ELECTRIC FORCES IN A TRANSPARENT DISPERSIVE MEDIUM

L. P. PITAEVSKIĬ

Institute for Terrestial Magnetism, Ionosphere, and Radio Transmission, Academy of Sciences, U.S.S.R.

Submitted to JETP editor July 27, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) **39**, 1450-1458 (November, 1960)

It is shown that the expression for the stress tensor of a variable electric field in a transparent dispersive medium is the same as the usual expression for a nondispersive medium. It is also shown that if the parameters of the medium are time-dependent an imaginary part appears in the dielectric susceptibility, which is a real quantity for time-independent parameters. An expression is derived for the stress tensor of a variable electric field in a transparent liquid located in a constant magnetic field.

1. THE STRESS TENSOR OF A VARIABLE ELECTRIC FIELD IN A TRANSPARENT LIQUID DISPERSIVE MEDIUM

In the present work we investigate the form of the stress tensor of a variable electric field in a transparent liquid dispersive medium. We are interested here, of course, in the average of the tensor over an oscillation period of the field. As is well known, the stress in a dielectric placed in a constant electric field is given by the Maxwell-Abraham stress tensor. For an isotropic liquid it has the form

$$\sigma_{ik} = -p_0\delta_{ik} - \frac{E^2}{8\pi}\left[\varepsilon - \rho\left(\frac{\partial\varepsilon}{\partial\rho}\right)_T\right]\delta_{ik} + \varepsilon\,\frac{E_iE_k}{4\pi}, \qquad (1)$$

Here $p_0 = p_0(\rho, T)$ is the pressure in the liquid in the absence of the electric field as a function of ρ and T, ϵ is the dielectric susceptibility of the liquid, and ρ is its density. The derivation of (1) is based on the expression for the energy density of a constant electric field in a dielectric,

$$U = U_0 + \frac{D^2}{8\pi\varepsilon} = U_0 + \frac{\varepsilon E^2}{8\pi}, \qquad (2)$$

Here U_0 is the energy density in the absence of the field.

Let us now consider a liquid dielectric whose dielectric permittivity is a function of the frequency of the field. Let it be located in an electric field which varies with a frequency at which the dielectric is transparent, i.e., such that $\epsilon(\omega)$ is a real quantity. A typical example of such a medium is an electron plasma at $\omega \gg \nu$ (ν is the effective collision frequency). The expression for the time average of the energy density of the electric field in such a medium is different from (2), and contains the derivative of ϵ with respect to ω (see reference 1):

$$U = U_0 + \frac{d\omega\varepsilon}{d\omega}\frac{\overline{E^2}}{8\pi}. \qquad (3)$$

One might naturally expect additional terms containing $d\epsilon/d\omega$ to appear in this case in the expression for the stress tensor. However, we shall see that this is not so, i.e., that the stress tensor in this case is given by expression (1), but is averaged over the time.

For a proof of this assertion we first give a very simple derivation of expression (1) for a constant field. Consider a capacitor filled with a dielectric of dielectric constant ϵ, and which carries on its plates a constant charge Q. The energy of such a capacitor is

$$\mathcal{U} = \mathcal{U}_0 + Q^2/2C = \mathcal{U}_0 + C\varphi^2/2. \qquad (4)$$

Here $\mathcal{U}_0$ is the energy of the uncharged capacitor, C its capacity, and φ the potential difference between its plates.

Let us now subject the capacitor to some continuous deformation. First we displace its plates or its lateral surfaces. The work performed on it by the external force will then be

$$\delta A = \sigma_{ik}\xi_i n_k S \qquad (i, k = 1, 2, 3), \qquad (5)$$

where n_k is the normal to the displaced surface, ξ_i its displacement, and S its area. However, since a capacitor with constant charge is an electrically closed system, the work (5) will simply be equal to the change of its energy:

$$\delta A = \sigma_{ik}\xi_i n_k S = \delta\mathcal{U}_0 - \frac{Q^2}{2C^2}\delta C = \delta\mathcal{U}_0 - \frac{1}{2}\varphi^2\delta C. \qquad (6)$$

If we now express δC in terms of the displacement of the plates (taking into account the change of the density of the dielectric during deformation), we immediately find from Eq. (6) the expression (1) for σ_{ik}.

