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PHASE SHIFT ANALYSIS OF SCATTERING OF 240–330 Mev PIONS BY HYDROGEN

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Results are presented of a phase-shift analysis of the experimental data on the scattering of pions by nucleons in the energy range from 240 to 330 Mev. Information on phase shifts for the interaction of pions in states with spin $T = \frac{1}{2}$ is of satisfactory accuracy.

1. PHASE-SHIFT ANALYSIS EQUATIONS

In the phase-shift analysis of our experimental data[1] we used the formalism of isotopic spin, based on the hypothesis of charge-independence of the nuclear forces. At energies near 300 Mev the states with orbital momenta $l > 1$ and primarily with $l = 2$ are expected to make a noticeable contribution to the scattering. If it is assumed that the phase shifts are real quantities, then the differential scattering cross sections for pions on hydrogen can be written as follows:[2]

$$\begin{aligned} k^2 (d\sigma/d\omega)_{\pi^+\to\pi^+} = {} & [\sin^2\alpha_3 + (2\sin^2\alpha_{33} + \sin^2\alpha_{31})\cos\vartheta \\ & + (3\sin^2\delta_{35} + 2\sin^2\delta_{33}) P_2(\cos\vartheta)]^2 + \tfrac{1}{4}[\sin 2\alpha_3 \\ & + (2\sin 2\alpha_{33} + \sin 2\alpha_{31})\cos\vartheta \\ & + (3\sin 2\delta_{35} + 2\sin 2\delta_{33}) P_2(\cos\vartheta)]^2 \\ & + \sin^2\vartheta\{[\sin^2\alpha_{33} - \sin^2\alpha_{31} + 3\cos\vartheta(\sin^2\delta_{35} - \sin^2\delta_{33})]^2 \\ & + \tfrac{1}{4}[\sin 2\alpha_{33} - \sin 2\alpha_{31} + 3\cos\vartheta(\sin 2\delta_{35} - \sin 2\delta_{33})]^2\}; \end{aligned} \quad (1)$$

$$\begin{aligned} 9k^2 (d\sigma/d\omega)_{\pi^-\to\pi^-} = {} & [\sin^2\alpha_3 + 2\sin^2\alpha_1 \\ & + (2\sin^2\alpha_{33} + 4\sin^2\alpha_{13} + \sin^2\alpha_{31} + 2\sin^2\alpha_{11})\cos\vartheta \\ & + (3\sin^2\delta_{35} + 6\sin^2\delta_{15} + 2\sin^2\delta_{33} \\ & + 4\sin^2\delta_{13}) P_2(\cos\vartheta)]^2 + \tfrac{1}{4}[\sin 2\alpha_3 + 2\sin 2\alpha_1 \\ & + (2\sin 2\alpha_{33} + 4\sin 2\alpha_{13} + \sin 2\alpha_{31} + 2\sin 2\alpha_{11})\cos\vartheta \\ & + (3\sin 2\delta_{35} + 6\sin 2\delta_{15} + 2\sin 2\delta_{33} \\ & + 4\sin 2\delta_{13}) P_2(\cos\vartheta)]^2 \\ & + \sin^2\vartheta\{[\sin^2\alpha_{33} + 2\sin^2\alpha_{13} - \sin^2\alpha_{31} - 2\sin^2\alpha_{11} \\ & + 3\cos\vartheta(\sin^2\delta_{35} + 2\sin^2\delta_{15} - \sin^2\delta_{33} - 2\sin^2\delta_{13})]^2 \\ & + \tfrac{1}{4}[\sin 2\alpha_{33} + 2\sin 2\alpha_{13} - \sin 2\alpha_{31} - 2\sin 2\alpha_{11} \\ & + 3\cos\vartheta(\sin 2\delta_{35} + 2\sin 2\delta_{15} - \sin 2\delta_{33} - 2\sin 2\delta_{13})]^2\}; \end{aligned} \quad (2)$$

$$\begin{aligned} \tfrac{9}{2} k^2 (d\sigma/d\omega)_{\pi^-\to\pi^0} = {} & [\sin^2\alpha_3 - \sin^2\alpha_1 \\ & + (2\sin^2\alpha_{33} - 2\sin^2\alpha_{13} + \sin^2\alpha_{31} - \sin^2\alpha_{11})\cos\vartheta \\ & + (3\sin^2\delta_{35} - 3\sin^2\delta_{15} + 2\sin^2\delta_{33} \\ & - 2\sin^2\delta_{13}) P_2(\cos\vartheta)]^2 \\ & + \tfrac{1}{4}[\sin 2\alpha_3 - \sin 2\alpha_1 + (2\sin 2\alpha_{33} - 2\sin 2\alpha_{13} \\ & + \sin 2\alpha_{31} - \sin 2\alpha_{11})\cos\vartheta \\ & + (3\sin 2\delta_{35} - 3\sin 2\delta_{15} + 2\sin 2\delta_{33} \\ & - 2\sin 2\delta_{13}) P_2(\cos\vartheta)]^2 \\ & + \sin^2\vartheta\{[\sin^2\alpha_{33} - \sin^2\alpha_{13} - \sin^2\alpha_{31} + \sin^2\alpha_{11} \\ & + 3\cos\vartheta(\sin^2\delta_{35} - \sin^2\delta_{15} - \sin^2\delta_{33} + \sin^2\delta_{13})]^2 \\ & + \tfrac{1}{4}[\sin 2\alpha_{33} - \sin 2\alpha_{13} - \sin 2\alpha_{31} + \sin 2\alpha_{11} \\ & + 3\cos\vartheta(\sin 2\delta_{35} - \sin 2\delta_{15} - \sin 2\delta_{33} + \sin 2\delta_{13})]^2\}. \end{aligned} \quad (3)$$

The total cross sections are written in the following manner:

For positive pions

$$\sigma_t^+ = \frac{4\pi}{k^2}(\sin^2\alpha_3 + 2\sin^2\alpha_{33} + \sin^2\alpha_{31} + 3\sin^2\delta_{35} + 2\sin^2\delta_{33}); \quad (4)$$

For negative pions

$$\begin{aligned} \sigma_t^- = \frac{4\pi}{3k^2} & (\sin^2\alpha_3 + 2\sin^2\alpha_1 + 2\sin^2\alpha_{33} + 4\sin^2\alpha_{13} \\ & + \sin^2\alpha_{31} + 2\sin^2\alpha_{11} + 3\sin^2\delta_{35} + 6\sin^2\delta_{15} \\ & + 2\sin^2\delta_{33} + 4\sin^2\delta_{13}). \end{aligned} \quad (5)$$

The symbols for the phase shifts are given in Table I.

