HIGH ENERGY ELECTRON SCATTERING PROCESSES

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A method is developed for calculating cross sections of various processes involving fast polarized electrons, which can be described by two-component spinors. Some helicity properties of the electrons are elucidated. The pair-production cross section in electron-positron collisions is found.

IN the limit of very high velocities polarized electrons can be described by two- rather than four-component spinors, as is done with the neutrino, except that the electron can be polarized either parallel or antiparallel to the direction of motion, while the neutrino is always polarized antiparallel.

The use of two-component spinors to describe the electrons greatly simplifies cross-section calculations for many processes involving high-energy polarized electrons. It is to such a description, together with an elucidation of some helicity properties of the electrons, that the present work is devoted.

1. Let φ and χ be two-component spinors describing fast electrons polarized parallel and antiparallel to the direction of motion. They satisfy the Weyl equation

$$\overset{+}{\sigma}_\mu \frac{\partial}{\partial x_\mu}\varphi = 0, \quad \overset{-}{\sigma}_\mu \frac{\partial}{\partial x_\mu}\chi = 0, \tag{1}$$

where the σ_μ are four independent two-by-two matrices, namely the three Pauli matrices and the unit matrix multiplied by i:

$$\overset{+}{\sigma}_\mu = (\boldsymbol{\sigma}, i), \quad \overset{-}{\sigma}_\mu = (\boldsymbol{\sigma}, -i).$$

These matrices satisfy the commutation relations

$$\overset{+}{\sigma}_\mu\overset{-}{\sigma}_\nu + \overset{+}{\sigma}_\nu\overset{-}{\sigma}_\mu = 2\delta_{\mu\nu}, \qquad \overset{-}{\sigma}_\mu\overset{+}{\sigma}_\nu + \overset{-}{\sigma}_\nu\overset{+}{\sigma}_\mu = 2\delta_{\mu\nu}. \tag{2}$$

The σ_μ matrices form a complete set of two-by-two matrices. The completeness condition can be written

$$\sum_\mu \left(\overset{+}{\sigma}_\mu\right)^{\alpha\dot\beta}\left(\overset{-}{\sigma}_\mu\right)_{\dot\rho\sigma} = 2\delta^\alpha_\sigma\delta^{\dot\beta}_{\dot\rho} \tag{3}$$

(tensor notation is used here for the spinor indices).

The $\overset{+}{\sigma}_\mu$ and $\overset{-}{\sigma}_\mu$ matrices are related by equations of the form

$$\varepsilon\overset{+}{\sigma}_\mu\varepsilon = \overset{-}{\sigma}{}^T_\mu, \quad \varepsilon\overset{-}{\sigma}_\mu\varepsilon = \overset{+}{\sigma}{}^T_\mu, \tag{4}$$

where σ^T is the transposed matrix, and ϵ is a metric tensor related to the antisymmetric of the Pauli matrices, $\epsilon = i\sigma_2$.

We shall call particles satisfying the first equation in (1) R-particles (for right-handed helicity), and those satisfying the second equation, L-particles (for left-handed helicity).

It is only in the high-energy limit that the equations for helical particles can be of first order, and then the helicity and polarization become identical. In the more general case which includes also relatively low velocities, the equations for helical particles can be only of second order[1] and there is no simple relation between the helicity and polarization.

2. We perform a plane-wave expansion[2] on φ and χ, writing

$$\varphi(x) = \frac{1}{\sqrt{V}}\sum_{\mathbf{p}}[a_R(\mathbf{p})\,u_R(\mathbf{p})\,e^{ipx} + b_R^+(\mathbf{p})\,v_R(-\mathbf{p})\,e^{-ipx}],$$

$$\chi(x) = \frac{1}{\sqrt{V}}\sum_{\mathbf{p}}[a_L(\mathbf{p})\,u_L(\mathbf{p})\,e^{ipx} + b_L^+(\mathbf{p})\,v_L(-\mathbf{p})\,e^{-ipx}],$$

