TRANSITION RADIATION AND CERENKOV RADIATION

V. E. PAFOMOV

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

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The angular distribution of the radiation produced by a charged particle which passes through the interface between a vacuum and an isotropic ferrite is considered; the case of a vacuum and a crystalline dielectric is also considered. It is shown that the radiation depends on the sign of the group velocity. The transition radiation is considered in connection with the characteristics of Cerenkov radiation in crystals and in isotropic media in the frequency region characterized by negative group velocities. The "inverse" Doppler effect is considered.

1. INTRODUCTION

TRANSITION radiation of a charged particle which moves perpendicularly to the boundary between two media with different dielectric constants was first considered by Ginzburg and Frank.[1] The problem has also been treated by other methods in later work.[2-5]

In the present paper we consider transition radiation, taking account of the magnetic permeability. It is shown that this radiation is affected significantly by the sign of the group velocity. We also consider transition radiation in connection with the characteristics of Cerenkov radiation in frequency regions associated with negative group velocities. In these regions the solutions of Maxwell's equations appear as advanced potentials which carry energy from the radiator; it is found that the Cerenkov ray forms an obtuse angle with the direction of motion of the particle. In this case a sharp intensity peak (in the transition radiation) due to the generation of Cerenkov radiation in the medium should be observed in the vacuum if the particle moves from vacuum into the medium.

We also consider the radiation of a charged particle which moves perpendicularly to the boundary between a vacuum and a uniaxial crystalline dielectric. This problem is of interest in connection with Cerenkov radiation in crystals because in certain frequency regions the radiation is described by an advanced potential characterized by a phase which moves along the particle trajectory. There is also a frequency region in which the Cerenkov ray forms an obtuse angle with the direction of motion of the charged particle (cf. reference 6).

In an isotropic medium these characteristics of the Cerenkov radiation are found in a single frequency region — the region in which the group velocity is negative.

In conclusion we consider the so-called "inverse" Doppler effect — low-frequency radiation in the forward direction. From the inverse Doppler effect it follows that in the case of a source of zero frequency which moves with a velocity greater than the phase velocity of light the Cerenkov radiation which is produced should form an obtuse angle with respect to the direction of motion.

2. RADIATION OF A CHARGED PARTICLE THAT PASSES THROUGH THE BOUNDARY BETWEEN A VACUUM AND AN ISOTROPIC FERRITE

The problem is solved conveniently in cylindrical coordinates. Taking the z axis in the direction of the particle velocity **v** we write the current density, as in reference 3, in the following form:

$$j = \frac{e}{(2\pi)^2} \int_{-\infty}^{+\infty} e^{i\omega(t - z/v)}\, d\omega \int_0^{\infty} \alpha J_0(\alpha r)\, d\alpha. \tag{1}$$

Introducing the Hertz vector in accordance with the expression

$$\mathbf{E}_\omega = \frac{1}{\varepsilon\mu} \operatorname{grad} \operatorname{div} \mathbf{\Pi}_\omega + \frac{\omega^2}{c^2} \mathbf{\Pi}_\omega, \qquad \mathbf{H}_\omega = \frac{i\omega}{\mu c} \operatorname{curl} \mathbf{\Pi}_\omega, \tag{2}$$

we obtain the following solution for Maxwell's equations:

$$\Pi = -\frac{ie}{\pi} \int_{-\infty}^{+\infty} \frac{\mu e^{i\omega t}}{\omega}\, d\omega \int_0^{\infty} \alpha J_0(\alpha r) \Big[\frac{e^{-i\omega z/v}}{\alpha^2 + (\omega/v)^2 (1 - \varepsilon\mu\beta^2)}$$
$$+ A \exp\left(z\sqrt{\alpha^2 - \varepsilon\mu\omega^2/c^2}\right)$$
$$+ B \exp\left(-z\sqrt{\alpha^2 - \varepsilon\mu\omega^2/c^2}\right) \Big] d\alpha, \tag{3}$$

where the last two terms satisfy the source free equation and are determined from the boundary conditions. The Hertz vector has only one non-vanishing component; this is along the z axis (in Eq. (3) we have dropped the z subscript).

