

THEORY OF THE NONLINEAR FIELD $(\square - \lambda\varphi^2)\varphi = 0$

D. F. KURDGELAIDZE

Moscow State University

Submitted to JETP editor June 9, 1958; resubmitted November 5, 1958

J. Exptl. Theoret. Phys. (U.S.S.R.) **36**, 842-849 (March, 1959)

The nonlinear field described by the equation $(\square - \lambda\varphi^2)\varphi = 0$ is considered. Starting from the exact wave-type solution of the field equation, the spectral analysis of the energy of the nonlinear field is obtained. The mass spectrum derived has the form $M_0^{(n)} = (2n + 1)M_0^{(0)}$, $n = 0, 1, 2, \dots$. The exact radially symmetric solution of the field equation is found. A general method of integrating the nonlinear field of Dirac is given, and it is shown that in some cases it is possible to go over to a two-component spinor equation of the second order.

1. THEORY OF THE NONLINEAR FIELD

$(\square - \lambda\varphi^2)\varphi = 0$

WE considered the nonlinear field $(\square - \lambda\varphi^2)\varphi = 0$ in previous papers,^{1,2} where a wave solution of the field equation was found and it was shown that a nonlinear state could appear as a superposition of linear states with the integral frequencies $(2n + 1)$, $n = 0, 1, 2, \dots$. Recently Heisenberg has obtained by different means the mass spectrum of particles, and in particular that of mesons, in a nonlinear theory.^{3,4} This spectrum obeys only approximately the $(2n + 1)$ rule; but, as Heisenberg shows, it has the property that the ratio $M_0^{(n)}/M_0^{(0)}$ is almost independent of the nonlinearity parameter, as in the case of the $(2n + 1)$ rule. In connection with this, the present paper gives a more detailed derivation of the $(2n + 1)$ rule for mass spectra by means of the spectral analysis of the energy of the nonlinear field.

(a) Spectral Decomposition of the Energy of the Nonlinear Field

The Lagrangian of our nonlinear field has the form

$$L = -\frac{1}{2} \left\{ \varphi_\mu^2 + k_0^2 \varphi^2 + \frac{1}{2} \lambda \varphi^4 \right\}, \quad \mu = 1, 2, 3, 4. \quad x_4(x_n, x_4 = it), \quad c = \hbar = 1, \quad (1.1)$$

where, for the sake of generality, the term $k_0^2 \varphi^2$ is included.

From Eq. (1.1) we find by the usual methods expressions for the energy and momentum and also obtain the field equation

$$H = \frac{1}{2V} \int \left\{ \varphi_n^2 - \varphi_4^2 + k_0^2 \varphi^2 + \frac{1}{2} \lambda \varphi^4 \right\} (dx), \quad (1.2)$$

$$G = -\frac{i}{V} \int \varphi_4 (\nabla \varphi) (dx), \quad (1.3)$$

$$-\varphi_{\mu\mu} + k_0^2 \varphi + \lambda \varphi^3 = 0. \quad (1.4)$$

In reference 2, the time-averaged energy and momentum densities of the field were calculated, starting from an exact wave-type solution

$$\varphi = \varphi_0 \operatorname{cn}(\sigma + C), \quad \sigma = k_\mu x_\mu, \quad k_\mu(k_n, k_4 = i\omega),$$

$$-k_\mu^2 = -k^2 + \omega^2 = k_0^2 + \lambda \varphi_0^2,$$

$$k_1^2 = \lambda \varphi_0^2 / 2 (k_0^2 + \lambda \varphi_0^2); \quad (1.5)$$

where φ_0 and C are arbitrary constants and k_1 is the modulus of an elliptic function.

