NONRELATIVISTIC SOLUTION OF THE BETHE-SALPETER EQUATION

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A nonrelativistic solution of the Bethe-Salpeter equation has been obtained which includes terms of the order of the relative velocity of the particles.

THE Bethe-Salpeter equation for two identical Dirac particles with mass m and charge e has the following form in momentum space:

$$\left(\tfrac{1}{2}\hat{K}^a+\hat{p}^a-m\right)\left(\tfrac{1}{2}\hat{K}^b-\hat{p}^b-m\right)\psi(p) = \frac{ie^2}{4\pi^3}\int\frac{\gamma_\nu^a\gamma_\nu^b}{k^2}\psi(p-k)\,d^4k, \tag{1}$$

where K is the total, and p the relative, momentum of the system. We always set $\hbar = c = 1$, and the scalar product of any two vectors q and τ is taken as $q\tau = q_\nu\tau_\nu = q_0\tau_0 - q_1\tau_1 - q_2\tau_2 - q_3\tau_3$. The index a refers to the Dirac matrices belonging to the first particle, and the index b, to those belonging to the second particle. We have $\gamma_0 = \beta$, $\gamma_{1,2,3} = \beta\alpha_{1,2,3}$ and $\hat{q} = q_\nu\gamma_\nu$. Finally,

$$(\hat{q}^a\psi)_{\rho_1\rho_2} = (\hat{q}^a)_{\rho_1\rho_1'}\psi_{\rho_1'\rho_2}, \qquad (\hat{q}^b\psi)_{\rho_1\rho_2} = (\hat{q}^b)_{\rho_2\rho_2'}\psi_{\rho_1\rho_2'} = \psi_{\rho_1\rho_2'}(\hat{q}^{bT})_{\rho_2'\rho_2}.$$

We multiply both sides of (1) by the operator

$$\left(\tfrac{1}{2}\hat{K}^a+\hat{p}^a+m\right)\left(\tfrac{1}{2}\hat{K}^b-\hat{p}^b+m\right),$$

and obtain in the coordinate system in which the two-partiole system as a whole is at rest ($\mathbf{K} = 0$, $K_0 = 2m - E$, where E is the binding energy)

$$\left[\left(m-\frac{E}{2}+p_0\right)^2-(\mathbf{p}^2+m^2)\right]\times\left[\left(m-\frac{E}{2}-p_0\right)^2-(\mathbf{p}^2+m^2)\right]\psi(p)$$
$$=\frac{ie^2}{4\pi^3}\left[(1+\beta^a)m-\frac{E}{2}+p_0+(\boldsymbol{\alpha}^a\mathbf{p})\right]\times\left[(1+\beta^b)m-\frac{E}{2}-p_0-(\boldsymbol{\alpha}^b\mathbf{p})\right]\times\int\frac{1-(\boldsymbol{\alpha}^a\boldsymbol{\alpha}^b)}{k_0^2-\mathbf{k}^2}\psi(p-k)\,d^4k. \tag{2}$$

In the nonrelativistic case, $E/m \ll 1$, and the wave function $\psi(p)$ differs from zero appreciably in the region of small momenta, $|\mathbf{p}/m| = v \ll 1$, where v is the relative velocity of the particles, and of small p_0 (the magnitude of p_0 is of the same order as E). In the following we shall be interested in the solution of equation (2) with an accuracy up to and including terms of order v. On the right hand side of (2) we can therefore neglect the terms k_0^2 and $(\boldsymbol{\alpha}^a\boldsymbol{\alpha}^b)$ under the integral, as this neglection is equivalent to replacing the retarded interaction by the instantaneous Coulomb interaction,[1] which is an effect of higher order [the retardation effect gives rise to terms which are smaller than the Coulomb potential by the factor v^2 (reference 2)]. Discarding all terms of higher order of smallness, we rewrite Eq. (2) in the form

$$\left(p_0-\frac{E}{2}-\frac{\mathbf{p}^2}{2m}\right)\left(p_0+\frac{E}{2}+\frac{\mathbf{p}^2}{2m}\right)\psi(\mathbf{p},p_0)$$
$$=\frac{ie^2}{2\pi^2}\left[\frac{1}{2}(1+\beta^a)+\frac{(\boldsymbol{\alpha}^a\mathbf{p})}{2m}\right]\left[\frac{1}{2}(1+\beta^b)-\frac{(\boldsymbol{\alpha}^b\mathbf{p})}{2m}\right]\times\int\frac{1}{\mathbf{k}^2}\psi(\mathbf{p}-\mathbf{k},\,t=0)\,d^3k$$