We shall assume that the boundary coincides with the plane $z = 0$ and that the ferrite occupies the semi-space $z < 0$. In the vacuum region ($z > 0$) the solution is expressed by Eq. (3) if we take $\epsilon = 1$ and $\mu = 1$. If

$$\mathrm{Re}\sqrt{\alpha^2 - \varepsilon_f\mu_f\omega^2/c^2} > 0$$

to satisfy the requirement that the solution be finite at infinity we take $B_f = 0$ and $A_v = 0$.

Using the boundary conditions*

$$\frac{1}{\varepsilon\mu}\frac{\partial \Pi_f}{\partial z} = \frac{\partial \Pi_v}{\partial z}, \quad \frac{1}{\mu}\Pi_f = \Pi_v, \tag{4}$$

we find the components A_f and B_v which differ from zero.

In what follows we shall be interested in the radiation energy in the vacuum region. The radiation field in the vacuum is completely described by the term which contains B_v, which does not become infinite if the ferrite has losses. In this case the integration over α can be carried out by the method of stationary phase. The important contribution in the integral is due to a small region of order $\sqrt{\omega/cR}$ close to $\alpha = (\omega/c)\sin\vartheta$, where R is the distance from the origin of coordinates to the point of observation. Hence, at large distances from the origin integration over α means approximately that we replace α by $(\omega/c)\sin\vartheta$. Carrying out this integration and computing the Poynting vector flux through an isolated hemisphere we obtain the vacuum radiation energy for a charged particle which moves from the ferrite into the vacuum:

$$W = \frac{2e^2v^2}{\pi c^3}\int_0^\infty d\omega \int_0^{\pi/2} \frac{\sin^3\vartheta\cos^2\vartheta}{(1-\beta^2\cos^2\vartheta)^2}$$
$$\times\left|\frac{(\varepsilon-1)(1-\beta\sqrt{\varepsilon\mu-\sin^2\vartheta})-\beta^2(\varepsilon\mu-1)}{(1-\beta\sqrt{\varepsilon\mu-\sin^2\vartheta})(\varepsilon\cos\vartheta+\sqrt{\varepsilon\mu-\sin^2\vartheta})}\right|^2 d\vartheta, \tag{5}$$

where $\beta = v/c$ is the ratio of the particle velocity to the velocity of light in vacuum and ϑ is the angle between the z axis and the direction of observation.

In the case in which the particle moves from vacuum into the ferrite, the vacuum radiation energy is obtained from Eq. (5) by replacing β by $-\beta$. When $\mu = 1$, Eq. (5) becomes the equation which has been obtained earlier (cf. references 1 — 5). Equation (5) describes the entire energy which appears in the vacuum region, including the Cerenkov radiation generated in the medium.*

The integration over α (to within a numerical factor) leads to the replacement of α by $(\omega/c)\sin\vartheta$. Whence it follows that the inequality which we have taken above ($\mathrm{Re}\sqrt{\alpha^2 - \epsilon\mu\omega^2/c^2} > 0$) corresponds to the following inequality:

$$\mathrm{Im}\sqrt{\varepsilon\mu - \sin^2\vartheta} < 0. \tag{6}$$

Equation (5), which is written under the assumption that (6) holds, is valid over the entire frequency range, including the range in which the group velocity is negative.