The final results of the calculations are

$$H = a(k^2 + K_0^2)/\omega, \quad G = ak, \quad a = \varphi_0^2 \omega l / 2,$$

$$K_0^2 = \frac{1}{l} \left(k_0^2 + \frac{1}{2} \lambda \varphi_0^2 \right),$$

$$l = \frac{2}{3} \left\{ 2 - (1 - k_1^2) \frac{1}{k_1^2} \left(1 - \frac{E(k_1)}{K(k_1)} \right) \right\}. \quad (1.6)$$

From this, in particular, it follows when $k = 0$ that

$$H = \frac{1}{2} \varphi_0^2 \left(k_0^2 + \frac{1}{2} \lambda \varphi_0^2 \right), \quad (1.7)$$

and, as we see, in the case where $k_0 = 0$ the quantity

$$\bar{k}_0^2 = \lambda \varphi_0^2 / 2. \quad (1.8)$$

corresponds to the square of the mass in the ordinary case. Now, setting the energy of the field in (1.7) equal to the meson mass \bar{k}_0 , with $k_0 = 0$, we get

$$\varphi_0^2 = 2/\bar{k}_0, \quad \lambda = \bar{k}_0^3. \quad (1.9)$$

We then examine the energy spectrum of the nonlinear field. To do this, we write the solution (1.5) in the form ($C = 0$)

$$\begin{aligned} \varphi/\varphi_0 &= \sum_{n=0}^{\infty} a_n \cos \sigma_n, \quad \sigma_n = (2n+1)B(\lambda)\sigma = k_\mu^{(n)}x_\mu, \\ B(\lambda) &= \pi/2K(k_1), \quad a_n = \pi/k_1K(k_1)\text{ch } \rho_n, \\ \rho_n &= (2n+1)\rho_0, \quad \rho_0 = \pi K'(k_1)/2K(k_1), \end{aligned} \quad (1.10)$$

where k_1 is the modulus of the elliptic function and $K(k_1)$ is the elliptical integral of the first kind. $K'(k_1) = K(k_1')$, $k_1^2 + k_1'^2 = 1$.

From (1.4) and (1.10) we find

$$\lambda\varphi^3/\varphi_0 = \sum_{n=0}^{\infty} b_n \cos \sigma_n, \quad (1.11)$$

where

$$\begin{aligned} b_n &= -(k_0^2 + k_\mu^{(n)2}) \\ &= -(k_0^2 - (k_0^2 + \lambda\varphi_0^2)(B(\lambda)(2n+1))^2)a_n. \end{aligned} \quad (1.12)$$

Putting (1.10) - (1.12) into (1.2) and (1.3), we get

$$H = \varphi_0^2 \sum_{n=0}^{\infty} H_n, \quad G = \varphi_0^2 \sum_{n=0}^{\infty} G_n, \quad (1.13)$$

where

$$\begin{aligned} H_n &= \frac{1}{2} a_n^2 \omega^{(n)2} \{1 + (k_0^2 + k_\mu^{(n)2})/4\omega^{(n)2}\}, \\ G_n &= \frac{1}{2} a_n^2 \omega^{(n)} k^{(n)}. \end{aligned} \quad (1.14)$$

We introduce the new amplitude N_n , defined by the formula

$$N_n = a_n^2 \omega^{(n)}/2. \quad (1.15)$$

Then we have

$$\begin{aligned} H_n &= N_n \omega^{(n)} \{1 + (k_0^2 + k_\mu^{(n)2})/4\omega^{(n)2}\}, \\ G_n &= N_n k^{(n)}. \end{aligned} \quad (1.16)$$

From this, for the case $\mathbf{k} = 0$, $k_0 = 0$, we find

$$H_n = \frac{3}{4} N_n \omega^{(n)}, \quad G_n = 0. \quad (1.17)$$

Further, according to (1.5) and (1.10), we have

$$\begin{aligned} k_1^2 &= \frac{1}{2}, \quad K'(k_1) = K(k_1) = 1.85, \\ B(\lambda) &= \pi/2 \cdot 1.85 = 0.84, \\ \rho_0 &= \frac{\pi}{2}, \quad a_n = \frac{\pi \sqrt{2}}{1.85} \frac{1}{\text{ch}[\pi(2n+1)/2]} = \frac{2.396}{\text{ch}[\pi(2n+1)/2]}, \\ (2\sqrt{2}a_0)^2 &= 7.29, \quad (\sqrt{3}B(\lambda)/4)^2 = 0.355. \end{aligned} \quad (1.18)$$