Now we show that the expression for the tensor in a variable field is determined by an equation that agrees exactly with the time average of Eq. (6). Consider a capacitor filled with a dielectric of dielectric constant $\epsilon(\omega)$; to its plates we apply an alternating potential,

$$\varphi = \frac{1}{2}(\varphi_0 e^{-i\omega t} + \varphi_0^* e^{i\omega t}) \tag{7}$$

of frequency ω such that $\mathrm{Im}\,\epsilon(\omega) = 0$. Such a capacitor, however, is no longer a closed system. In order to avoid calculating the work of the current source we therefore consider a simpler, electrically closed, oscillating system, consisting of a capacitor and an inductance, i.e., an oscillating circuit with natural frequency ω.

The energy of a capacitor filled with a dispersive dielectric is given by the following expression, which takes the place of Eq. (4),

$$\mathcal{U} = \mathcal{U}_0 + \frac{d\omega C}{d\omega}\frac{\overline{\varphi^2}}{2}, \tag{8}$$

This is easily obtained by comparing Eqs. (2) and (3) and taking it into account that $C \sim \epsilon$. The energy in the inductance is $L\overline{I^2}/2c^2 = C\overline{\varphi^2}/2$, where c is the speed of light, and I the current through the inductance (here we assume that the inductance does not depend on frequency, and we have used the well known relation between current and voltage in an oscillating circuit). Therefore the total energy of our closed system is equal to

$$\mathcal{U} = \mathcal{U}_0 + \frac{d\omega C}{d\omega}\frac{\overline{\varphi^2}}{2} + \frac{C\overline{\varphi^2}}{2} = \mathcal{U}_0 + \frac{1}{\omega}\frac{d\omega^2 C}{d\omega}\frac{\overline{\varphi^2}}{2}. \tag{9}$$

Now let us subject the capacitor in our circuit to an infinitesimal deformation. The accompanying work of the stress forces will be, as before,

$$\delta A = \sigma_{ik}\xi_i n_k S.$$

To calculate the change of energy in the system, which equals δA, we note that, as is known from mechanics, the so-called adiabatic invariant remain constant during an infinitesimal change of the parameters of any linear oscillating system. For linear systems this invariant is defined as the ratio of the oscillation energy to the natural frequency of the system:

$$\mathcal{U}_{\mathrm{osc}}/\omega = \mathrm{const.} \tag{10}$$

We note that Eq. (10) has a simple quantum-mechanical basis. In fact, the energy levels of every linear quantum-mechanical system have (after subtracting the zero-point energy) the form

$$E_n = \hbar\omega n \qquad (n = 0, 1, \ldots),$$

where ω is the eigenfrequency and n the quantum number. Therefore

$$\mathcal{U}/\omega = E_n/\omega = \hbar n.$$

But during an infinitesimal change of external conditions the quantum state of the system does not change, i.e., n remains constant.

From (10) it is clear that the energy changes in direct proportion to the frequency, and in our case we obtain

$$\delta\mathcal{U} = \delta\mathcal{U}_0 + \frac{d\omega^2 C}{\omega\, d\omega}\frac{\overline{\varphi^2}}{2}\frac{\delta\omega}{\omega}, \tag{11}$$

where $\delta\omega$ is the change in natural frequency of the circuit due to the deformation of the capacitor. Since $\omega = c/\sqrt{LC}$ we find

$$\delta\omega/\omega = -\frac{1}{2}\delta C/C, \tag{12}$$

where δC is the change in capacitance.

We must, however, take into account that the capacitance changes not only by deformation, but also because of the change in the oscillation frequency of the circuit, since the capacitance depends on frequency. Therefore,

$$\delta C = \delta C_{\mathrm{st}} + (dC/d\omega)\,\delta\omega, \tag{13}$$

where δC_{st} is expressed in terms of the displacement ξ_i like in the static case.

