DAMPING OF OSCILLATIONS OF A DISC IN ROTATING HELIUM II

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The interaction of an oscillating disc with rotating helium II is examined. An expression is obtained for the torsional moment acting on the surface of the disc, taking into account the presence of vortex lines, mutual friction between the normal and superfluid components, and the possibility of sliding of the vortex lines along the solid surface. A calculation of the damping of oscillations, which neglects sliding and is in linear approximation with respect to the mutual friction coefficients, is in qualitative agreement with the experimental data. It seems that quantitative agreement can only be obtained if sliding is taken into account.

1. The aim of the present work is the explanation of the dependence of the damping of torsional oscillations of a disc on the speed of rotation for helium II,[1,2] which is completely different from the behavior of a classical liquid.[3] A preliminary investigation showed that this dependence cannot be explained by taking account of mutual friction alone.[4] The corresponding effect is less than that observed by at least an order of magnitude.

2. We use Hall's relation[5] to describe the rotating superfluid component

$$\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s\nabla)\mathbf{v}_s + \nu_s\left[\boldsymbol{\omega}\times\operatorname{curl}\frac{\boldsymbol{\omega}}{\omega}\right] = \alpha_n\boldsymbol{\omega}\times[\mathbf{v}_n-\mathbf{v}_s] + \beta_n\left[\frac{\boldsymbol{\omega}}{\omega}\,\boldsymbol{\omega}\times[\mathbf{v}_n-\mathbf{v}_s]\right] + \nabla\Phi. \tag{1}$$

In this the existence of Onsager-Feynman vortices is taken into account by the additional (third) term which distinguishes this equation from the usual one for the velocity of the superfluid component. In equation (1) $\mathbf{v}_s$ denotes the velocity of the superfluid component, averaged over a volume of linear dimensions considerably greater than the distance between the vortices: $\nu_s = \epsilon/\rho_s\Gamma$, where ϵ is the energy per unit length of vortex line and Γ is the circulation; $\boldsymbol{\omega} = \operatorname{curl}\mathbf{v}_s$; α_n and β_n are the coefficients of mutual friction, related to the coefficients of Hall and Vinen[4] by the equations $\alpha_n = 0.5(\rho_n/\rho)B'$ and $\beta_n = -0.5 \times (\rho_n/\rho)B$; the expression $\nabla\Phi$ contains all gradient terms.*

The normal component is described by the relation

$$\frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n\nabla)\mathbf{v}_n = \nu_n\nabla^2\mathbf{v}_n - \alpha_s\boldsymbol{\omega}\times[\mathbf{v}_n-\mathbf{v}_s] - \beta_s\omega^{-1}\boldsymbol{\omega}\times[\boldsymbol{\omega}\times[\mathbf{v}_n-\mathbf{v}_s]] + \nabla\Psi. \tag{2}$$

Here ν_n is the kinematic viscosity of the normal component, $\alpha_s = 0.5(\rho_s/\rho)B'$, $\beta_s = -0.5(\rho_s/\rho)B$, and $\nabla\Psi$ contains the gradient terms.

As the relevant effect has been observed at constant temperature, the independent equations of continuity have been used for the superfluid and normal components:

$$\operatorname{div}\mathbf{v}_s = 0, \quad \operatorname{div}\mathbf{v}_n = 0. \tag{3}$$

3. The system (1) – (3) has been linearized for small oscillatory additions to a velocity distribution of the type $\mathbf{v}_s = \mathbf{v}_n = \boldsymbol{\omega}_0\times\mathbf{r} = \mathbf{v}_0$ ($\boldsymbol{\omega}_0$ is the angular velocity of rotation) and solved for the boundary conditions corresponding to adhesion of the normal component to the surface of the disc. The boundary conditions for $\mathbf{v}_s$ were obtained in the following way.* Equation (1) is rewritten in the form of an equation of conservation of vortices

$$\partial\mathbf{v}_s/\partial t = \mathbf{v}_L\times\boldsymbol{\omega} + \text{gradient terms} \tag{4}$$

where $\mathbf{v}_L$ is the velocity of the vortex lines. It can easily be shown that

$$\mathbf{v}_L\times\boldsymbol{\omega} = \left\{\mathbf{v}_s + \nu_s\operatorname{curl}\frac{\boldsymbol{\omega}}{\omega} - \alpha_n(\mathbf{v}_n-\mathbf{v}_s) - \beta_n\frac{\boldsymbol{\omega}}{\omega}\times[\mathbf{v}_n-\mathbf{v}_s]\right\}\times\boldsymbol{\omega}. \tag{5}$$