Then we get on rearranging

$$\begin{aligned} \omega^{(n)} &= \sqrt{2}B(\lambda)(2n+1)\bar{k}_0 = \sqrt{32/3}M_0^{(n)}, \\ N_n &= \frac{1}{2} a_n^2 \omega^{(n)} = \frac{1}{16} (2\sqrt{2}a_0)^2 \left(\frac{a_n}{a_0}\right)^2 \omega^{(n)} \\ &= \frac{1}{4} \sqrt{\frac{2}{3}} \left(\frac{K_0}{\bar{k}_0}\right) \left(\frac{a_n}{a_0}\right)^2 M_0^{(n)}, \\ H_n &= \frac{3}{4} N_n \omega^{(n)} = \frac{1}{2} (K_0/\bar{k}_0) (a_n/a_0)^2 M_0^{(n)}, \end{aligned} \quad (1.19)$$

where

$$\begin{aligned} M_0^{(n)} &= (2n+1)M_0^{(0)}, \quad K_0 = (2\sqrt{2}a_0)^2 \bar{k}_0 = 7.29\bar{k}_0, \\ M_0^{(0)} &= (\sqrt{3}B(\lambda)/4)^2 \bar{k}_0 = 0.3554\bar{k}_0. \end{aligned} \quad (1.20)$$

The total energy of the field then has the form

$$H = \varphi_0^2 (K_0/\bar{k}_0) \sum_{n=0}^{\infty} \frac{1}{2} (a_n/a_0)^2 M_0^{(n)2} \quad (1.21)$$

Calculating (1.9), we find

$$H = \bar{k}_0 \left\{ (K_0/\bar{k}_0) \sum_{n=0}^{\infty} (a_n/a_0)^2 (M_0^{(n)}/\bar{k}_0)^2 \right\}. \quad (1.22)$$

Equating (1.22) and (1.7) for $k_0 = 0$, we get

$$\begin{aligned} (K_0/\bar{k}_0) \sum_{n=0}^{\infty} (a_n/a_0)^2 (M_0^{(n)}/\bar{k}_0)^2 &= \frac{3}{2} B^2(\lambda) \sum_{n=0}^{\infty} [(2n+1)a_n]^2 \\ &= \frac{3}{8} \left(\frac{\pi}{\sqrt{k_1}K(k_1)}\right)^4 \sum_{n=0}^{\infty} \left(\frac{2n+1}{\text{ch}(\pi(2n+1)/2)}\right)^2 = 1. \end{aligned} \quad (1.23)$$

The reader can easily convince himself of the validity of (1.23).

The mesonic mass spectrum we have derived

$$M_0^{(n)}/\bar{k}_0 \approx (2n+1) \cdot 0.36 \quad (1.24)$$

is listed below. For comparison, Heisenberg's results are also shown

n	0	1	2	3	4	5
From (1.24)	0.36	1.08	1.80	2.52	3.24	3.96
Heisenberg	0.33	0.94	1.74	—	3.32	—

(b) The Radially Symmetric Solution of the Wave Equation

If, for $k_0 = 0$, we examine the solution of (1.4) in the form

$$\psi = \psi(s), \quad s = \sqrt{x_\mu^2}, \quad \mu = 1, 2, 3, 4, \quad (1.25)$$

the field equation becomes

$$\frac{d^2\psi}{ds^2} + \frac{3}{s} \frac{d\psi}{ds} + \lambda\psi^3 = 0, \quad (1.26)$$

the solution of which is

$$\psi = \sqrt{\frac{2k_1^2}{\lambda s_0^2(2k_1^2-1)}} \frac{1}{(s/s_0)} \text{cn} \left[\frac{1}{\sqrt{2k_1^2-1}} \ln(s/s_0) \right], \quad (1.27)$$

where s_0 is an arbitrary constant and k_1 is the modulus of an elliptic function. For $k_1^2 = 1$, we get

$$\psi = 2 \sqrt{\frac{2}{\lambda s_0^2}} \left(1 + \left(\frac{s}{s_0}\right)^2 \right).$$