We may note that in principle the group velocity in a ferrite ($\epsilon \neq 1$, $\mu \neq 1$) can be negative.† In order to show this, we substitute a plane wave in Maxwell's equations ($\mathbf{E}, \mathbf{H} \sim \exp[i(\omega t - \mathbf{kr})]$) in which case the following vector relation holds:

$$\mathbf{k}E^2 = (\omega\mu/c)[\mathbf{E}\times\mathbf{H}] = (\omega\mu/4\pi)\mathbf{S}, \qquad k = \omega\sqrt{\varepsilon\mu}/c,$$

where $\mathbf{S}$ is the Poynting vector. Undamped electromagnetic waves can exist both for $\epsilon > 0$, $\mu > 0$ as well as for $\epsilon < 0$, $\mu < 0$ since the index of refraction $n = \sqrt{\epsilon\mu}$ is a real quantity in both of these cases. It is apparent from the last relation that in the first case the Poynting vector coincides in direction with the wave vector $\mathbf{k}$; in the second case $\mathbf{S}$ and $\mathbf{k}$ point in opposite directions, i.e., the group velocity is negative. In the frequency region in which the following relation is not satisfied: $\mathrm{Re}(\partial\omega/\partial\mathbf{k}) \gg \mathrm{Im}(\partial\omega/\partial\mathbf{k})$ the concept of group velocity has no meaning. In this case the term "positive (negative) group velocity" is no longer meaningful in the above sense.

In the frequency region in which the group ve-

*The subscript "f" corresponds to quantities taken in the region $z < 0$, occupied by the ferrite, while the subscript "v" corresponds to vacuum quantities ($z > 0$).

*In this connection we wish to point out an error in the interpretation of the corresponding result in reference 4, where an analysis is made of the radiation of a charged particle which moves through a vacuum-dielectric boundary and it is asserted that the Cerenkov waves do not make a contribution in the first integral in Eq. (27). At the large distances for which the first integral in Eq. (27) is computed in reference 4 the cylindrical Cerenkov wave is transformed into a spherical wave and the second integral can be neglected. To convince ourselves of this it is only necessary to take account of the fact that the residue at the pole in the integration over $\varkappa$ in Eq. (23) is to be taken only when the pole in the complex plane lies between the real axis and the line of steepest descent (if there is damping, in Eq. (27) of reference 4 the quantity ε should be used instead of ε'.

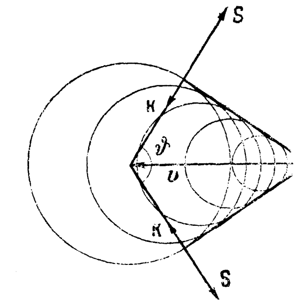
†The group velocity can also be negative in a medium with spatial dispersion (cf. reference 7). The features associated with a negative group velocity which are being considered in the present paper are quite general.

locity is negative the solution is given by advanced potentials, with phases which do not move away from the radiator but approach it. In this case one expects peculiar features for the radiation in moving systems. In the figure is shown a diagram of the propagation of waves which are excited by a uniformly moving particle which interacts with the electromagnetic field. If the velocity of the particle exceeds the phase velocity of light ($n\beta > 1$) then in the direction given by ϑ, which satisfies the relation $n\beta \cos\vartheta = 1$, all the elementary waves are propagated in the same phase. The Cerenkov waves formed in this way exhibit a surface of uniform phase in the form of a cone with the vertex in the forward direction. The phase velocity forms an acute angle ϑ with the direction of motion of the particle. However, the energy flow is in the opposite direction. As a result a narrow peak in the transition radiation intensity, due to the Cerenkov radiation, should be observed in vacuum when the particle moves from vacuum into the medium. One is easily convinced of this if one introduces damping $\epsilon = \epsilon' - i\epsilon''$, $\mu = \mu' - i\mu''$, where $\epsilon'' > 0$ and $\mu'' > 0$. In this case, in the region of negative group velocity (6) corresponds to the inequality $\operatorname{Re}\sqrt{\epsilon\mu - \sin^2\vartheta} < 0$. Taking account of this situation and replacing β by $-\beta$ we see that Eq. (5) yields a narrow maximum ($\beta^2(\epsilon\mu - \sin^2\vartheta_C) = 1$) in the direction of the refracted Cerenkov angle. Under these conditions, in the vacuum at small distances from the medium there will be a converging cylindrical Cerenkov wave which, after intersecting the normal to the boundary, diverges and fills the inner part of the Cerenkov cone. One is easily convinced of this result from an analysis of the diffraction of the plane waves at the boundary.