The boundary conditions for the r and φ components of $\mathbf{v}_s$ can be found from (5) by substituting the boundary conditions for the φ and r components of $\mathbf{v}_L$. Taking account of partial sliding of

*L. D. Landau has shown that (1) follows from the laws of conservation of energy and momentum. The value of η_s, equal to $dE/d\omega$ (where E is the energy per unit volume of rotating helium), is related to ν_s by the relation $\eta_s = \rho_s\nu_s$.

*This method of obtaining the boundary conditions for $\mathbf{v}_s$ is due to L. D. Landau.

the vortex lines along the surface of the disc, we obtain*

$$w_{sr}(0) - 2\omega_0\varphi_0 = \frac{a}{i\Omega}(dw_{sr}/dz)_{z=0},$$

$$w_{s\varphi}(0) = \frac{a}{i\Omega}(dw_{s\varphi}/dz)_{z=0}. \quad (6)$$

In these equations $\mathbf{w}_S(z)$ denotes $(\mathbf{v}_S - \mathbf{v}_0)/r \times \exp(i\Omega t)$; Ω and φ_0 are the frequency and amplitude of the oscillations and a is the coefficient of sliding. We assume that the z axis is in a direction at right angles to the surface of the disc, which corresponds to $z = 0$. We have, finally, the natural boundary condition

$$w_{sz}(0) = 0. \quad (7)$$

4. If we consider the oscillation of an infinite disc in an unbounded liquid, the solution of the system (1) — (3) consists of the sum of four plane waves with wave numbers determined by the relations

$$k_{1,2}^2 = \frac{1}{2}(k_{n0}^{(+)2} + k_{s0}^{(+)2} + q_1^{(+)} + q_2^{+)})$$
$$\pm \frac{1}{2}[(k_{s0}^{(+)2} - k_{n0}^{(+)2} + q_1^{+)} - q_2^{+)})^2 + 4q_1^{(+)}q_2^{(+)}]^{1/2},$$

$$k_{3,4}^2 = \frac{1}{2}(k_{n0}^{(-)2} + k_{s0}^{(-)2} + q_1^{(-)} + q_2^{(-)})$$
$$\pm \frac{1}{2}[(k_{s0}^{(-)2} - k_{n0}^{-)2} + q_1^{(-)} - q_2^{(-)})^2 + 4q_1^{(-)}q_2^{(-)}]^{1/2};$$

$$k_{n0}^{(\pm)2} = -i(\Omega \pm 2\omega_0)/\nu_n, \qquad k_{s0}^{(\pm)2} = \mp(\Omega \pm 2\omega_0)/\nu_s,$$

$$q_1^{(\pm)} = \mp 2\omega_0 i(\beta_n \mp i\alpha_n)/\nu_s, \qquad q_2^{(\pm)} = 2\omega_0(\beta_s \mp i\alpha_s)/\nu_n. \quad (8)$$

The penetration depths corresponding to these wave numbers are sufficiently small for the walls of the beaker to be considered to be located at infinity.† In addition to the boundary conditions already given, there must then be no oscillatory addition to $\mathbf{v}_0$ at infinity.

5. The moment of the force, acting on the surface of the disc is given by the formula‡

$$M = \pi R^4[\eta_n(dw_{n\varphi}/dz)_{z=0} + \eta_s(dw_{sr}/dz)_{z=0}]\exp(i\Omega t), \quad (9)$$

with the values

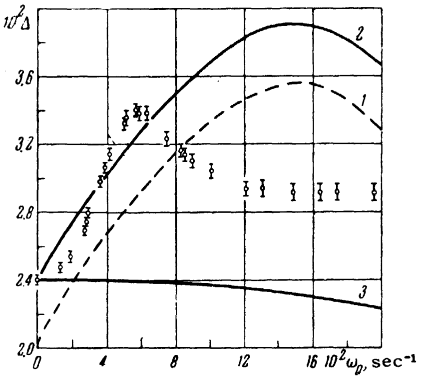
$$\frac{2}{\varphi_0}\left(\frac{dw_{n\varphi}}{dz}\right)_{z=0} = \frac{\Omega(\varkappa_1k_2 - \varkappa_2k_1) + 2\omega_0q_2^{(+)}(k_2 - k_1) + ak_1k_2(\varkappa_2 - \varkappa_1)}{\varkappa_2 - \varkappa_1 - a(\varkappa_2k_2 - \varkappa_1k_1)/\Omega}$$
$$+ \frac{\Omega(\varkappa_3k_4 - \varkappa_4k_3) - 2\omega_0q_2^{(-)}(k_4 - k_3) + ak_3k_4(\varkappa_4 - \varkappa_3)}{\varkappa_4 - \varkappa_3 - a(\varkappa_4k_4 - \varkappa_3k_3)/\Omega},$$