2. INTEGRATION OF THE NONLINEAR DIRAC EQUATION

In the linear theory, one can go over from the Dirac equation for a free field to the Klein-Gordon

equation and vice versa. Furthermore, there exists a connection between the solutions of the Dirac equation and those of the corresponding Klein-Gordon equation: if a solution of the Klein-Gordon equation is known, it is then possible to construct a solution of the Dirac equation.

It is shown below that in some sense an analogous situation occurs even in the nonlinear theory. However, as a rule, this dependence has a more complicated nature.

The nonlinear Dirac equation has the following general form:

$$\{\gamma_\mu (\partial/\partial x_\mu + B_\mu(\bar{\psi}, \psi)) + A(\bar{\psi}, \psi)\} \psi = 0, \quad (2.1)$$

where

$$x_\mu (x_n, x_4 = it), \quad \gamma_\mu^T = \gamma_\mu, \\ \gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4, \quad \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}. \quad (2.2)$$

The functions $A(\bar{\psi}, \psi)$ and $\gamma_\mu B_\mu(\bar{\psi}, \psi)$ are scalars constructed from $\bar{\psi}, \psi$, and the γ_μ matrices in line with invariance requirements. As a rule, they have the form

$$f(\Gamma_\mu(\bar{\psi}\Gamma_\mu\psi)), \quad \Gamma_\mu = 1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5. \quad (2.3)$$

The equation adjoint to (2.1) is

$$\bar{\psi} \{(\partial/\partial x_\mu + \bar{B}_\mu(\bar{\psi}, \psi))\gamma_\mu - \bar{A}(\bar{\psi}, \psi)\} = 0, \quad (2.4)$$

where

$$\bar{\psi} = \psi^* \gamma_4, \quad \bar{A}(\bar{\psi}, \psi) = \gamma_4 A^T(\bar{\psi}, \psi) \gamma_4, \quad \bar{\psi} \partial/\partial x_\mu \equiv \partial \bar{\psi}/\partial x_\mu, \\ \bar{B}_n(\bar{\psi}, \psi) = \gamma_4 B_n^T(\bar{\psi}, \psi) \gamma_4, \\ \bar{B}_4(\bar{\psi}, \psi) = -\gamma_4 B_4^T(\bar{\psi}, \psi) \gamma_4. \quad (2.5)$$

We introduce the operators

$$\hat{p} = \gamma_\mu (\partial/\partial x_\mu + B_\mu(\bar{\psi}, \psi)), \\ \hat{p} = (\partial/\partial x_\mu + \bar{B}_\mu(\bar{\psi}, \psi))\gamma_\mu, \quad (2.6) \\ \hat{D} = -(m\tilde{A}/A\tilde{A})\hat{p}, \quad \hat{D} = -\hat{p}(m\tilde{A}/\tilde{A}\tilde{A}), \quad (2.7)$$

where $A\tilde{A}$ is a quantity not dependent on the matrices. For example, for $A = \lambda_1(\bar{\psi}\psi) + \lambda_2\gamma_5(\bar{\psi}\gamma_5\psi)$ we would have

$$\tilde{A} = \lambda_1(\bar{\psi}\psi) - \lambda_2\gamma_5(\bar{\psi}\gamma_5\psi), \quad A\tilde{A} = (\lambda_1(\bar{\psi}\psi))^2 - (\lambda_2\gamma_5(\bar{\psi}\gamma_5\psi))^2.$$

Then Eqs. (2.1) and (2.4) can be written in the form

$$(\hat{D} - m)\psi = 0, \quad \bar{\psi}(\hat{D} + m) = 0. \quad (2.8)$$

We introduce the new functions

$$\psi = \frac{1}{m}(\hat{D} + m)\varphi, \quad \bar{\psi} = \bar{\varphi}(-\hat{D} + m)\frac{1}{m}. \quad (2.9)$$

Then, substituting (2.9) into (2.8), we find

$$(\hat{D}\hat{D} - m^2)\varphi = 0, \quad \bar{\varphi}(\hat{D}\hat{D} - m^2) = 0. \quad (2.10)$$

As we see, (2.8) and (2.10) have exactly the

same form with respect to the operator \hat{D} as in the linear case.