In actual ferrites there is no frequency region in which $\epsilon < 0$ and $\mu < 0$; however these materials do not represent the only possibility. As we have noted above, the group velocity can be negative in a medium characterized by spatial dispersion.*

3. RADIATION OF A CHARGED PARTICLE WHICH MOVES THROUGH A VACUUM-CRYSTAL BOUNDARY

As has been indicated in reference 6, the Cerenkov ray generated in a crystalline dielectric may



form an obtuse angle with the direction of motion of the charged particle. It is of interest to consider the radiation of a point charge which passes through a vacuum-crystal boundary in connection with this feature of Cerenkov radiation.

We take the velocity to be along the z axis and introduce the Hertz vector:

$$E_\omega = \frac{1}{\varepsilon_0}\,\mathrm{grad}\,\frac{\partial \Pi_\omega}{\partial z} + \frac{\omega^2}{c^2}\,\Pi_\omega, \qquad H_\omega = i\frac{\omega}{c}\,\mathrm{curl}\,\Pi_\omega, \tag{7}$$

for a uniaxial crystal with axis in the z direction we obtain the following solution for Maxwell's equations:

$$\Pi = -\frac{ie}{\pi}\int_{-\infty}^{+\infty}\frac{e^{i\omega t}}{\omega}\,d\omega\int_0^\infty \alpha J_0(\alpha r)\left\{\frac{e^{-i\omega z/v}}{\alpha^2 + (\varepsilon_e\omega^2/\varepsilon_0 v^2)(1-\varepsilon_0\beta^2)}\right.$$
$$+ A\exp\left[z\sqrt{(\alpha^2 - \varepsilon_e\omega^2/c^2)\,\varepsilon_0/\varepsilon_e}\right]$$
$$\left.+ B\exp\left[-z\sqrt{(\alpha^2 - \varepsilon_e\omega^2/c^2)\,\varepsilon_0/\varepsilon_e}\right]\right\}d\alpha, \tag{8}$$

where ϵ_0 and ϵ_e are the transverse (to the crystal axis) and longitudinal components of the dielectric primitivity tensor. The Hertz vector has one non-vanishing component (along the z-axis) so that we drop the index (also the vector nature of Π).

We assume that the crystal fills the half-space $z < 0$ and that the region for which $z > 0$ is a vacuum. Assuming that

$$\operatorname{Re}\sqrt{(\alpha^2 - \varepsilon_e\omega^2/c^2)\,\varepsilon_0/\varepsilon_e} > 0, \tag{9}$$

we determine the solution completely by requiring that the solution be finite at infinity (the vacuum wave diverges from the origin), using the continuity conditions on the tangential components of the electric and magnetic fields; these are equivalent to the relations

$$\frac{1}{\varepsilon_0}\frac{\partial \Pi_v}{\partial z} = \frac{\partial \Pi_c}{\partial z}, \qquad \Pi_c = \Pi_v$$

where the subscript "c" denotes quantities taken in the crystal.

Carrying out the calculations as in the preceding section we obtain the vacuum radiation energy for a charged particle which moves from the crystal into the vacuum:

*In reference 8 the Cerenkov radiation has been considered for isotropic gyrotropic media in which spatial dispersion is taken into account. The energy in reference 8 which corresponds to the Cerenkov radiation in the region of negative group velocity moves at an obtuse angle with respect to the direction of motion of the particle.