where $\psi(\mathbf{p}) = \psi(\mathbf{p}, p_0)$. We also used the relation

$$\int\psi(\mathbf{p},p_0)\,dp_0 = 2\pi\psi(\mathbf{p},\,t=0).$$

Multiplying both sides of Eq. (3) by the expression

$$\frac{\exp(-ip_0t)}{2\pi(p_0-E/2-\mathbf{p}^2/2m)(p_0+E/2+\mathbf{p}^2/2m)} \tag{4}$$

and integrating over p_0 under the assumption that m in (1) has an infinitesimal negative imaginary part (this gives rise to the infinitesimal terms $+i\delta$ and $-i\delta$, where $\delta > 0$, in the first and second parentheses of the denominator of (4), respectively), we obtain, for $t > 0$,

$$\psi(\mathbf{p},t)=\frac{e^2}{2\pi^2}\frac{\exp\{-i(E/2+\mathbf{p}^2/2m)t\}}{E+p^2/m}\left[\frac{1}{2}(1+\beta^a)+\frac{(\boldsymbol{\alpha}^a\mathbf{p})}{2m}\right]\times\left[\frac{1}{2}(1+\beta^b)-\frac{(\boldsymbol{\alpha}^b\mathbf{p})}{2m}\right]\int\frac{1}{\mathbf{k}^2}\psi(\mathbf{p}-\mathbf{k},\,t=0)\,d^3k. \tag{5}$$

For the following it is convenient to introduce the notation

$$\psi(p)=\begin{pmatrix}\psi^{\sigma}(p) & \psi^{M_2}(p)\\ \psi^{M_1}(p) & \psi^{M_3}(p)\end{pmatrix}, \tag{6}$$

where $\psi^\sigma(p)$, $\psi^{M1}(p)$, $\psi^{M2}(p)$, and $\psi^{M3}(p)$ are two-rowed functions. Furthermore we set

$$\psi(\mathbf{p},\ t=0)=\varphi(\mathbf{p}). \qquad (7)$$

Using (5), (6), and (7), we find the following equations for the determination of the components of the wave function (7):

$$\left(E+\frac{\mathbf{p}^2}{m}\right)\varphi^{\sigma}(\mathbf{p})=\frac{e^2}{2\pi^2}\int\frac{1}{\mathbf{k}^2}\left[\varphi^{\sigma}(\mathbf{p}-\mathbf{k})+\frac{(\sigma^a\mathbf{p})}{2m}\varphi^{M_1}(\mathbf{p}-\mathbf{k})-\frac{(\sigma^b\mathbf{p})}{2m}\varphi^{M_2}(\mathbf{p}-\mathbf{k})-\frac{(\sigma^a\mathbf{p})(\sigma^b\mathbf{p})}{4m^2}\varphi^{M_3}(\mathbf{p}-\mathbf{k})\right]d^3k, \qquad (8)$$

$$\left(E+\frac{\mathbf{p}^2}{m}\right)\varphi^{M_1}(\mathbf{p})=\frac{e^2}{2\pi^2}\int\frac{1}{\mathbf{k}^2}\left[\frac{(\sigma^a\mathbf{p})}{2m}\varphi^{\sigma}(\mathbf{p}-\mathbf{k})-\frac{(\sigma^a\mathbf{p})(\sigma^b\mathbf{p})}{4m^2}\varphi^{M_2}(\mathbf{p}-\mathbf{k})\right]d^3k, \qquad (9)$$

$$\left(E+\frac{\mathbf{p}^2}{m}\right)\varphi^{M_2}(\mathbf{p})=-\frac{e^2}{2\pi^2}\int\frac{1}{\mathbf{k}^2}\left[\frac{(\sigma^b\mathbf{p})}{2m}\varphi^{\sigma}(\mathbf{p}-\mathbf{k})+\frac{(\sigma^a\mathbf{p})(\sigma^b\mathbf{p})}{4m^2}\varphi^{M_1}(\mathbf{p}-\mathbf{k})\right]d^3k, \qquad (10)$$

$$\left(E+\frac{\mathbf{p}^2}{m}\right)\varphi^{M_3}(\mathbf{p})=-\frac{e^2}{2\pi^2}\int\frac{1}{\mathbf{k}^2}\frac{(\sigma^a\mathbf{p})(\sigma^b\mathbf{p})}{4m^2}\varphi^{\sigma}(\mathbf{p}-\mathbf{k})\,d^3k. \qquad (11)$$