For the transition from the Dirac equation to the Klein-Gordon equation to be complete, the functions $\bar{\psi}$ and ψ entering into the nonlinear operators \hat{D} and \hat{D} must be expressed in terms of $\bar{\varphi}$ and φ . Since it is difficult to carry through the indicated procedure in the general case, we examine a few individual cases

$$1. \quad B_\mu(\bar{\psi}, \psi) = 0, \quad A(\bar{\psi}, \psi) = \lambda(\bar{\psi}\psi). \quad (2.11)$$

We then get

$$\hat{D} = -\frac{m}{\lambda(\bar{\psi}\psi)}\gamma_\mu \frac{\partial}{\partial x_\mu}, \quad \hat{D} = -\frac{\partial}{\partial x_\mu}\gamma_\mu \frac{1}{\lambda(\bar{\psi}\psi)} \quad (2.12)$$

and, correspondingly,

$$\psi = \varphi - \gamma_\mu \frac{\partial \varphi}{\partial x_\mu} \frac{1}{\lambda(\bar{\psi}\psi)}, \quad \bar{\psi} = \bar{\varphi} + \frac{\partial \bar{\varphi}}{\partial x_\mu} \gamma_\mu \frac{1}{\lambda(\bar{\psi}\psi)}, \quad (2.13)$$

From this we find

$$(\bar{\psi}\psi) = \bar{\varphi}\varphi + \frac{1}{\lambda(\bar{\psi}\psi)}(\bar{\varphi}_\mu\gamma_\mu\varphi - \bar{\varphi}\gamma_\mu\varphi_\mu) - (\bar{\varphi}_\mu\gamma_\mu\gamma_\nu\varphi_\nu) \frac{1}{(\lambda(\bar{\psi}\psi))^2}, \quad (2.14)$$

that is

$$Y^3 - \lambda(\bar{\varphi}\varphi)Y^2 - \lambda(\bar{\varphi}_\mu\gamma_\mu\varphi - \bar{\varphi}\gamma_\mu\varphi_\mu)Y + \lambda(\bar{\varphi}_\mu\gamma_\mu\gamma_\nu\varphi_\nu) = 0,$$

$$Y \equiv \lambda(\bar{\psi}\psi), \quad (2.15)$$

whose solution gives

$$Y = f(\bar{\varphi}\theta\varphi), \quad (2.16)$$

where θ is some operator. The corresponding Klein-Gordon equation is

$$\left\{ \frac{\partial^2}{\partial x_\mu^2} - Y^2 \left[1 - \frac{1}{2} \left(\frac{\partial}{\partial x_\mu} \frac{1}{Y^2} \right) \gamma_\mu \gamma_\nu \frac{\partial}{\partial x_\nu} \right] \right\} \varphi = 0. \quad (2.17)$$

In particular, if we look at only wave-type solutions, as we did in Sec. 1, we arrive at the results of references 2 and 6. There, for this special case, the reverse problem is also solved and it is shown that the equation $(\square - \lambda\varphi^2)\varphi = 0$ can be "linearized" to yield the equations

$$\gamma_\mu \frac{\partial \psi}{\partial x_\mu} + \left[c_1 + \frac{\lambda}{2} (\bar{\psi}\psi)^2 \right]^{1/2} \chi(s) = 0, \quad (2.18)$$

where

$$\psi = \chi(s)\varphi(\sigma), \quad \bar{\psi} = \bar{\varphi}(\sigma)\bar{\chi}(\sigma), \quad \bar{\chi}\chi = 1,$$

$\chi(s)$ being a constant spinor; s , the spin coordinate; and c_1 , an arbitrary constant.