2. METHOD OF CALCULATING PHASE SHIFTS AND THEIR ERRORS

Using the least-squares method, we find an assembly of phase shifts such as to minimize the sum of the weighted squares of deviations

$$M = \sum [(\sigma_{i\,\mathrm{calc}} - \sigma_{i\,\mathrm{exp}})/\varepsilon_i]^2, \quad (6)$$

TABLE I

State	$S_{1/2}$	$P_{1/2}$	$P_{3/2}$	$D_{3/2}$	$D_{5/2}$
$T = {}^3/_2$	α_3	α_{31}	α_{33}	δ_{33}	δ_{35}
$T = {}^1/_2$	α_1	α_{11}	α_{13}	δ_{13}	δ_{15}

where $\sigma_{i\,calc}$ are the differential and total cross sections, calculated from the phase shifts with the aid of Eqs. (1) – (3), $\sigma_{i\,exp}$ are the corresponding experimental values of the cross sections, and ϵ_i are the experimental errors in the measured cross sections.

All the calculations were made with the high-speed "Strela" computer.

To reduce the computation time, the minimum value of M was obtained by the gradient method. Corresponding to the minimum value $M = M_{min}$ is a definite point in the phase-shift hyperspace. The functions $M = \text{const}$ form for all values $M > M_{min}$ a system of hypersurfaces, which are level surfaces of the scalar field $M(\alpha)$. It is obvious that by moving along the gradient of this field it is possible to arrive most rapidly to the point of minimum. Accordingly, we determined the value of M_1 and the normal to the hypersurface $M_1 = \text{const}$ at the initial point of the phase-shift hyperspace. All phase shifts were then increased sufficiently to move in the direction of this normal by approximately 1 deg. The value $M = M_2$ at the new point was then calculated and the new value M_2 compared with the initial one. If $M_2 < M_1$, the motion along the initial gradient was repeated until the new value of M_2 was found to be greater than the preceding one. We then determined a new gradient direction at the point next to the last and the motion then proceeded along this direction at the same spacing as before. This was repeated until the first step no longer led to an increase in M after calculating the gradient. The magnitude of the step was then reduced one half and an analogous calculation performed. The step was then again reduced one half, etc. The calculations were completed after the required step amounted to 2^{-4} deg.

After finding the solution ($M = M_{min}$) we calculated the errors in the phase shifts. An expansion of the function $M(\alpha_1, \alpha_2, \ldots)$ in a Taylor series about M_0 yields

$$M = M_0 + \sum_i \frac{\partial M}{\partial \alpha_i} \Delta\alpha_i + \frac{1}{2} \sum_i \sum_j \frac{\partial^2 M}{\partial \alpha_i\, \partial \alpha_j} \Delta\alpha_i\, \Delta\alpha_j + \ldots . \quad (7)$$

At the point of the minimum, the second term vanishes and

$$M - M_{min} \approx \sum_i \sum_j S_{ij}\, \Delta\alpha_i\, \Delta\alpha_j; \quad S_{ij} = \frac{1}{2} \frac{\partial^2 M}{\partial \alpha_i\, \partial \alpha_j}. \quad (8)$$

Expression (8) defines a hyperellipsoid at the minimum point, that is, the matrix of the coefficients $\langle S_{ij} \rangle$ determines the sensitivity M_{min} to deviations of the phase shifts from the optimum values. Accordingly, the terms of the reciprocal matrix $\langle S_{ij} \rangle^{-1}$ determine the errors in the phase shifts (when $i = j$) and the connection between the phase shifts (when $i \neq j$).[3]

The coefficients S_{ij} can be calculated directly from (8), but this is too cumbersome. We used an approximate finite-difference method. To determine the diagonal element S_{ii} we increased the phase shift α_i by an amount $\Delta\alpha_i$ and determined the value of M. Then, according to (8),

$$S_{ii} = (M - M_{min}) / (\Delta\alpha_i)^2. \quad (9)$$

The calculations were carried out both for positive and for negative increments of α_i and the results averaged.

To find the non-diagonal elements, increments were applied to both phase shifts. Increments of all possible signs were taken and the result was averaged. If we denote the quantities M corresponding to the possible combinations of the signs of the increments $\Delta\alpha_i$ and $\Delta\alpha_j$ as M_{++}, M_{--}, M_{+-}, and M_{-+}, we have

$$\begin{aligned} M_{++} - M_{min} &= S_{ii}(\Delta\alpha_i)^2 + S_{jj}(\Delta\alpha_j)^2 + 2S_{ij}(\Delta\alpha_i)(\Delta\alpha_j), \\ M_{--} - M_{min} &= S_{ii}(-\Delta\alpha_i)^2 + S_{jj}(-\Delta\alpha_j)^2 \\ &\quad + 2S_{ij}(-\Delta\alpha_i)(-\Delta\alpha_j), \\ M_{+-} - M_{min} &= S_{ii}(\Delta\alpha_i)^2 + S_{jj}(-\Delta\alpha_j)^2 + 2S_{ij}(\Delta\alpha_i)(-\Delta\alpha_j), \\ M_{-+} - M_{min} &= S_{ii}(-\Delta\alpha_i)^2 + S_{jj}(\Delta\alpha_j)^2 + 2S_{ij}(-\Delta\alpha_i)(\Delta\alpha_j). \end{aligned} \quad (10)$$

Subtracting the third and fourth equation of (10) from the first and second we obtain (for $\Delta\alpha_i = \Delta\alpha_j = \Delta\alpha$)

$$S_{ij} = \frac{[(M_{++} + M_{--}) - (M_{-+} + M_{+-})]}{8(\Delta\alpha)^2}.$$

In all cases, the increment $\Delta\alpha$ amounted to 1 deg.