$$\frac{2}{\varphi_0}\left(\frac{dw_{sr}}{dz}\right)_{z=0} = i\frac{2\omega_0(\varkappa_2k_2 - \varkappa_1k_1) - \Omega q_1^{(+)}(k_2 - k_1)}{\varkappa_2 - \varkappa_1 - a(\varkappa_2k_2 - \varkappa_1k_1)/\Omega}$$
$$+ i\frac{2\omega_0(\varkappa_4k_4 - \varkappa_3k_3) + \Omega q_1^{(-)}(k_4 - k_3)}{\varkappa_4 - \varkappa_3 - a(\varkappa_4k_4 - \varkappa_3k_3)/\Omega}. \quad (10)$$

In (10) we have introduced the notation

$$\varkappa_{1,2} = -k_{1,2}^2 + k_{n0}^{(+)2} + q_2^{(+)}, \qquad \varkappa_{3,4} = -k_{3,4}^2 + k_{n0}^{(-)2} + q_2^{(-)}. \quad (11)$$

6. In view of the unwieldiness of (10) for making comparison with experimental data, we will only consider the case $a = 0$ (perfectly rough surface) and carry out the calculation to linear approximation with respect to the mutual friction coefficients (which is justifiable for relatively small values of ω_0). The results of this calculation are shown in the figure together with the experimental results. The ordinate is $\Delta = \delta - \Omega_0\delta_0/\Omega$, where δ and δ_0 are the logarithmic decrements in helium and in vacuum respectively, and Ω and Ω_0 are the corresponding frequencies of oscillation. We are now carrying out the calculations without the restrictions mentioned.



Dependence of Δ on the speed of rotation of helium II. 1 – Result calculated from Eq. (9) with the coefficients corresponding to T = 1.78°K, not taking account of sliding; 2 – The same with a correction for edge effects to first order [Eq. (9)];[3] 3 – Damping due to the viscosity of the normal component; the experimental points were obtained at 1.78°K with a "heavy" disk ($\Omega \approx \Omega_0$), the surface of which was covered with granules.[6]

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*The physical interpretation of the boundary conditions imposed on v_L is that the difference in velocity between a vortex line and the surface is proportional to the force exerted on the vortex: $F = \varepsilon\sigma$, σ is the tangential unit vector of the vortex line at the point of contact with the surface of the disk.

†There are, therefore, no resonance effects connected with the accommodation of a whole number of half wavelengths into the distance between the disk and the top or bottom of the beaker.

‡Equation (9) is derived by using the expression for the flow momentum tensor, given by L. D. Landau in the derivation of (1), which has already been mentioned. To explain the physical meaning of the second term of (9) we point out that it can be obtained by a direct consideration of the force acting on the surface of the disk by the vortex lines attached to it, which have a tension ε.

[1] Andronikashvili, Tsakadze, Mamaladze, and Matinyan, 5th All-Union Conference on the Physics of Low Temperatures, Tbilisi, 1958.

[2] É. L. Andronikashvili and D. S. Tsakadze, Сообщения АН ГрузССР (Reports, Acad. Sci. Georgian S.S.R. **20**, 667 (1958).

[3] Yu. G. Mamaladze and S. G. Matinyan, Физика металлов и металловедение (Phys. of Metals and Metal Res.) in press.

[4] H. E. Hall and W. F. Vinen, Proc. Roy. Soc. **A238**, 215 (1956).

[5] H. E. Hall, Proc. Roy. Soc. **A245**, 546 (1958).

[6] É. L. Andronikashvili and D. S. Tsakadze, JETP **37**, 322 and 562 (1959), Soviet Phys. JETP **10**, 227 and 397 (1960).

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