If c_1 is given in terms of φ_0 , the amplitude of (1.5), by $c_1 = -\kappa^2\varphi_0^2/2$ where $\kappa^2 = \lambda\varphi_0^2$, we get for the solution of (2.18) $\psi = \chi(s)\varphi(\sigma)$, where $\varphi(\sigma)$ is given by (1.5) and $\chi(s)$ is a solution of the equation $(\gamma_\mu k_\mu - i\kappa)\chi(s) = 0$. Consequently,

the nonlinear field ψ can be represented as a particular combination of the linear states (1.10) with the integral frequencies $(2n+1)$.

It is easy to see that in the given case both (2.1) and (2.17) are invariant with respect to the transformations

$$\varphi \rightarrow \gamma_5 \varphi, \quad \bar{\varphi} \rightarrow -\bar{\varphi} \gamma_5, \quad (2.19)$$

$$\psi \rightarrow \gamma_5 \psi, \quad \bar{\psi} \rightarrow -\bar{\psi} \gamma_5, \quad (2.20)$$

respectively. According to (2.13), the transformation (2.19) induces (2.20) and inversely. Taking this circumstance into account we can, following Feynman, choose the solutions $\psi \equiv \psi_I$, $\varphi \equiv \varphi_I$, satisfying the conditions

$$(1 - \gamma_5) \varphi_I = 0, \quad 1/2 (1 + \gamma_5) \varphi_I = \varphi_I, \quad \bar{\varphi}_I (1 + \gamma_5) = 0, \\ 1/2 \bar{\varphi}_I (1 - \gamma_5) = \bar{\varphi}_I \quad (2.21)$$

$$(1 - \gamma_5) \psi_I = 0, \quad 1/2 (1 + \gamma_5) \psi_I = \psi_I, \\ \bar{\psi}_I (1 + \gamma_5) = 0, \quad 1/2 \bar{\psi}_I (1 - \gamma_5) = \bar{\psi}_I. \quad (2.22)$$

Then from (2.13) we have $\varphi_I = \psi_I$, $\bar{\varphi}_I = \bar{\psi}_I$. However, since $(1 - \gamma_5)(1 + \gamma_5) = 0$, we find

$$(\bar{\psi}\psi) = (\bar{\psi}_I\psi_I) = (\bar{\varphi}\varphi) = (\bar{\varphi}_I\varphi_I) = (\bar{\varphi}_I(1 - \gamma_5)1/4(1 + \gamma_5)\varphi) = 0.$$

Thus a two-component nonlinear equation is not possible in this case.*

2. Now let us consider the case⁵

$$A(\bar{\psi}, \psi) = m = \text{const}, \\ B_\mu(\bar{\psi}, \psi) = \beta(\bar{\psi}\gamma_\mu\psi), \quad \beta = \text{const}. \quad (2.23)$$

Then we have

$$\hat{D} = -\gamma_\mu \left(\frac{\partial}{\partial x_\mu} + \beta(\bar{\psi}\gamma_\mu\psi) \right) \frac{1}{m}, \\ \hat{D} = -\frac{1}{m} \left(\frac{\partial}{\partial x_\mu} - \beta(\bar{\psi}\gamma_\mu\psi) \right) \gamma_\mu \quad (2.24)$$

and, correspondingly

$$\psi = \varphi - \gamma_\mu \left(\frac{\partial}{\partial x_\mu} + \beta(\bar{\psi}\gamma_\mu\psi) \right) \varphi \frac{1}{m}, \\ \bar{\psi} = \bar{\varphi} + \frac{1}{m} \bar{\varphi} \left(\frac{\partial}{\partial x_\mu} - \beta(\bar{\psi}\gamma_\mu\psi) \right) \gamma_\mu. \quad (2.25)$$

From this we find the equation for $(\bar{\psi}\gamma_\mu\psi)$

$$\frac{1}{\beta} Y_\nu + a_{\nu\rho} Y_\rho + b_{\mu\nu\rho} Y_\mu Y_\rho = g_\nu, \quad (2.26)$$

