J = 33.6$ kev, and $F = 0.015$ kev for W^{182}, we obtain $\Delta_\beta = 0.110$ (we shall call the quantity Δ_β the "β-deformability" of the nucleus). If we now assume $C_\gamma \approx C_\beta\beta_0^2$ for W^{182}, we get $|(\partial/\partial\beta)\hbar^2/2J| = 65$ kev. A similar value is obtained for the derivative if the moment of inertia depends linearly on β.

We now perform analogous calculations for W^{184}, using the values $C_\gamma = 18$ Mev, $\hbar^2/J = 30$ kev, and $F = 0.073$ kev. It then follows from (24) that $\Delta_\beta = 0.31$. The three-fold increase over the value for W^{182} can naturally not be attributed to a corresponding decrease in $\sqrt{C_\beta}$, for this would necessitate a ten-fold decrease in C_β. On the other hand, the moments of inertia and the value of the parameter β_0 for these two isotopes are the same, within the accuracy of measurement. To verify whether we deal, in the case of W^{184}, with fluctu-

TABLE II. Parameters β_0, A, and F in the rare earth region. The parameters of equilibrium deformation have been experimentally determined from the Coulomb excitation. The parameters A and F determine the energy of the rotation-band levels of the ground state: $E = AI(I+1) - FI^2(I+1)^2$

Nucleus	β_0	A, kev	F, kev
Sm^{152}	0.28	21.18	0.141
Gd^{154}	0.30	21.34 (20.20±0.60)	0.139 (0.072 ±0.015)
Gd^{156}	0.41	15.02 (14.81±0.12)	0.0324 (0.022 ±0.003)
Gd^{158}	0.46	13.25	0.0102
Dy^{160}	0.35	14.53	0.019
Dy^{162}	0.36	13.55 (13.54±0.23)	0.0133 (0.010 ±0.006)
Er^{166}	0.33	13.63	0.0194
Er^{168}	0.33	13.37 (13.48±0.05)	0.0060 (0.0119±0.0012)
Yb^{170}	0.30	14.11	0.0110
Yb^{172}	0.31	13.16	0.0076
Hf^{176}	0.29	14.93 (14.94±0.05)	0.0166 (0.0166±0.012)
Hf^{178}	0.31	15.61 (15.64±0.14)	0.0132 (0.0132±0.0021)
Hf^{180}	0.27	15.60 (15.54±0.08)	0.0079 (0.0067±0.0013)
W^{182}	0.26	16.78	0.0156
W^{184}	0.24	16.78	0.0729
Os^{190}	0.15	32.59 (30.1±1.1)	0.264 (0.12±0.03)

ations, we consider the behavior of the derivative in isotopes of Hf. The isotope Hf^{180} has the least value of F, namely 0.0067 kev. It is therefore reasonable to propose that, in the case of Hf^{180}, the contribution to F from the β-deformability term is negligibly small. We then obtain from (24), by putting $\hbar^2/2J = 15.54$ kev, a value $C_\gamma =$ 24 Mev. Comparing this value with $C_\gamma(W^{184}) =$ 18 Mev and $C_\gamma(W^{182}) = 21$ Mev, we see that a value $C_\gamma(Hf^{180}) = 24$ Mev is quite reasonable.* Quite reasonable, too, is the assumption that C_γ cannot be less than 24 Mev in the case of Hf^{178} and Hf^{176}. Then, according to (24), using the data of Tables I and II, we get

$$\Delta_\beta \geqslant \begin{cases} 0.11 \text{ for } Hf^{178}, \\ 0.14 \text{ for } Hf^{176}. \end{cases}$$

If we take for Hf^{180} the same value for the β-deformability parameter as for Hf^{178}, C_γ will be several times greater than that obtained. On the other hand, it is difficult to assume that C_γ or $(\partial/\partial\beta)\hbar^2/2J$ changes abruptly in going from one isotope to another. One can propose that such an abrupt change in F is due to violation of adiabaticity for rotational levels. To check such an assumption, we calculated F first using data on two levels, and then from data on four levels (the values of F in parentheses in Table II). The good agreement between the results argues against the assumed violation of adiabaticity. One is left with the assumption that the rotational band is perturbed by single-particle levels that lie close to the oscillational levels. Such a perturbation can be proportional to $I^2(I+1)^2$ and lead to differing increases in F for different isotopes, if the single-particle levels are different. So far there are no detailed information on the levels in the 1 to 2 Mev region, and therefore the latter assumption cannot be verified.

It must be noted in conclusion that an experimental investigation of the high levels of the Hf isotopes would yield new information on the character of the oscillations and rotation of nuclei.

*The reasonable value of C_γ obtained for Hf^{180} indicates that the parameter referred to in the preceding footnote should be close to unity.

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