3. PHASE-SHIFT ANALYSIS WITH ALLOWANCE FOR THE S AND P WAVES (SP ANALYSIS)

In carrying out the phase-shift analysis we used a total of 25 experimental points for each value of the pion energy: 8 differential cross sections for elastic scattering of positive pions by protons, 7 differential elastic-scattering cross sections for negative pions, and 8 differential cross sections for exchange scattering of negative pions by hydrogen and the total cross sections for the scattering of positive and negative pions by hydrogen.

The differential cross sections for elastic scattering of positive pions by hydrogen and the total cross sections for the interaction between positive pions and hydrogen, as well as the initial values of the phase shifts for states with $T = 3/2$, were taken from the work by Mukhin et al.[4-7]

The differential cross sections for the scattering of positive pions by hydrogen at 330 Mev were determined by interpolating the data of Mukhin, Ozerova, Pontecorvo,[6] and Grigor'ev and Mitin.[8] In this connection, the result of the phase-shift analysis at 333 Mev cannot be considered as fully independent.

The initial values of the phase shifts for the states with $T = \frac{1}{2}$ were obtained by the graphic method of Ashkin.[9] This method gave two possible sets of phase shifts (variants 1 and 3 of Tables II – VI). Along with these, we used two initial phase-shift assemblies for each energy, in which the phase shift α_1 had a sign opposite that obtained graphically (variants 2 and 4) and two assemblies with negative phase shifts of P waves with $T = \frac{1}{2}$ (variants 6 and 7). Such assemblies should agree better with certain data on photoproduction (private communication from A. M. Baldin).

In addition to the data of reference 1, data for 220 Mev, obtained by Ashkin et al.,[10] were also processed.

The initial and calculated values of the phase shifts are given in Tables II – VI.

It is seen from these tables that there are only two stable variants of this solution, observed at all energies. In each of these variants, the same phase shifts differ little from each other at different energies. These are solutions a and b. The

TABLE II. Phase shifts (in degrees) at pion energies of 220 Mev (data by Ashkin et al.[10])

Variant		3	6	1	2	4	7
Initial values	α_3	−12.0	−19.0	−12.0	−12.0	−12.0	−19.0
	α_{31}	0.0	−5.0	0.0	0.0	0.0	−5.0
	α_{33}	107.0	125.0	107.0	107.0	107.0	125.0
	α_1	10.0	10.0	10.0	−10.0	−10.0	−10.0
	α_{11}	8.0	−10.0	−6.0	−6.0	8.0	−10.0
	α_{13}	−3.0	−10.0	5.0	5.0	−3.0	−10.0
	M	36.0	687.0	109.0	88.0	122.0	1559.0
Solution		a		b			
Values at the minimum	α_3	−15.8±1.5		−17.0			
	α_{31}	−2.0±2.9		−0.1			
	α_{33}	111.0±1.8		112.0			
	α_1	14.0±4.3		−7.0			
	α_{11}	6.2±3.3		16.1			
	α_{13}	−5.2±0.8		3.6			
	M	16.1		14.5			

TABLE III. Phase shifts (in degrees) at pion energies of 240 Mev

Variant		1	3	5	4	2	6	7
Initial values	α_3	−18.1	−18.1	−14.7	−18.1	−18.1	−19.0	−19.0
	α_{31}	−2.6	−2.6	−3.2	−2.6	−2.6	−5.0	−5.0
	α_{33}	114.7	114.7	109.5	114.7	114.7	125.0	125.0
	α_1	10.0	10.0	8.0	−10.0	−10.0	10.0	−10.0
	α_{11}	−6.5	8.0	−5.4	8.0	−6.5	−10.0	−10.0
	α_{13}	4.5	−2.5	3.3	−2.5	4.5	−10.0	−10.0
	M	63.0	20.0	85.0	140.0	121.0	468.0	832.0
Solution		a			c	b		
Values at the minimum	α_3	−18.1±1.3			−18.5	−19.6		
	α_{31}	−3.0±2.5			−0.5	−1.4		
	α_{33}	115.0±1.0			115.0	115.0		
	α_1	11.2±4.4			4.8	−0.3		
	α_{11}	10.0±5.4			16.7	−2.6		
	α_{13}	−2.4±1.4			1.7	14.0		
	M	18.8			18.8	29.7		

TABLE IV. Phase shifts (in degrees) at pion energies of 270 Mev

Variant		3	6	7	4	1	2	5
Initial values	α_3	—20.2	—19.0	—19.0	—20.2	—20.2	—20.2	—24.1
	α_{31}	—6.7	—5.0	—5.0	—6.7	—6.7	—6.7	—10.3
	α_{33}	129.3	125.0	125.0	129.3	129.3	129.3	132.8
	α_1	11.0	10.0	—10.0	—11.0	11.0	—11.0	9.6
	α_{11}	10.5	—10.0	—10.0	10.5	—17.5	—17.5	—10.0
	α_{13}	—7.5	—10.0	—10.0	—7.5	10.5	10.5	+10.0
	M	76.0	404.0	727.0	410.0	104.0	180.0	216.0
Solution		*a*			*c*	*b*		
Values at the minimum	α_3	—20.1±1.3			—19.9	—20.3		
	α_{31}	—7.0±1.8			—6.1	—8.2		
	α_{33}	129.0±0.9			129.2	129.7		
	α_1	25.7±2.5			3.0	0.2		
	α_{11}	5.3±3.0			27.3	—9.4		
	α_{13}	—1.2±1.2			4.1	20.1		
	M	9.6			15.3	27.0		

remaining variants either coincide with the aforementioned solutions, or give phase-shift values that are observed only at certain energies (for example, variant 4 at 240 and 270 Mev). Apparently these last solutions are connected with the mathematical aspect of the problem and not with the physical one, i.e., they are accidental.