Eliminating $\delta C/C$ from (12) and (13) we find

$$\frac{\delta\omega}{\omega}\left(\frac{1}{\omega}\frac{d\omega^2 C}{d\omega}\right) = -\delta C_{\mathrm{st}}. \tag{14}$$

Substituting $\delta\omega/\omega$ from (14) into (11) we finally obtain

$$\delta\mathcal{U} = \delta\mathcal{U}_0 - \frac{1}{2}\overline{\varphi^2}\delta C_{\mathrm{st}}, \tag{15}$$

i.e., an expression which agrees with the average of Eq. (6). If we now note that δC_{st} is expressed in terms of the deformation (like in the static case) we immediately conclude without additional calculation that the tensor is also given by the average of Eq. (1),

$$\sigma_{ik} = -p_0\delta_{ik} - \frac{\overline{E^2}}{8\pi}\left[\varepsilon - \rho\left(\frac{\partial\varepsilon}{\partial\rho}\right)_T\right]\delta_{ik} + \varepsilon\frac{\overline{E_i E_k}}{4\pi} \tag{16}$$

even in the presence of dispersion.

Formula (15) can be presented from a somewhat different point of view. We assume that the capacitance changes only because of changes in permittivity of the dielectric. Then

$$\delta C = (S / 4\pi d)\, \delta\varepsilon$$

(S is the surface area of the plates and d their separation), and the infinitesimal work of the external forces is

$$\delta A = \delta \mathcal{U} = \delta \mathcal{U}_0 - \frac{1}{8\pi} \frac{\overline{\varphi^2}}{d} S \delta\varepsilon = \delta \mathcal{U}_0 - \frac{\overline{E^2}}{8\pi} V \delta\varepsilon \qquad (17)$$

(V is the volume of the capacitor). If we now consider instead of the capacitor an arbitrary volume filled with dielectric, then instead of Eq. (17) we obtain

$$\delta \mathcal{U} = \delta \mathcal{U}_0 - \int \frac{\overline{E^2}}{8\pi} \delta\varepsilon\, dV. \qquad (18)$$

Formula (18) is correct for changes occurring at a constant entropy of the medium. However, since at constant entropy and temperature small additions to the energy and free energy, respectively, are equal, exactly the same formula applies for the change in the free energy $\mathcal{F}$ at a given temperature and at zero work of the sources of the field:

$$\delta \mathcal{F} = \delta \mathcal{F}_0 - \int \frac{\overline{E^2}}{8\pi} \delta\varepsilon\, dV. \qquad (18a)$$

This formula agrees with the formula for the change in the free energy for fixed field sources in the static case [see reference (1), formula 14.1)].

As is well known, the components σ_{ik}, together with U and the components of the energy-flux vector $\mathbf{S} = (c/4\pi)\, \mathbf{E} \times \mathbf{H}$, form the four-dimensional energy-momentum tensor $T_{\alpha\beta}$ of the system. From our results it follows that in a static medium the only component of $T_{\alpha\beta}$ containing the derivative of ϵ with respect to ω is T_{44}. Knowing $T_{\alpha\beta}$ in a static medium, we can find its value in a moving medium from the Lorentz transformation formula. Then $d\epsilon/d\omega$ will enter also into the other components of the tensor. We write down as an example the equation for the energy flux of the field in a medium moving with velocity v [with accuracy up to terms $\sim (v/c)^2$]:

$$\mathbf{S} = \frac{c}{4\pi} \overline{[\mathbf{E} \times \mathbf{H}]} + \frac{\overline{E^2}}{8\pi} \left(\omega \frac{\partial\varepsilon}{\partial\omega} - \rho \frac{\partial\varepsilon}{\partial\rho} \right) \mathbf{v}. \qquad (19)$$

2. PERMITTIVITY OF MEDIA WITH TIME-DEPENDENT PARAMETERS

Let the state of our liquid be described by some parameter λ. (In particular, this may be the density or the pressure of the liquid.) In this case the dielectric permittivity will of course be a function of this parameter $[\epsilon = \epsilon(\omega, \lambda)]$. Now let the parameter λ depend on time. Then we can introduce the idea of a time-dependent permittivity. Since also in this case the connection between $\mathbf{E}$ and $\mathbf{D}$ for weak field remains linear we may write

$$\mathbf{D}(t) = \mathbf{E}(t) + \int_0^\infty f(\tau, t)\, \mathbf{E}(t - \tau)\, d\tau. \qquad (20)$$

(If the properties of the medium did not depend on time, the function f would depend only on τ and not on t.)