where $p = (\mathbf{p}, i\epsilon)$, $\epsilon = |\mathbf{p}|$, and $a_R(\mathbf{p})$ and $\overset{+}{b}_R(\mathbf{p})$ are the destruction operators for an R-electron and the creation operator for a R-positron with momentum $\mathbf{p}$ (and $a_L(\mathbf{p})$ and $\overset{+}{b}_L(\mathbf{p})$ are the corresponding L-particle operators). We assume the amplitudes u and v to be normalized according to

$$u_R(\mathbf{p})\overline{u}_R(\mathbf{p}) = v_R(-\mathbf{p})\overline{v}_R(-\mathbf{p}) = -\overline{p}/2\varepsilon,$$
$$u_L(\mathbf{p})\overline{u}_L(\mathbf{p}) = v_L(-\mathbf{p})\overline{v}_L(-\mathbf{p}) = \overset{+}{p}/2\varepsilon,$$
$$u_R(\mathbf{p})\overline{u}_L(\mathbf{p}) = v_R(-\mathbf{p})\overline{v}_L(-\mathbf{p}) = 0, \tag{5}$$

where $\overline{u}$ is the Hermitian conjugate spinor, and $\overline{p} = \overset{-}{\sigma}_\mu p_\mu$, $\overset{+}{p} = \overset{+}{\sigma}_\mu p_\mu$. One then easily obtains the following commutation relations for the φ and χ fields:

$$\{\varphi(x), \overline{\varphi}(x')\} = -i\sigma_-(x-x'),$$

$$\{\chi(x), \overline{\chi}(x')\} = -i\sigma_+(x-x'), \quad \{\varphi(x), \overline{\chi}(x')\} = 0. \tag{6}$$

Here we have written

$$\sigma_-(x) = -i\bar{\sigma}_\mu \frac{\partial}{\partial x_\mu}\Delta_0(x), \qquad \sigma_+(x) = i\overset{+}{\sigma}_\mu \frac{\partial}{\partial x_\mu}\Delta_0(x),$$

$$\Delta_0(x) = \frac{i}{(2\pi)^3}\int e^{ipx}\,\mathrm{sgn}\,(p_0)\,\delta(p^2)\,d^4p.$$

The remaining commutators vanish. The last of equations (6) indicates the kinematic independence of the φ and χ fields.

The Green's functions for the fields are given by the relations

$$\langle 0\,|\,T\varphi(x)\bar{\varphi}(x')\,|\,0\rangle = \sigma_-^c(x-x'),$$

$$\langle 0\,|\,T\chi(x)\bar{\chi}(x')\,|\,0\rangle = \sigma_+^c(x-x');$$

$$\sigma_-^c(x) = -i\bar{\sigma}_\mu \frac{\partial}{\partial x_\mu}\Delta_0^c(x), \qquad \sigma_+^c(x) = i\overset{+}{\sigma}_\mu \frac{\partial}{\partial x_\mu}\Delta_0^c(x),$$

$$\Delta_0^c(x) = -\frac{i}{(2\pi)^4}\int \frac{e^{ipx}}{p^2 - i\delta}\,d^4p.$$

In momentum space these Green's functions may be written

$$\sigma_-^c(p) = -i\bar{p}/(p^2 - i\delta), \qquad \sigma_+^c(p) = i\overset{+}{p}/(p^2 - i\delta).$$

We write the Heisenberg-representation equations for the interacting electron-positron and electromagnetic fields in the form

$$i\overset{+}{\sigma}_\mu \frac{\partial}{\partial x_\mu}\varphi + \overset{+}{\sigma}_\mu A_\mu \varphi = 0, \qquad i\bar{\sigma}_\mu \frac{\partial}{\partial x_\mu}\chi + \bar{\sigma}_\mu A_\mu \chi = 0,$$

$$\Box A_\mu = -(j_\mu^R + j_\mu^L), \tag{7}$$

where the current densities of the R and L particles are defined as

$$j_\mu^R(x) = -\tfrac{1}{2}e\,[\bar{\varphi}(x),\ \overset{+}{\sigma}_\mu\varphi(x)], \quad j_\mu^L(x) = \tfrac{1}{2}e\,[\bar{\chi}(x),\ \bar{\sigma}_\mu\chi(x)]. \tag{8}$$