$$W = \frac{2e^2v^2}{\pi c^3} \int_0^\infty d\omega \int_0^{\pi/2} \frac{\sin^3\vartheta \cos^2\vartheta}{(1-\beta^2\cos^2\vartheta)^2} \left| B(\omega, \vartheta) \right|^2 d\vartheta,$$

$$B(\omega, \vartheta) = \frac{\varepsilon_0}{\varepsilon_e} \left[\beta \left(1 - \beta^2 - \frac{\varepsilon_e}{\varepsilon_0} + \beta^2 \varepsilon_e\right) \sqrt{\varepsilon_0 - \frac{\varepsilon_0}{\varepsilon_e}\sin^2\vartheta} + (\beta^2\sin^2\vartheta + 1 - \beta^2 - \beta^2\varepsilon_0\sin^2\vartheta - \varepsilon_e + \beta^2\varepsilon_0\varepsilon_e) \right] \left[\varepsilon_0\cos\vartheta + \sqrt{\varepsilon_0 - \frac{\varepsilon_0}{\varepsilon_e}\sin^2\vartheta} \right]^{-1} \left[1 - \beta^2\left(\varepsilon_0 - \frac{\varepsilon_0}{\varepsilon_e}\sin^2\vartheta\right)\right]^{-1}. \tag{10}$$

The vacuum radiation energy for the case of motion of a particle from vacuum into a crystal is obtained from Eq. (10) by replacing β by $-\beta$. When $\epsilon_0 = \epsilon_e = 1$ the quantity W vanishes as is to be expected and when $\epsilon_0 = \epsilon_e = \epsilon$ Eq. (10) gives the familiar result (cf. references 1 — 5).

In the integration over α, as in the preceding section, the important region is the small region in the vicinity of the point of stationary phase and in integrating over α we replace α by $\frac{\omega}{c}\sin\vartheta$. Hence the relation in (9) becomes:

$$\operatorname{Im}\sqrt{\varepsilon_0 - (\varepsilon_0/\varepsilon_e)\sin^2\vartheta} < 0. \tag{11}$$

In Eq. (10) it is assumed that this inequality is satisfied.

As has already been shown in reference 6, the Cerenkov ray forms an obtuse angle in the frequency region in which $\epsilon_0 < 0$ and $\epsilon_e > 0$. By introducing a small damping factor ($\epsilon_{0,e} = \epsilon'_{0,e} - i\epsilon''_{0,e}$, $\epsilon''_{0,e} > 0$) we can show that when $\epsilon_0 < 0$ and $\epsilon_e > 0$ the following inequality is satisfied

$$\operatorname{Re}\sqrt{\varepsilon_0 - (\varepsilon_0/\varepsilon_e)\sin^2\vartheta} < 0, \tag{12}$$

where ϑ is the refracted Cerenkov angle which, as has been shown, satisfies the equation

$$\beta^2\left(\varepsilon_0 - \frac{\varepsilon_0}{\varepsilon_e}\sin^2\vartheta\right) = 1. \tag{13}$$

Making use of Eq. (12) and replacing β by $-\beta$, from Eq. (10) we can show that in the direction of the refracted Cerenkov ray there should be a sharp radiation maximum because in this frequency region the Cerenkov ray forms an obtuse angle with the direction of motion.

4. "INVERSE" DOPPLER EFFECT

In the frequency region in which the Cerenkov ray forms an obtuse angle with the direction of motion one also expects peculiar properties for the radiation of a moving oscillator. This is apparent from the Doppler formula

$$|\omega_\vartheta - \mathbf{k}\cdot\mathbf{v}| = \omega_0, \tag{14}$$

where ω_0 is the source frequency in the rest system, $\mathbf{v}$ is the velocity, $\mathbf{k}$ is the wave vector: $k = \omega_\vartheta n(\omega_\vartheta, \vartheta)/c$ (ϑ is the angle between the vectors $\mathbf{k}$ and $\mathbf{v}$). It is apparent from Eq. (14) that in the frequency region in which the projections of the wave vector and the group velocity on the direction of motion have different signs lower frequencies are radiated in the forward direction ("inverse" Doppler effect).* From the "inverse" Doppler effect it follows that for zero frequency of a source which moves with a velocity exceeding the phase velocity of light the Cerenkov radiation forms an obtuse angle with the direction of motion.†

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*The "inverse" Doppler effect in anisotropic gyrotropic media has been considered in reference 9.

†The role of the group velocity in the radiation of a moving oscillator has been discussed by Frank.[10]