Putting $\mathbf{E}(t) = \mathbf{E}_0 e^{-i\omega t}$ we obtain

$$\mathbf{D}(t) = \varepsilon(\omega, t)\, \mathbf{E}_0 e^{-i\omega t}, \qquad (21)$$

where it is natural to call the quantity

$$\varepsilon(\omega, t) = 1 + \int_0^\infty f(\tau, t)\, e^{i\omega\tau}\, d\tau \qquad (22)$$

the permittivity of the medium with time-dependent parameters.

The function $\epsilon(\omega, t)$, however, will by no means agree with the function $\epsilon_0(\omega, t) = \epsilon(\omega, \lambda(t))$, i.e., with the function of the static value of λ, in which we put $\lambda = \lambda(t)$. This is connected with the fact that the function $\epsilon(\omega, t)$ in general will not be determined by the value of λ at the same moment of time, but will depend also on the time derivatives of λ. If λ changes slowly (compared to atomic frequencies) then the dependence of ϵ can be limited to $d\lambda/dt$ only, and we can take only the first term of the expansion in this quantity. Thus

$$\varepsilon(\omega, t) = \varepsilon_0(\omega, t) + a(\omega, \lambda(t))\, d\lambda / dt. \qquad (23)$$

(ϵ must of course tend toward ϵ_0 as $d\lambda/dt \rightarrow 0$).

The second term is usually taken to be only a negligibly small correction to the first. However, if the medium is transparent when the parameters are time-independent, i.e., if $\operatorname{Im} \epsilon(\omega, \lambda) = 0$, then, generally speaking, $\operatorname{Im} a \neq 0$. Therefore the dependence of λ on time leads to the appearance of an imaginary part of ϵ, which represents a qualitatively new effect.

We now show that if $\operatorname{Im} \epsilon(\omega, \lambda) = 0$, then $\operatorname{Im} a$ can be found in closed form (the real part of a gives only a small correction to ϵ_0 and is uninteresting). For the derivation consider a capacitor filled with a dielectric whose parameters vary slowly with time. We shall consider that a potential difference

$$\varphi = \tfrac{1}{2} (\varphi_0 e^{-i\omega t} + \varphi_0^* e^{i\omega t}),$$

is maintained between the plates of the capacitor. Here φ_0 is a complex constant.

The current through the capacitor now has the form

$$I(t)=\frac{1}{2}[I_0(t)e^{-i\omega t}+I_0^*(t)e^{i\omega t}], \quad (24)$$

where $I_0(t)$ is a slow function of the time. The capacitance is given by the formula

$$C(\omega, t)=\varepsilon(\omega, t)S/4\pi d. \quad (25)$$

Then the charge on the capacitor plates is

$$Q=\frac{1}{2}[C(\omega, t)\varphi_0 e^{-i\omega t}+C^*(\omega, t)\varphi_0^* e^{i\omega t}]. \quad (26)$$

Since $I = dQ/dt$, we find

$$I_0=[-i\omega C(\omega, t)+\partial C(\omega, t)/\partial t]\varphi_0. \quad (27)$$

Let us now apply the law of conservation of energy to our condensor. In this case it means that the change of energy of the capacitor equals the work needed to change its dielectric constant at a fixed current source, plus the work of the source to maintain the given potential across the capacitor. According to (15) the work for the change $\delta\epsilon$ (per unit time) equals $\frac{1}{2}\overline{\varphi^2}\,\partial C/\partial t$, and the work of the current source is $\overline{I\varphi}$, i.e.,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\frac{d\omega C}{d\omega}\overline{\varphi^2}\right)=-\frac{\overline{\varphi^2}}{2}\frac{\partial C}{\partial t}+\overline{I\varphi}. \quad (28)$$

With accuracy up to terms of higher order of smallness, we can regard here C as a real quantity in all expressions containing $\partial C/\partial t$, since Im C is of the order $d\lambda/dt \sim \partial C/\partial t$.