*It should be noticed that the indicated special solutions φ_I and ψ_I are obtained in a natural manner from (2.13). Indeed, if we write the solution of (2.13) in the form $\varphi = c\psi$, we find that $\psi = (c + \bar{c})\psi$, where $c\gamma_\mu = \gamma_\mu \bar{c}$; that is, $c + \bar{c} = 1$, $c = 1/2(1 + l\gamma_5)$, $\bar{c} = 1/2(1 - l\gamma_5)$, where l is number. Now, if we require that $cc = c$, we get $l = 1$.

where

$$Y_\nu = \beta(\bar{\psi}\gamma_\nu\psi), \\ a_{\nu\rho} = \frac{2}{m} (\bar{\varphi}\delta_{\nu\rho}\varphi) + \frac{1}{m^2} (\bar{\varphi}_\mu\gamma_\mu\gamma_\nu\gamma_\rho\varphi - \bar{\varphi}\gamma_\rho\gamma_\nu\gamma_\mu\varphi_\mu), \\ b_{\mu\nu\rho} = -\frac{\beta}{m^2} (\bar{\varphi}\gamma_\mu\gamma_\nu\gamma_\rho\varphi), \\ g_\nu = (\bar{\varphi}\gamma_\nu\varphi) + \frac{1}{m} (\bar{\varphi}_\mu\gamma_\mu\gamma_\nu\varphi - \bar{\varphi}\gamma_\nu\gamma_\mu\varphi_\mu) \\ - \frac{1}{m^2} (\bar{\varphi}_\mu\gamma_\mu\gamma_\nu\gamma_\rho\varphi_\rho), \quad (2.27)$$

and the Klein-Gordon equation takes the form

$$\left\{ \left(\frac{\partial}{\partial x_\mu} + B_\mu \right) \left(\frac{\partial}{\partial x_\mu} + B_\mu \right) - \sigma_{\mu\nu} H_{\mu\nu} - m^2 \right\} \varphi = 0, \quad (2.28)$$

where

$$\sigma_{\mu\nu} = 1/2 (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), \quad H_{\mu\nu} = \partial B_\nu / \partial x_\mu - \partial B_\mu / \partial x_\nu, \quad (2.29)$$

$B_\mu = \beta(\bar{\psi}\gamma_\mu\psi) \equiv Y_\mu$, the solution of (2.26). It is easy to verify that although the operator of (2.28) does commute with γ_5 , nevertheless $\gamma_5\varphi$ is not a solution of (2.28). Equation (2.26) is not invariant under the transformations (2.19), which implies the noninvariance of (2.28) under these transformations.

The transformation (2.19) leads to that of (2.20) [in virtue of (2.13)], only provided that $m \rightarrow -m$, just as the invariance of (2.1) relative to (2.20) requires the change $m \rightarrow -m$. Therefore, if ψ and φ are solutions of (2.1) and (2.28) for $+m$, $\gamma_5\psi$ and $\gamma_5\varphi$ are solutions for $-m$. We therefore cannot get the relations (2.21) and (2.22), and it is impossible to go over to the two-component spinor equation.*

The problem is greatly simplified when $m = 0$. Then instead of (2.3) and (2.25) one must take

$$\psi = \hat{D}\varphi = -\gamma_\mu \left(\frac{\partial}{\partial x_\mu} + \beta(\bar{\psi}\gamma_\mu\psi) \right) \varphi, \\ \bar{\psi} = -\bar{\varphi}\hat{D} = \bar{\varphi} \left(\frac{\partial}{\partial x_\mu} - \beta(\bar{\psi}\gamma_\mu\psi) \right) \gamma_\mu \quad (2.30)$$