Only solution a describes the experimental data sufficiently well. Actually, the expected value of M in the SP analysis is 19 (25 experimental points minus 6 phase shifts). In the case of solution b, the value of M is approximately 30 for almost all energies. The probability of obtaining such a value of M is approximately 5%.[11] At 307 Mev, the value of M for solution b reaches 71. The probability of observing such a value of M is negligibly small. At the same time, the values of M obtained in solution a are in good agreement with the expected values.

The angular distributions of the negative pions and photons in elastic and exchange scattering, calculated from the phase shifts of solution a, are shown in Figs. 1 — 4.

One must point out still another feature of so-

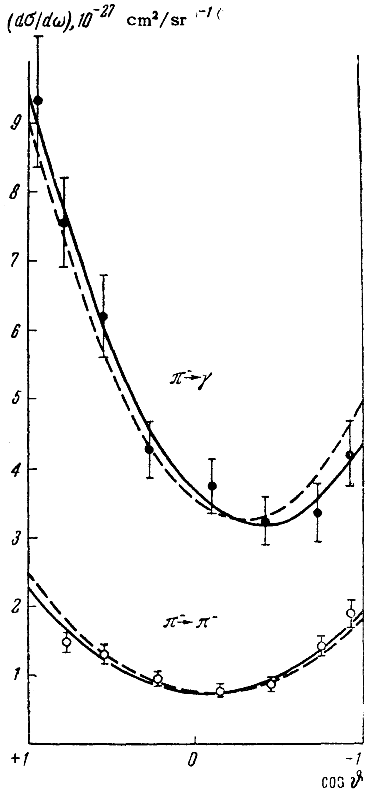


FIG. 1. Differential cross sections (c.m.s.) of the elastic and exchange scattering of 240-Mev pions by hydrogen. Solid curve – distribution of the form $[A_0 + A_1P_1(\cos\vartheta) + A_2P_2(\cos\vartheta)]$; dotted curve – distribution calculated from the SP analysis (solution a).

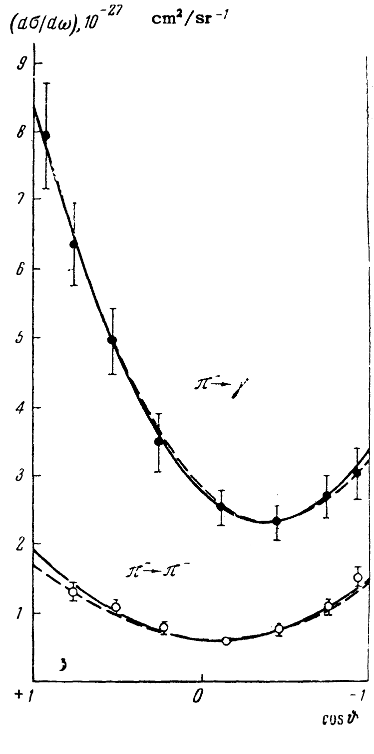


FIG. 2. Differential cross section (c.m.s.) of elastic and exchange scattering of negative pions with 270-Mev energy on hydrogen. Solid curve – distribution of the form $[A_0 + A_1P_1 \times (\cos\vartheta) + A_2P_2(\cos\vartheta)]$; the dotted curve represents distribution calculated from the SP analysis (solution a).

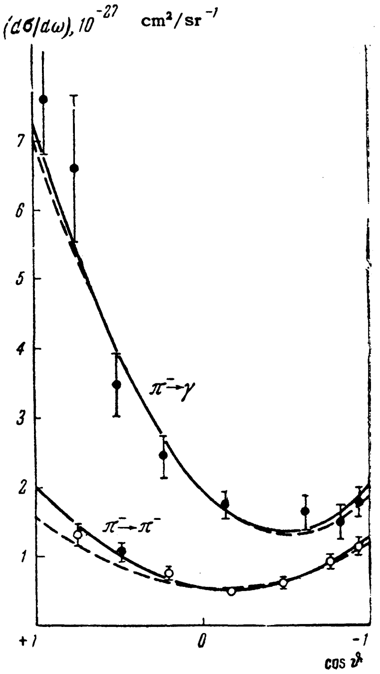
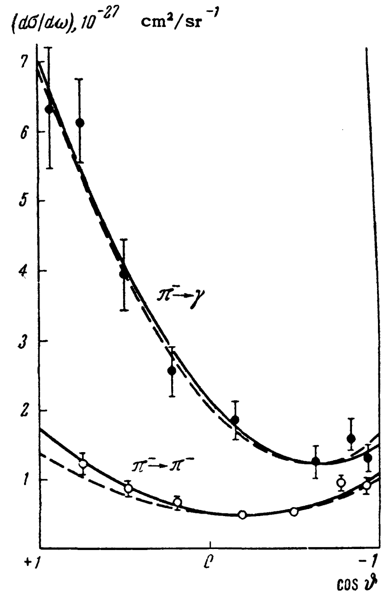


FIG. 3. Differential cross sections (c.m.s.) of elastic and exchange scattering of 307-Mev negative pions by hydrogen. Solid curve – distribution of the form $[A_0 + A_1P_1(\cos\vartheta) + A_2P_2(\cos\vartheta)]$; dotted curve – distribution calculated from SP analysis (solution a).