Conservation of j_μ^R and j_μ^L follows from Eqs. (7). Under a space reflection j_μ^R goes over into J_μ^L, and vice versa [which is related to the sign difference between the two equations of (8)]. The scattering matrix is of the usual form

$$S = T\exp\left(i\int(\mathscr{L}_i^R(x) + \mathscr{L}_i^L(x))\,d^4x\right), \tag{9}$$

where $\mathscr{L}_i^R(x)$ and $\mathscr{L}_i^L(x)$ are the interaction Lagrangian densities for the R and L particles in the interaction representation, namely

$$\mathscr{L}_i^R(x) = A_\mu(x)\,j_\mu^R(x), \qquad \mathscr{L}_i^L(x) = A_\mu(x)\,j_\mu^L(x).$$

3. Proceeding by perturbation theory, we may expand the S-matrix as a power series in the coupling constant e, and represent the individual terms of the series by Feynman diagrams. Then in the momentum representation we have the following rules for constructing the mathematical expressions corresponding to these diagrams.

To every incoming free electron line belonging to an R particle of momentum **p** there corresponds a two-component spinor amplitude $u_R(\mathbf{p})$ or $v_R(-\mathbf{p})$ for an electron in the initial state or a positron in the final state, respectively. The amplitudes $\bar{u}_R(\mathbf{p})$ and $\bar{v}_R(-\mathbf{p})$ correspond to outgoing lines belonging to an electron in the final state and a positron in the initial state. For L particles the same free electron lines correspond to the amplitudes $u_L(\mathbf{p})$, $v_L(-\mathbf{p})$, $\bar{u}_L(\mathbf{p})$, and $\bar{v}_L(-\mathbf{p})$. Free photon lines with momentum k, on the other hand, correspond as usual to the amplitudes $e_\mu/\sqrt{2\omega}$ (where e_μ is a polarization unit vector satisfying $e_\mu^2 = 1$).

2) To internal electron lines for R particles with 4-momentum p correspond propagators $\sigma_-^c(p)$, while to internal L particles correspond propagators $\sigma_+^c(p)$. Internal photon lines correspond to the functions $D_{\mu\nu}^c(k) = -i\delta_{\mu\nu}/(k^2 - i\delta)$.

3) Vertex parts at which R-particle electron lines converge correspond to the matrix $e\overset{+}{\sigma}_\mu\delta(p - p' + k)$, while those for L particles correspond to $e\bar{\sigma}_\mu\delta(p - p' + k)$.

The rules for constructing the expressions for the matrix elements are the usual ones, the expressions for the individual elements of the diagrams being ordered in the direction opposite to that of the electron line. The expression obtained for a Feynman diagram according to these rules is then multiplied by $(-1)^{n+l}\delta_{\mathbf{p}}(2\pi)^{4(n-k)}$, where n, k, and l are the number of vertices, internal lines, and closed electrons loops, respectively, and δ_p is 1 or −1 and arises because of the antisymmetry of the electron states (cf. Akhiezer and Berestetskiĭ.[2])

4. As an example of a cross section calculation involving high-energy helical particles, consider electron-electron scattering in the center-of-mass (c.m.) system of the colliding particles.

We note first that the vector nature of the electromagnetic interaction preserves helicity along the electron lines. Let us use the rules given above to write the expression for the matrix element of the direct scattering process of electrons with different helicity:

$$M_{RL} = -i\frac{(2\pi)^4}{(p_1 - q_1)^2}\bar{u}_R(q_1)\overset{+}{\sigma}_\mu u_R(p_1)\,\bar{u}_L(q_2)\,\bar{\sigma}_\mu u_L(p_2)$$

which, using Eq. (3), can be written

$$M_{RL} = -2i\frac{(2\pi)^4}{(p_1 - q_1)^2}\bar{u}_R(q_1)\,u_L(p_2)\,\bar{u}_L(q_2)\,u_R(p_1).$$

Now making use of (5) and calculating some simple traces, we obtain the following expression for the square of the absolute value of the matrix element:

$$|M_{RL}|^2 = \frac{(2\pi)^8}{(p_1 - q_1)^4}\frac{1}{\varepsilon^4}(p_1q_2)(q_1p_2). \tag{10}$$

The expression for the corresponding exchange diagram is obtained from (10) by interchanging q_1 and q_2. In view of (5), the cross-product term between the direct and exchange diagrams vanishes in this case.