and we find

$$Y_\nu / \beta = (\bar{\varphi}\gamma_\mu\gamma_\nu\gamma_\rho\varphi) Y_\mu Y_\rho - (\bar{\varphi}_\mu\gamma_\mu\gamma_\nu\gamma_\rho\varphi - \bar{\varphi}\gamma_\rho\gamma_\nu\gamma_\mu\varphi_\mu) \\ \times Y_\rho - (\bar{\varphi}_\mu\gamma_\mu\gamma_\nu\gamma_\rho\varphi_\rho), \quad (2.31)$$

where

$$Y_\nu \equiv \beta(\bar{\psi}\gamma_\nu\psi). \quad (2.32)$$

The quadratic equation coincides with (2.28) and

*If we introduce the new functions $\varphi = 1/2(1 + \gamma_5)\varphi'$ and $\varphi' = \bar{\varphi}'1/2(1 - \gamma_5)$, Eq. (2.28) for φ' remains unchanged, and in (2.26) the term $\sim 1/m$ drops out. This last circumstance makes (2.28) invariant under the transformation $\varphi' \rightarrow \gamma_5\varphi'$, $\bar{\varphi}' \rightarrow -\bar{\varphi}'\gamma_5$. However, from the point of view of φ this last transformation is the identity transformation.

(2.29) for $m = 0$, but now $Y_\nu = \beta(\bar{\psi}\gamma_\nu\psi)$ is a solution of (2.31). It is easily seen from (2.30) that the transformation (2.19) leads to (2.20) and that (2.28) is invariant under these transformations. In addition, (2.1) is invariant under (2.20); consequently, the solutions $\psi = \psi_I$ and $\varphi = \varphi_I$ can be chosen with the aid of (2.21) and (2.22).

Following Feynman, we use the relations

$$\varphi_I = \frac{1}{2}(1 + \gamma_5)\psi = \begin{pmatrix} \Phi \\ -\Phi \end{pmatrix},$$

$$\psi = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \Phi = \frac{1}{2}(a - b), \quad (2.33)$$

where a and b are two-component spinors. We then get to the two-component nonlinear spinor equations (now $\bar{\psi}\gamma_\mu\psi = \bar{\psi}_I\gamma_\mu\psi_I \neq 0$) of the first and second orders, respectively.*

We can consider analogously the cases

$$A = \lambda_1(\bar{\psi}\psi) + \lambda_2\gamma_5(\bar{\psi}\gamma_5\psi), \quad B_\mu = 0 \quad (2.34)$$

*Note (added November 5, 1958). After the present work had gone to press we learned about an article by Ascoli, in which the author considers a first-order two-component nonlinear spinor equation, but under more stringent conditions. The two-component equation contains all the solutions of the fundamental equation; at the same time, as in our case, it contains only a part of the solutions we picked out of the fundamental four-component equation.

or

$$A = m, \quad B_\mu = \beta_1(\bar{\psi}\gamma_\mu\psi) + \beta_2\gamma_5(\bar{\psi}\gamma_\mu\gamma_5\psi). \quad (2.35)$$

Several topics touched upon in this article were discussed with Professor D. D. Ivanenko.

¹D. D. Ivanenko and D. F. Kurdgelaidze, Dokl. Akad. Nauk SSSR **88**, 39 (1954). D. F. Kurdgelaidze, Вестн. МГУ, (Bull. Moscow State University) **8**, 81 (1954), and dissertation, Moscow State University, 1954.

²D. F. Kurdgelaidze, J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 1156 (1957), Soviet Phys. JETP **5**, 941 (1957).

³W. Heisenberg, Revs. Modern Phys. **29**, 269 (1957).

⁴W. Heisenberg, Z. Naturforsch. **9a**, 292 (1954).

⁵D. D. Ivanenko and A. M. Brodskii, Dokl. Akad. Nauk SSSR **84**, 683 (1952) and J. Exptl. Theoret. Phys. (U.S.S.R.) **24**, 383 (1953).

⁶D. F. Kurdgelaidze, J. Exptl. Theoret. Phys. **34**, 1587 (1958), Soviet Phys. JETP **7**, 1093 (1958).

⁷R. Ascoli, Nuovo cimento **8**, 501 (1958).