FIG. 4. Differential cross sections (c.m.s.) of elastic and exchange scattering of 333-Mev negative pions by hydrogen. Solid curve – distribution of type $[A_0 + A_1P_1(\cos\vartheta) + A_2P_2(\cos\vartheta)]$; dotted curve – distribution calculated from SP analysis (solution a).



lution b, namely that on going from 240 to 270 Mev the signs of the phases α_1 and α_{11} become reversed. It appears to be little likely that the phase shift α_1, which is positive at small energies, reverses its sign twice between 150 and 270 Mev. An additional criterion, which makes it possible to choose between the different solutions, is the result of the experiments on the polarization of recoil protons is the interaction between pions and hydrogen. These experiments were made by Ashkin et al.[12] at a negative-pion energy of 223 Mev and by Vasilevskiĭ and Vishnyakov[13] (preliminary data) at negative-pion energies of 300 Mev. It follows from the experiments that solution a is the more probable.

All the foregoing statements lead to the conclusion that the scattering of pions by protons in the energy range above "resonant" is most probably described by solution a. The error matrices for this solution are given in Tables VII – XI. The errors in the phase shifts of solution a, which are given in Tables II – VI, are the diagonal elements of the corresponding matrices.

The phase shifts in the states with isotopic spin $T = 3/2$, given by solution a, actually do not differ from those obtained in experiments with positive mesons.[7]

As regards phase shift in states with isotopic spin $T = 1/2$, we can state the following. The phase shift of the S state is positive. The dependence of the value of this phase on the momentum is shown in Fig. 5. It is seen that, accurate to errors

TABLE V. Phase shifts (in degrees) at pion energies of 370 Mev

Variant		3	4	6	7	1	2	5
Initial values	α_3	−25.0	−25.0	−25.0	−25.0	−25.0	−25.0	−24.1
	α_{31}	−10.0	−10.0	−12.0	−12.0	−10.0	−10.0	−10.3
	α_{33}	133.0	133.0	135.0	135.0	133.0	133.0	132.8
	α_1	9.0	−9.0	10.0	−10.0	9.0	−9.0	9.6
	α_{11}	10.0	10.0	−10.0	−10.0	−18.5	−18.5	−10.0
	α_{13}	−9.0	−9.0	−10.0	−10.0	10.0	10.0	10.0
	M	56.0	217.0	379.0	505.0	151.0	159.0	228.0
Solution		a				b		
Values at the minimum	α_3	−23.9±1.2				−23.6		
	α_{31}	−10.0±2.0				−15.4		
	α_{33}	132.4±0.9				135.2		
	α_1	17.1±5.2				4.1		
	α_{11}	11.4±3.3				−22.4		
	α_{13}	−5.0±1.2				14.6		
	M	29.0				71.0		

TABLE VI. Phase shifts (in degrees) at pion energies of 333 Mev

	Variant	3	4	1	2	5	6	7
Initial values	α_3	—30.0	—30.0	—30.0	—30.0	—24.1	—25.0	—25.0
	α_{31}	—12.0	—12.0	—12.0	—12.0	—10.3	—12.0	—12.0
	α_{33}	139.0	139.0	139.0	139.0	132.8	135.0	135.0
	α_1	10.0	—10.0	10.0	—10.0	9.6	10.0	—10.0
	α_{11}	10.0	10.0	—18.0	—18.0	—10.0	—10.0	—10.0
	α_{13}	—9.0	—9.0	10.0	10.0	10.0	—10.0	—10.0
	M	87	324	179	224	251	3875	6706
	Solution	*a*		*b*			*d*	
Значения в минимуме	α_3	—26.5±1.4		—25.0			—28.0	
	α_{31}	—10.6±2.1		—15.7			—10.7	
	α_{33}	137.2±1.1		140.1			139.5	
	α_1	29.2±1.8		11.7			31.4	
	α_{11}	8.1±2.9		—24.0			—16.3	
	α_{13}	—2.0±1.3		16.0			10.0	
	M	22.6		33.1			94.4	

TABLE VII; Matrix of phase-shift errors (radians^2), E_π = 220 Mev

	$10^4\alpha_3$	$10^4\alpha_{31}$	$10^4\alpha_{33}$	$10^4\alpha_1$	$10^4\alpha_{11}$	$10^4\alpha_{13}$
α_3	6.16	—5.40	—0.08	0.68	7.46	1.28
α_{31}		22.34	—0.56	3.80	—12.66	1.36
α_{33}			8.54	16.38	— 5.16	1.14
α_1				49.38	—21.22	3.88
α_{11}					29.84	—0.32
α_{13}						0.17

TABLE VIII. Matrix of phase-shift errors (radians^2), E_π = 240 Mev

	$10^4\alpha_3$	$10^4\alpha_{31}$	$10^4\alpha_{33}$	$10^4\alpha_1$	$10^4\alpha_{11}$	$10^4\alpha_{13}$
α_3	4.92	—1.80	—0.42	—3.16	7.24	1.76
α_{31}		17.54	—0.38	2.56	—10.90	0.80
α_{33}			2.56	4.46	—2.52	—0.82
α_1				52.34	—47.10	—8.52
α_{11}					79.72	14.64
α_{13}						5.32

TABLE IX. Matrix of phase-shift errors (radians^2), E_π = 270 Mev

	$10^4\alpha_3$	$10^4\alpha_{31}$	$10^4\alpha_{33}$	$10^4\alpha_1$	$10^4\alpha_{11}$	$10^4\alpha_{13}$
α_3	3.70	—0.20	—0.40	—1.86	2.38	0.44
α_{31}		10.28	—0.42	11.98	—11.40	2.74
α_{33}			1.94	5.20	—1.04	0.80
α_1				73.04	—23.46	13.64
α_{11}					28.96	—4.78
α_{13}						3.98