From (5) we may arrive at the following general conclusion: the product of two matrix elements will vanish if they correspond to diagrams which differ in the helicity of at least one electron line. This follows from conservation of helicity along electron lines.

Direct calculation will show that if the particles have the same helicity, the square of the matrix element is

$$|M_{LL}|^2 = \frac{(2\pi)^8}{(p_1 - q_1)^4} \frac{1}{\varepsilon^4} (q_1 q_2)(p_1 p_2). \qquad (10')$$

Having obtained the product of matrix elements for the direct and exchange diagrams, we can arrive at the cross section in the usual way.

The final result for the cross section is

$$d\sigma_{RL} = \frac{\alpha^2}{4\varepsilon^2} do \left(\tan^4 \frac{\theta}{2} + \cot^4 \frac{\theta}{2}\right), \quad d\sigma_{LL} = \frac{4\alpha^2}{\varepsilon^2} do \sin^{-4}\theta.$$

Similarly, one easily obtains the cross section for electron-positron scattering in the c.m. system:

$$d\sigma_{RL} = \frac{\alpha^2}{4\varepsilon^2} do \left(\sin^4 \frac{\theta}{2} + \sin^{-4} \frac{\theta}{2}\right),$$
$$d\sigma_{LL} = \frac{\alpha^2}{4\varepsilon^2} do \cos^4 \frac{\theta}{2} \cot^4 \frac{\theta}{2}.$$

We remark that electrons and positrons of different helicity do not undergo scattering by means of virtual annihilation.

Equation (10′) can be obtained from (10) by interchanging p_1 and $-q_1$, and in general we have the following rules.

1) Consider an electron line no parts of which are described by a propagation function, and assume we have an expression for a quadratic combination of matrix elements of any process involving such an electron line with given helicity and with initial and final momenta p and q. If we interchange p and $-q$, we obtain the corresponding expression for a similar process, but involving an electron line of opposite helicity.

2) Consider an electron line containing sections described by a propagation function, and assume we have an expression for a quadratic combination of matrix elements of any process involving such an electron line with given helicity and with initial and final momenta p and q. If we interchange p and $-q$ and change the sign of the momentum in the electron propagator, we obtain the corresponding expression for the time-reversed process, but involving an electron line of opposite helicity. Here by time-reversed process we mean the process in which the order of events along the electron line is reversed.

Both of these rules follow from CP invariance of the theory.

The second rule simplifies the calculation of the Compton cross section. In the c.m. system, we obtain the following expression for the scattering cross section of polarized photons by helical electrons:

$$d\sigma_R = \frac{\alpha^2}{16\varepsilon^2} do \Big[\Big(\cos^2 \frac{\theta}{2} + \cos^{-2} \frac{\theta}{2}\Big) e_1^2 e_2^2 + 2e_1^2 e_2^2 - 4(e_1 e_2)^2 + \frac{(k_1 e_2)^2 (k_2 e_1)^2}{\varepsilon^4 \cos^4 (\theta/2)} + 2 \frac{e_1^2 (k_1 e_2)^2 + 2(e_1 e_2)(k_1 e_2)(k_2 e_1)}{\varepsilon^2 \cos^2 (\theta/2)}\Big],$$

where θ is the scattering angle, and k_1, e_1, k_2, and e_2 are the photon momenta and polarization before and after collision.

We note that reversal of all particle helicities leaves quadratic combinations of matrix elements invariant. This is because the $\overset{+}{\sigma}$ and $\bar{\sigma}$ matrices satisfy the same commutation relations, namely those of (2).