Substituting (27) into (28) we find

$$\frac{\overline{\varphi^2}}{2}\frac{\partial C}{\partial t}+\frac{\overline{\varphi^2}}{2}\frac{\partial^2 C}{\partial\omega\,\partial t}=\frac{\overline{\varphi^2}}{2}\frac{\partial C}{\partial t}+\frac{\overline{\varphi^2}}{2}\operatorname{Im} C, \quad (29)$$

$$\operatorname{Im} C=\frac{1}{2}\frac{\partial^2 C}{\partial\omega\,\partial t}=\frac{1}{2}\frac{\partial^2 C}{\partial\lambda\,\partial\omega}\frac{d\lambda}{dt}. \quad (30)$$

Substituting (23) and (25) into (30) we finally find

$$\operatorname{Im} a(\omega, \lambda)=\frac{1}{2}\frac{\partial^2\varepsilon(\omega, \lambda)}{\partial\omega\,\partial\lambda}, \quad (31)$$

$$\varepsilon(\omega, t)=\varepsilon_0(\omega, t)+\frac{i}{2}\frac{\partial^2\varepsilon_0(\omega, t)}{\partial\omega\,\partial t}. \quad (32)$$

(The real part of a, as already indicated, is negligible and can be omitted). The formula (32) clearly is correct also in the case when ϵ depends on several parameters.

We note in conclusion that formula (32), like formula (15), can be derived not only by applying the theorem on adiabatic invariants, but also directly from the fundamental formulas of quantum mechanics, if one expresses the left and right sides of these formulas by the corresponding matrix elements of quantum-mechanical perturbation theory. However, we shall not dwell upon these rather cumbersome calculations. (These calculations were first performed by I. E. Dzyaloshinskiĭ to derive formulas equivalent to (15).)

3. THE STRESS TENSOR OF A VARIABLE ELECTRIC FIELD IN A LIQUID LOCATED IN A STRONG MAGNETIC FIELD

Consider a liquid placed in a strong constant magnetic field and a weak electric field varying with a frequency at which the liquid is transparent. Here we shall suppose that when there is no electric field the liquid does not have any magnetic properties, i.e., that its magnetic permeability is $\mu = 1$.

The permittivity of the liquid in the magnetic field in the absence of absorption is a Hermitian tensor of the form

$$\varepsilon_{ik}(\omega, \mathbf{H})=\varepsilon_1(\omega, H^2)\delta_{ik}+\varepsilon_2(\omega, H^2)H_iH_k + i\varepsilon_3(\omega, H^2)e_{ikl}H_l, \quad (33)$$

where $\mathbf{H}$ is the constant magnetic field intensity, and e_{ikl} a totally antisymmetric unit tensor. Here ϵ_1, ϵ_2, and ϵ_3 are functions of H^2 and ω which tend toward some finite limit as $H^2 \to 0$. In particular, for an electron plasma,

$$\varepsilon_1=1-\frac{\omega_0^2}{\omega^2-\omega_H^2}, \qquad \varepsilon_2=\frac{e^2}{m^2c^2}\frac{\omega_0^2}{(\omega^2-\omega_H^2)\omega^2},$$

$$\varepsilon_3=-\frac{e}{mc}\frac{\omega_0^2}{(\omega^2-\omega_H^2)\omega},$$

$$\omega_H^2=e^2H^2/m^2c^2, \qquad \omega_0^2=4\pi ne^2/m, \quad (34)$$

(n is the number of electrons per unit volume, e the charge, and m the mass of the electron).

Consider some volume filled by a dielectric of permittivity (33). By generalizing some arguments of the preceding section, it is not difficult to show that in this case the following relation holds for the variation of the free energy [instead of formula (18a)]

$$\delta\mathcal{F}=\int\left\{\delta F_0-\frac{1}{16\pi}E_{0i}^*E_{0k}\delta\varepsilon_{ik}\right\}dV, \quad (35)$$

where F_0 is the density of the free energy in the absence of an electric field, and $\mathbf{E}_0$ is the complex amplitude of the electric field:*

$$\mathbf{E}=\frac{1}{2}(\mathbf{E}_0 e^{-i\omega t}+\mathbf{E}_0^* e^{i\omega t}).$$

We note first that although our medium is by assumption nonmagnetic ($\mathbf{B} = \mathbf{H}$) when there is