TABLE X. Matrix of phase-shift errors (radians^2), E_π = 307 Mev

	$10^4\alpha_3$	$10^4\alpha_{31}$	$10^4\alpha_{33}$	$10^4\alpha_1$	$10^4\alpha_{11}$	$10^4\alpha_{13}$
α_3	4.48	—0.28	0.10	1.00	1.52	1.26
α_{31}		8.62	—0.28	0.68	—5.28	0.82
α_{33}			2.16	3.40	—1.98	1.08
α_1				16.40	—13.60	6.28
α_{11}					24.36	—5.88
α_{13}						3.94

TABLE XI. Matrix of phase-shift errors (radians^2), E_π = 333 Mev

	$10^4\alpha_3$	$10^4\alpha_{31}$	$10^4\alpha_{33}$	$10^4\alpha_1$	$10^4\alpha_{11}$	$10^4\alpha_{13}$
α_3	5.56	—2.38	—1.22	0.48	5.62	—0.04
α_{31}		11.88	—2.86	3.66	—10.66	5.18
α_{33}			3.44	1.84	2.40	0.06
α_1				8.68	—8.40	4.94
α_{11}					22.68	—7.66
α_{13}						4.60

in the determination of its magnitude, the behavior of the phase shift α_1 does not contradict the linear dependence observed at small energies ($\alpha_1 = 9.9\,\eta$, where η is the momentum of the pion in the c.m.s.).

The value of the phase shift α_{11} does not exceed 10 deg, and its sign is apparently positive.

The phase shift α_{13}, as can be seen from Tables II — VI, is small (1 — 5 deg) and apparently negative. It should be noted here that, according to the theory of Chew,[14-16] the phase shift

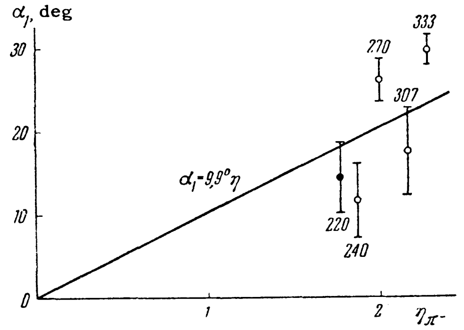


FIG. 5. Dependence of the value of the phase shift α_1 on the pion momentum in the c.m.s.

α_{13} should be negative and approximately equal to α_{31}. Within the limit of errors, the phase shifts of solution a satisfy this condition. A published paper by Chiu and Lomon[17] gives a phase-shift analysis of the data of Ashkin et al. and of our preliminary data at 370 Mev. In view of the fact that the solution which they consider to be the most reliable coincides with our solution a, we have made some additional calculations. In these calculations we took as the starting data the phase shifts of the solution given by Chiu and Lomon. These led to solution b at all energies (except 240 Mev).

At 307 and 333 Mev, the cross sections for inelastic processes are sizable, and therefore the scattering phase shifts, strictly speaking, can no longer be considered real. In general form, the differential cross sections for the scattering of pions by nucleons in a state with definite isotopic spin can be written in the following manner:

$$\left(\frac{d\sigma}{d\omega}\right)_{\pi^+\to\pi^+} = \left|\sum_{l=0}^{\infty} \frac{a_l}{2ki} P_l(\cos\vartheta)\right|^2 + \left|\sum_{l=0}^{\infty} \frac{b_l}{2ik} P_{l1}(\cos\vartheta)\right|^2, \tag{12}$$

where k is the momentum of the pion in the c.m.s., while a_l and b_l are given by

$$a_l = (l+1)\exp 2i\alpha^l_{3,\,2l+1} + l\exp 2i\alpha^l_{3,\,2l-1} - (2l+1),$$
$$b_l = \exp 2i\alpha^l_{3,\,2l+1} - \exp 2i\alpha^l_{3,\,2l-1}. \tag{13}$$

The presence of an imaginary part of the phase shift causes expressions of the form $\beta_l \exp(2i\alpha_l)$ to appear in Eq. (13) instead of expressions of the form $\exp(2i\alpha_l)$, where β_l and α_l are real numbers. Using the experimentally obtained cross sections for inelastic processes,[18] we can estimate β_l from the general quantum-mechanical theory of inelastic scattering.[19] It is found that for any state, even at 333 Mev, β_l differs from unity by not more than 2% or 3%, corresponding to an imaginary part not greater than 0.5 deg. The fact that β_l differs little from unity makes maximum phase-shift variation, due to taking the inelastic processes into account (in accordance with the estimates made) not greater than approximately ±3 deg, that is, the accuracy in the determination of the phases. Thus, up to 330 Mev, a phase-shift analysis based on the assumption that the phases are real numbers is apparently a sufficiently good approximation.

4. PHASE-SHIFT ANALYSIS WITH ALLOWANCE FOR S, P AND D WAVES (SPD ANALYSIS)

At pion energies near 300 Mev, one can expect a noticeable contribution from D waves ($l = 2$) to the scattering. In this connection, we made a phase-shift analysis with allowance for the D waves for all pion energies. The initial phase shifts for the S and P waves were taken in the SPD analysis to be the phase shifts of solutions a and b, obtained in the SP analysis. The initial values of the phase shifts of the D waves in states with $T = 3/2$, in accordance with the results obtained in experiments with positive pions,[7] were taken to be $\delta_{33} = 6$ deg and $\delta_{35} = -6$ deg. The values of the initial phase shifts of the D waves in states with isotopic spin $T = 1/2$ were taken to have an absolute value of 8 deg. The signs of these phase shifts were taken in all possible combinations, that is, ++, −−, +−, and −+. Thus, we obtained eight versions of the initial values of the phase shift for each energy, four for solution a and four for solution b.