It is seen from equations (8) to (11) that $\varphi^{M_1}(\mathbf{p})$ and $\varphi^{M_2}(\mathbf{p})$ are of the order of the relative velocity v, and the function $\varphi^{M_3}(\mathbf{p})$ is of the order v^2. In contrast to this, the function $\varphi^{\sigma}(\mathbf{p})$ contains "large components" which do not reduce to zero for $v = |\mathbf{p}/m| \to 0$. Since we are interested in the wave function (7) with an accuracy up to and including terms of order v, we shall discard the terms containing φ^{M_1} and φ^{M_2} on the right hand sides of Eqs. (8), (9), and (10), and set the function φ^{M_3} equal to zero, as it is of order v^2. Furthermore, we replace the integral over the variable **k** Eqs. (9) and (10) by $(E+\mathbf{p}^2/m)\varphi^{\sigma}(\mathbf{p})$, using Eq. (8). As a result we find that the large components of the wave function (7) satisfy the Schrödinger equation, which in **x** space has the usual form*

$$\left(\frac{1}{m}\nabla^2+\frac{e^2}{|\mathbf{x}|}\right)\varphi^{\sigma}(\mathbf{x})=E\varphi^{\sigma}(\mathbf{x}), \qquad (12)$$

The small components are given by

$$\varphi^{M_1}(\mathbf{x})=\frac{(\sigma^a\nabla)}{2mi}\varphi^{\sigma}(\mathbf{x}),\qquad \varphi^{M_2}(\mathbf{x})=-\frac{(\sigma^b\nabla)}{2mi}\varphi^{\sigma}(\mathbf{x}),$$
$$\varphi^{M_3}(\mathbf{x})=0. \qquad (13)$$

In the mixed representation $\psi(\mathbf{p},t)$, obtained from $\psi(\mathbf{x},t)$ by a Fourier transformation with respect to the spatial coordinates **x** only, the wave function corresponding to the Bethe-Salpeter equation (1) has, according to (5), the following form for $t > 0$ (with an accuracy up to and including terms of order v):

$$\psi(\mathbf{p},t)=\begin{pmatrix}\varphi^{\sigma}(\mathbf{p}) & -\frac{(\sigma^b\mathbf{p})}{2m}\varphi^{\sigma}(\mathbf{p})\\ \frac{(\sigma^a\mathbf{p})}{2m}\varphi^{\sigma}(\mathbf{p}) & 0\end{pmatrix}\exp\left\{-i\left(\frac{E}{2}+\frac{\mathbf{p}^2}{2m}\right)t\right\} \qquad (14)$$

For $t < 0$ the exponent in (14) takes the opposite sign. It must be emphasized that the two-rowed (four-component) function $\varphi^{\sigma}(\mathbf{p})$ in (14) satisfies the Schrödinger equation and is also an eigenfunction of the operators of the total angular momentum of the system and its projection. This means that the function $\varphi^{\sigma}(\mathbf{p})$ satisfies, in the spinor indices, the corresponding operator equations of nonrelativistic quantum mechanics. Expression (14) is also valid for the mesonic interaction of the nucleons; however, in this case the Coulomb potential in the Schrödinger equation (12) must be replaced by the Yukawa potential or its generalization.

With the wave function (14) we can calculate the annihilation probability for bound particles (positronium) in the P state.[3,4] With the help of formula (14) it is possible to take account of the Coulomb interaction between the electron and the positron in the pair production by photons, such that the expression for the differential cross section contains all terms of the order of the relative velocity between the pair of particles. If Eq. (1) is to be solved with an accuracy up to and including terms of order v^2, the expression for the interaction energy between the electron and the positron will differ from the known result[5] by the absence of the exchange interaction, since Eq. (1) does not include exchange forces. To obtain the correct result, Eq. (1) has to be replaced by an equation of the Bethe-Salpeter type which contains, together with the ordinary interaction, a specific exchange interaction between the electron and the positron arising from their virtual annihilation.[6]

*Equation (12) was obtained by Bethe and Salpeter[1] as the nonrelativistic limit of Eq. (1).

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