*To derive (35), strictly speaking, it is not enough to consider only the field in a capacitor, since we cannot produce in that way a field of the most general form, say a rotating one. This difficulty can easily be avoided by considering, instead of an oscillating circuit, a cavity resonator of arbitrary form. Simple calculations lead to the same formula (35) in this case, too.

no electric field, this is no longer so when the electric field is taken into account. Therefore we determine first the connection between **B** and **H**, accurate to terms $\sim E^2$. For this we use the well known thermodynamic equation

$$\mathbf{H} = 4\pi \partial F / \partial \mathbf{B}.$$

Carrying out the differentiation with the help of formula (35), and taking into account that $\mu = 1$ by assumption, i.e.,

$$4\pi \partial F_0 / \partial \mathbf{B} = \mathbf{B}, \qquad F_0 = F_{00} + B^2/8\pi$$

(F_{00} is the density of free energy in the absence of both the electric and magnetic fields) we find

$$\mathbf{H} = \mathbf{B} - \tfrac{1}{4} E^*_{0i} E_{0k} \partial \varepsilon_{ik} / \partial \mathbf{B} \tag{36}$$

or, to the same accuracy,

$$\mathbf{B} = \mathbf{H} + \tfrac{1}{4} E^*_{0i} E_{0k} \partial \varepsilon_{ik} / \partial \mathbf{H}. \tag{36a}$$

We note that since $\partial \epsilon_{ik}/\partial H \neq 0$ when $\mathbf{H} = 0$, $\mathbf{B} \neq 0$ even when $\mathbf{H} = 0$, i.e., a variable electric field can create in the medium an average magnetic moment proportional to E^2. It is easy to see that for this the electric field must rotate, i.e., it must not be linearly polarized.

Now we subject our dielectric to a deformation at constant induction **B**, the displacement vector of its points being $\delta \mathbf{r}(\mathbf{r})$. Then we must recognize that when a volume element is displaced its permittivity changes, and consequently the magnetic field changes from point to point. Taking this into account we obtain

$$\delta \varepsilon_{ik} = -(\delta \mathbf{r} \nabla) \varepsilon_{ik} - \frac{\partial \varepsilon_{ik}}{\partial \rho} \rho \operatorname{div} \delta \mathbf{r} + \frac{\partial \varepsilon_{ik}}{\partial B_l} (\delta \mathbf{r} \nabla) B_l. \tag{37}$$

Since the deformation takes place at constant **B**, we have

$$\delta F_0 = \delta F_{00} = -(\delta \mathbf{r} \nabla) F_{00} - \frac{\partial F_{00}}{\partial \rho} \rho \operatorname{div} \delta \mathbf{r}.$$

Substituting into (35) and integrating by parts we obtain

$$\delta \mathcal{F} = \int \Big\{ \nabla \Big(\rho \frac{\partial F_{00}}{\partial \rho} - F_{00} \Big) - \frac{1}{16\pi} \nabla \Big(E^*_{0i} E_{0k} \rho \frac{\partial \varepsilon_{ik}}{\partial \rho} \Big) - \frac{E^*_{0i} E_{0k}}{16\pi} \Big(\nabla \varepsilon_{ik} + \frac{\partial \varepsilon_{ik}}{\partial B_l} \nabla B_l \Big) \Big\} \delta \mathbf{r} \, dV. \tag{38}$$

On the other hand, the change in free energy equals the work of the forces

$$\delta \mathcal{F} = -\int \mathbf{f} \delta \mathbf{r} \, dV, \tag{39}$$

where **f** is the force acting on a unit volume of the dielectric. Equating (38) and (39) and recalling that

$$F_{00} - \rho \partial F_{00} / \partial \rho = - p_{00}$$

[$p_{00} = p_{00}(\rho, T)$ is the pressure in the absence of the fields as a function of ρ and T], we find the final expression for the force

$$\mathbf{f} = -\operatorname{grad} p_{00} + \frac{1}{16\pi} \operatorname{grad} \Big[E^*_{0i} E_{0k} \Big(\frac{\partial \varepsilon_{ik}}{\partial \rho} \Big)_T \rho \Big] - \frac{E^*_{0i} E_{0k}}{16\pi} \Big[\operatorname{grad} \varepsilon_{ik} - \frac{\partial \varepsilon_{ik}}{\partial B_l} \operatorname{grad} B_l \Big] \tag{40}$$