As a result of the calculations performed, the initial versions, which differed only in the signs of the phase shifts δ_{13} and δ_{15}, yielded one and the same solution. In other words, we obtained two sets of final versions of solution, one for initial solution a and the other for solution b. The values of the phase shifts corresponding to solutions a and b are listed in Tables XII and XIII. These two versions of the solutions will henceforth be called the "a_{SPD} solution" and "b_{SPD} solution."

For 333 Mev we obtained three different variants for solution a. Two of these are listed in Table XII. These solutions give phase-shift values corresponding to those obtained at lower energies.

It is seen from Table XII that when the D waves are taken into account the phase shifts α_1, α_{11}, and α_{13} of solution a change by not more than a few degrees, that is, they remain unchanged within the limit of errors. This is expected, for the S and P phases alone have already described the experimental data sufficiently well.

The phase shifts δ_{33} and δ_{35} of the D waves in states with isotopic spin $T = 3/2$ do not differ

TABLE XII. Phase shifts (degrees) at pion energies 240 — 333 Mev. Solution a_{SPD}

Pion energy, Mev	240	270	307	333 (1)	333 (2)
α_3	—20.4±1,9	—15.0±6.2	—10.2±3.7	—22.0±1,6	—10,5±1.1
α_{31}	—4.2±3.6	—4.0±3.1	—3.6±2.2	—6.3±2.1	—2.8±1.7
α_{33}	116.7±2.8	128.4±1.0	134.7±1.9	136.2±1.1	139.7±1.5
δ_{33}	—0.9±3.4	3.9±6.6	11.1±3.0	2.4±1.3	12.1±1.8
δ_{35}	4.9±5.5	—5.3±5.1	—12.0±2.1	—4.3±1.0	—13.2±1.0
α_1	8.1±3.5	25.8±2,2	23.8±3.3	28.9±2,0	31,8±1,6
α_{11}	11.2±3.2	4,2±3.8	8.0±4.0	9.4±3.4	4,3±3.5
α_{13}	—1.4±1.45	—0.4±1.5	—2.2±1.9	—2.2±1.0	1.0±1.9
δ_{13}	3.0±2.1	0.8±2.2	3.1±1.4	2.0±1.1	0.8±1.5
δ_{15}	2,3±2.5	1,3±2.2	1.9±2.0	1.0±1.0	1.8±1,2
M_{min}	9.7	5.9	7,0	16.2	9.6

TABLE XIII. Phase shifts (degrees) at pion energies 240 — 333 Mev. Solution b_{SPD}

Pion energy, Mev	240	270	307	333	Pion energy, Mev	240	270	307	333
α_3	—16.8	—16.0	—10.1	—19.8	α_1	— 0.3	— 4.8	— 3.5	— 3.0
α_{31}	— 1.7	— 6.3	— 3.2	— 9.6	α_{11}	— 6.9	—12.9	0.0	—21.9
α_{33}	115.5	128.0	134.7	136.3	α_{13}	7.2	9.2	6.6	15.2
δ_{33}	7.1	3.6	11.3	3.6	δ_{13}	6.2	9.0	15.0	2.7
δ_{35}	— 2.4	— 5.7	—11.9	— 6.1	δ_{15}	3.8	3.6	4.2	4.5
					M_{min}	13.9	10.1	5.3	17.5

essentially from those obtained earlier by Mukhin and Pontecorvo.[7]

The phase shifts δ_{13} and δ_{15} of D waves in states with isotopic spin $T = 1/2$ are small, as can be seen from Table XII. The accuracy with which they have been determined does not permit any definite conclusions to be drawn concerning the signs of these phase shifts.

Nothing definite can be said about the solution of b_{SPD}. It cannot be discarded completely merely because the corresponding solution b of the SP analysis does not agree with the experimental data. Nor can one exclude the possibility that the poor agreement given by solution b is connected with the need for taking the D waves into account in the analysis. It appears to be little probable, however, that the large D-phase values (up to 15 deg) obtained in the solution b_{SPD} would occur already at 240 — 307 Mev. The dispersion relations described below give additional indications that the solution b_{SPD} is apparently incorrect.

5. COMPARISON OF THE PHASE-SHIFT ANALYSIS RESULTS WITH THE DISPERSION RELATIONS

It is well known that the dispersion relations yield the dependence of the real part of the zero-degree scattering amplitude D(0) on the pion energy.

An expression for $D_-(0)$ can be obtained from Eq. (2) for the cross section of elastic scattering of negative pions by hydrogen. This expression represents the sum of the squares of the real and imaginary parts of the scattering amplitude. Then

$$D_-(0) = \frac{1}{6} k^{-1} (\sin 2\alpha_3 + \sin 2\alpha_{31} + 2 \sin 2\alpha_{33} + 2 \sin 2\delta_{33} + 3 \sin 2\delta_{35} + 2 \sin 2\alpha_1 + 2 \sin 2\alpha_{11} + 4 \sin 2\alpha_{13} + 4 \sin 2\delta_{13} + 6 \sin 2\delta_{15}). \quad (14)$$

The values of $D_-(0)$ calculated from the phase shifts are shown in Fig. 6. It is seen that solution a is in satisfactory agreement with the curve given by Pontecorvo,[13] with a coupling constant $f^2 = 0.08$ for the meson-nucleon interaction.