or, substituting $E^*_{0i} E_{0k} \partial \epsilon_{ik} / \partial B_l$ from (36),

$$\mathbf{f} = -\nabla p_{00} + \frac{1}{16\pi} \nabla \Big[E^*_{0i} E_{0k} \Big(\frac{\partial \varepsilon_{ik}}{\partial \rho} \Big) \rho \Big] - \frac{E^*_{0i} E_{0k}}{16\pi} \nabla \varepsilon_{ik} + \frac{(B_l - H_l) \nabla B_l}{4\pi}. \tag{41}$$

To find the tensor σ_{ik} we must transform the right side of (41) into the form $\partial \sigma_{ik} / \partial x_k$. For this we note that in virtue of the Maxwell equations

$$E^*_{0l} E_{0m} \frac{\partial \varepsilon_{lm}}{\partial x_i} = \frac{\partial}{\partial x_i} (E^*_{0l} E_{0m} \varepsilon_{lm}) - \Big(E^*_{0l} \varepsilon_{lm} \frac{\partial E_{0m}}{\partial x_i} + \text{compl. conj.} \Big) = \frac{\partial}{\partial x_i} (E^*_{0l} E_{0m} \varepsilon_{lm}) - \Big(\frac{\partial E_{0i}}{\partial x_m} D^*_{0m} + \text{compl. conj.} \Big) = \frac{\partial}{\partial x_i} (E^*_{0l} E_{0m} \varepsilon_{lm}) - \Big(\frac{\partial E_{0i} D^*_{0m}}{\partial x_m} + \text{compl. conj.} \Big),$$

$$(B_k - H_k) \frac{\partial B_k}{\partial x_i} = \frac{\partial}{\partial x_i} \Big(\frac{B^2}{2} - \mathbf{BH} \Big) - \frac{\partial H_k}{\partial x_i} B_k = \frac{\partial}{\partial x_i} \Big(\frac{B^2}{2} - \mathbf{BH} \Big) + \frac{\partial H_i B_k}{\partial x_k} = \frac{\partial}{\partial x_i} \Big\{ -\frac{H^2}{2} - \frac{(\mathbf{B} - \mathbf{H})^2}{2} \Big\} + \frac{\partial H_i B_k}{\partial x_k} \approx -\frac{1}{2} \frac{\partial H^2}{\partial x_i} + \frac{\partial H_i B_k}{\partial x_k}.$$

We obtain

$$\sigma_{ik} = -p_{00} \delta_{ik} + \frac{1}{8\pi} \Big\{ \frac{E^*_{0l} E_{0m}}{2} \Big(\rho \frac{\partial \varepsilon_{lm}}{\partial \rho} - \varepsilon_{lm} \Big) - H^2 \Big\} \delta_{ik} + \frac{1}{4\pi} H_i B_k + \frac{1}{16\pi} (E^*_{0i} D_{0k} + \text{compl. conj.}) \tag{42}$$

or, expressing **D** by **E** and **B** by **H**,

$$\sigma_{ik} = -\Big(p_{00} + \frac{H^2}{8\pi} \Big) \delta_{ik} + \frac{1}{4\pi} H_i H_k + \frac{1}{16\pi} \Big\{ E^*_{0l} E_{0m} \Big(\rho \frac{\partial \varepsilon_{lm}}{\partial \rho} - \varepsilon_{lm} \Big) \Big\} \delta_{ik} + \frac{1}{16\pi} \Big\{ H_l \frac{\partial \varepsilon_{lm}}{\partial H_k} E^*_{0l} E_{0m} + (\varepsilon_{km} E^*_{0i} E_{0m} + \text{compl. conj.}) \Big\}. \tag{43}$$

The tensor σ_{ik} in the liquid must satisfy two conditions. First, the tangential components of the forces acting on the surface must be continuous on the interface between two adjoining liquids. This is necessary in order that the liquids be in mechanical equilibrium. If we recall that the tangential components of **E** and **H**, and the normal components of **D** and **B**, are continuous in virtue of the boundary conditions, then it follows at once from (42) that this condition is fulfilled, i.e., that

$$\sigma_{ik} n_k = \sigma'_{ik} n_k$$

(n is the normal to the surface).

Secondly, in virtue of the law of conservation of angular momentum, the tensor σ_{ik} must be symmetric. To verify that this condition is fulfilled we express ϵ_{ik} through ϵ_1, ϵ_2, and ϵ_3 in (43). We obtain

$$\begin{aligned}\sigma_{ik} = & -\left(p_{00} + \frac{H^2}{8\pi}\right)\delta_{ik} + \frac{1}{4\pi} H_i H_k \\ & + \frac{1}{16\pi}\left\{E^*_{0l} E_{0m}\left(\rho \frac{\partial \varepsilon_{lm}}{\partial \rho} - \varepsilon_{lm}\right)\right\}\delta_{ik} \\ & + \frac{1}{16\pi}\left\{[\varepsilon_1 E^*_{0i} E_{0k} + \varepsilon_2 (\mathbf{E}_0 \times \mathbf{H})(E^*_{0i} H_k + E^*_{0k} H_i) \right. \\ & + i\varepsilon_3 E^*_{0i} [\mathbf{E}_0 \times \mathbf{H}]_k + \text{compl. conj.}] + 2 H_i H_k \left[\frac{\partial \varepsilon_1}{\partial H^2} \mathbf{E}_0 \times \mathbf{E}^*_0 \right. \\ & \left.\left. + \frac{\partial \varepsilon_2}{\partial H^2} |[\mathbf{E}_0 \times \mathbf{H}]|^2 + i \frac{\partial \varepsilon_3}{\partial H^2} [\mathbf{E}^*_0 \times \mathbf{E}_0] \times \mathbf{H}\right] + i\varepsilon_3 H_i [\mathbf{E}^*_0 \times \mathbf{E}_0]_k\right\}. \end{aligned} \quad (44)$$

We can verify the correctness of the identity

$$iE^*_{0i} [\mathbf{E}_0 \times \mathbf{H}]_k + \text{compl. conj.} = -iH_i [\mathbf{E}^*_0 \times \mathbf{E}_0]_k + i [\mathbf{E}^*_0 \times \mathbf{E}_0]\, \mathbf{H} \delta_{ik}, \quad (45)$$

with the help of which we now write σ_{ik} in the manifestly symmetric form

$$\begin{aligned}\sigma_{ik} = & -\left(p_{00} + \frac{H^2}{8\pi}\right)\delta_{ik} + \frac{1}{4\pi} H_i H_k + \frac{1}{16\pi}\left\{E^*_{0l} E_{0m}\left(\rho \frac{\partial \varepsilon_{lm}}{\partial \rho} \right.\right. \\ & \left.\left. - \frac{\varepsilon_{lm} + \varepsilon^*_{lm}}{2}\right)\right\}\delta_{ik} + \frac{1}{16\pi}\{\varepsilon_1 E^*_{0i} E_{0k} + \varepsilon_2 (\mathbf{E}_0 \mathbf{H})(E^*_{0i} H_k \\ & + H_i E^*_{0k}) + \text{compl. conj.}\} + \frac{H_i H_k}{8\pi}\left\{\frac{\partial \varepsilon_1}{\partial H^2} \mathbf{E}_0 \mathbf{E}^*_0 \right. \\ & \left. + \frac{\partial \varepsilon_2}{\partial H^2} |[\mathbf{E}_0 \times \mathbf{H}]|^2 + i \frac{\partial \varepsilon_3}{\partial H^2} [\mathbf{E}^*_0 \times \mathbf{E}_0]\, \mathbf{H}\right\}. \end{aligned} \quad (46)$$

The expression (46) can be generalized without difficulty to the case where the medium possesses magnetic properties even when the electric field vanishes.

In conclusion I express my thanks to L. D. Landau for advice and discussion, and V. L. Ginzburg and I. E. Dzyaloshinskiĭ for a discussion of the results. I also thank M. A. Leontovich who called to my attention the question considered in Sec. 3 of the present paper.

[1] L. D. Landau and E. M. Lifshitz, Электродинамика сплошных сред (Electrodynamics of Continuous Media), Gostekhizdat, 1957.

Translated by D. Brill
262