In conclusion, the authors consider it their pleasant duty to express sincere gratitude to B. Pontecorvo for constant attention and help with the work; to A. I. Mukhin, L. I. Lapidus, S. N. So-

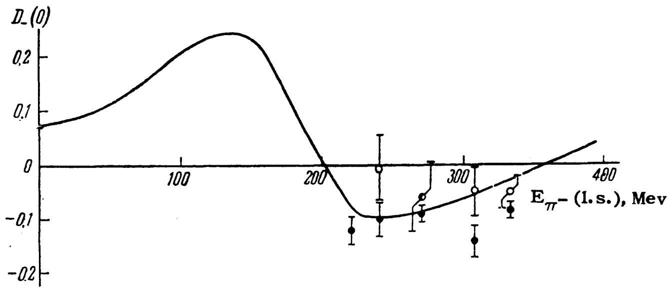


FIG. 6. Dependence of the real part of the amplitude of scattering at zero degrees, $D_-(0)$, in $h/\mu c$ units, on the negative-pion energy: ● – SP analysis (solution a), ○ – SPD analysis (solution a_{SPD}). The solid curve is taken from reference 13.

kolov, and N. P. Klepikov for numerous and fruitful discussions, and I. V. Popova and L. A. Chudov for valuable aid in programming the electronic computer.

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Transl. by J. G. Adashko
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ERRATA TO VOLUME 10

page	reads	should read
Article by A. S. Khaĭkin		
1044, title	. . . resonance in lead	. . . resonance in tin
6th line of article	~ 1000 oe	~ 1 oe
Article by V. L. Lyuboshitz		
1223, Eq. (13), second line	$\ldots - \mathrm{Sp}_{1,2}\,\mathscr{E}(\mathbf{e}_1)$	$\ldots - \mathrm{Sp}_{1,2}\,\mathscr{E}(\mathbf{e}_2)\ldots$
1226, Eq. (26), 12th line	$\cdots\{(\mathbf{p}+\mathbf{q},\ \mathbf{p}$	$\cdots\{(\mathbf{p}+\mathbf{q},\mathbf{p}) - (\mathbf{p}+\mathbf{q},\mathbf{n})\cdot\cdot(\mathbf{pn})\}$
1227, Eqs. (38), (41), (41a) numerators and denominators	$(\mathbf{p}^2 - \mathbf{q})$	$(\mathbf{p}^2 - \mathbf{q}^2)^2$
1228, top line	$\mathbf{m}_2 = \begin{matrix}\mathbf{q}_1 - \mathbf{p}_1 \\ \mathbf{q}_1 - \mathbf{p}_1\end{matrix}$	$\mathbf{m}_2 = [\mathbf{m}_3\ \mathbf{m}_1]$

ERRATA TO VOLUME 12

page	reads	should read
Article by Dzhelepov et al.		
205, figure caption	54	5.4
Article by M. Gavrila		
225, Eq. (2), last line	$-2\gamma\Theta^{-4}\,{}^1\!/_8$	$-2\gamma\Theta^{-4} - {}^1\!/_8$
Article by Dolgov-Savel'ev et al.		
291, caption of Fig. 5, 4th line	$p_0 = 50 \times 10^{-4}$ mm Hg	$p_0 = 5 \times 10^{-4}$ mm Hg.
Article by Belov et al.		
396, Eq. (24) second line	$\ldots - (4 - 2\eta)\,\sigma_1 + \ldots$	$\ldots + (4 - 2\eta)\,\sigma_1 + \ldots$
396, 17th line (r) from top	. . less than 0.7	. . less than 0.07
Article by Kovrizhnykh and Rukhadze		
615, 1st line after Eq. (1)	$\omega_{0e}^2 = 2\pi e^2 n_e/m_e,$	$\omega_{0e}^2 = 4\pi e^2 n_e/m_e,$
Article by Belyaev et al.		
686, Eq. (1), 4th line	$\cdots b_{\rho_2 m_2}{}^{\prime}(s_2^{\prime}) + \cdots$	$\cdots b_{\rho_1 m_1}{}^{\prime}(s_1^{\prime}) + \cdots$
Article by Zinov and Korenchenko		
798, Table X, heading of last column	$\sigma_{\pi^- \to \pi^+} =$	$\sigma_{\pi^- \to \pi^-} =$
Article by V. M. Shekhter		
967, 3d line after Eq. (3)	$\varepsilon \equiv 2m_pE + m_p^2$	$\varepsilon \equiv (2m_pE + m_p^2)^{1/2}$
967, Eq. (5), line 2	$+ (B_V^2 + B_A^2) \ldots$	$+ (B_V^2 + B_A^2)\,Q \ldots$
968, Eq. (7)	$\ldots (C_V^2 + C_A^2).$	$\ldots C_V^2 + C_A^2 - Q^2(B_V^2 + B_A^2).$
968, line after Eq. (7)	for $C_V^2 + C_A^2 \equiv \ldots$	for $C_V^2 + C_A^2 - Q^2(B_V^2 + B_A^2) \equiv \ldots$
Article by Dovzhenko et al.		
983, 11th line (r)	$\gamma = 1.8 \pm$	$\gamma = 1{,}8 \pm 0{,}2$
Article by Zinov et al.		
1021, Table XI, col. 4	$-1{,}22$	$1{,}22$
Article by V. I. Ritus		
1079, line 27 (1)	$-\Lambda_{\pm}(t),$	$\Lambda_{\pm}(t),$
1079, first line after Eq. (33)	$\frac{1}{2}(1 + \beta).$	$\frac{1}{2}(1 \pm \beta).$
1079, 3d line (1) from bottom	$\ldots\mathfrak{N}(q'p;\ pq')\ldots$	$\ldots\mathfrak{N}(p'q;\ pq')\ldots$
Article by R. V. Polovin		
1119, Eq. (8.2), fourth line	$U_{0x}u_xg(\gamma) - [\gamma\cdots$	$-U_{0x}u_xg(\gamma)\,[\gamma\cdots$
1119, Eq. (8.3)	$\cdots\mathrm{sign}\ u.$	$\cdots\mathrm{sign}\ u_g.$
Article by V. P. Silin		
1138, Eq. (18)	$\cdots + \frac{4}{5}c^2k^2$	$\cdots + \frac{6}{5}c^2k^2.$