5. Let us now consider inelastic electron-positron scattering with the emission of a photon, in the c.m. system. The matrix elements corresponding to the Feynman diagrams for an R electron and an L positron are

$$M_1^{RL} = \frac{-i}{\sqrt{2\omega}} \frac{(2\pi)^4}{r_1^2 \alpha_1^2} \bar{u}_R(q_1) \overset{+}{\sigma}_\mu u_R(p_1) \bar{v}_L(-p_2) \bar{\sigma}_\mu \overset{+}{\alpha}_1 \bar{e} v_L(-q_2),$$

$$M_2^{RL} = \frac{-i}{\sqrt{2\omega}} \frac{(2\pi)^4}{r_1^2 \alpha_2^2} \bar{u}_R(q_1) \overset{+}{\sigma}_\mu u_R(p_1) \bar{v}_L(-p_2) \bar{e} \overset{+}{\alpha}_2 \bar{\sigma}_\mu v_L(-q_2),$$

$$M_3^{RL} = \frac{-i}{\sqrt{2\omega}} \frac{(2\pi)^4}{r_2^2 \alpha_3^2} \bar{u}_R(q_1) \overset{+}{e} \bar{\alpha}_3 \overset{+}{\sigma}_\mu u_R(p_1) \bar{v}_L(-p_2) \bar{\sigma}_\mu v_L(-q_2),$$

$$M_4^{RL} = \frac{-i}{\sqrt{2\omega}} \frac{(2\pi)^4}{r_2^2 \alpha_4^2} \bar{u}_R(q_1) \overset{+}{\sigma}_\mu \bar{\alpha}_4 \overset{+}{e} u_R(p_1) \bar{v}_L(-p_2) \bar{\sigma}_\mu v_L(-q_2),$$

where $r_1 = p_1 - q_1$, $r_2 = p_2 - q_2$, $\alpha_1 = -q_2 - k$, $\alpha_2 = -p_2 + k$, $\alpha_3 = q_1 + k$, $\alpha_4 = p_1 - k$, and p_1, p_2, q_1, and q_2 are the electron and positron momenta before and after collision, while k and e are the momentum and polarization of the emitted photon. We are not considering annihilation diagrams, since their contribution at high energies and small scattering and emission angles is negligible because of the large denominators corresponding to virtual photon lines.

Using relations (3) and (5) and calculating the traces, we obtain expressions for quadratic combinations of the matrix elements in the form

$$|M_1^{RL}|^2 = \frac{(2\pi)^8}{2\omega} \frac{S_1^{RL}}{r_1^4 \alpha_1^4} = \frac{(2\pi)^8}{2\omega r_1^4 \alpha_1^4} \frac{1}{\varepsilon^2 \varepsilon_1 \varepsilon_2} [2p_1 p_2 q_2 e (2q_1 \alpha_1 \alpha_1 e - \alpha_1^2 q_1 e) - e^2 p_1 p_2 (2q_1 \alpha_1 q_2 \alpha_1 - \alpha_1^2 q_1 q_2)],$$

$$|M_2^{RL}|^2 = \frac{(2\pi)^8}{2\omega} \frac{S_2^{RL}}{r_1^4 \alpha_2^4} = \frac{(2\pi)^8}{2\omega r_1^4 \alpha_2^4} \frac{1}{\varepsilon^2 \varepsilon_1 \varepsilon_2} [2q_1 q_2 p_2 e (2p_1 \alpha_2 \alpha_2 e - \alpha_2^2 p_1 e) - e^2 q_1 q_2 (2p_1 \alpha_2 p_2 \alpha_2 - \alpha_2^2 p_1 p_2)],$$

$$2\mathrm{Re}\,(M_1^{RL}M_2^{RL*}) - \frac{(2\pi)^8}{2\omega}\frac{S_{12}^{RL}}{r_1^4\alpha_1^2\alpha_2^2} = 2\frac{(2\pi)^8}{2\omega r_1^4\alpha_1^2\alpha_2^2}\frac{1}{\varepsilon^2\varepsilon_1\varepsilon_2}$$
$$\times (p_1\alpha_2p_2e + p_1p_2\alpha_2e - p_1ep_2\alpha_2)(q_1q_2\alpha_1e$$
$$+ q_1\alpha_1q_2e - q_2\alpha_1q_1e).$$

The corresponding expressions for matrix elements 3 and 4 are obtained from (11) by interchanging q_2 and $-p_1$ and p_2 and $-q_1$. The cross terms between the matric elements are calculated just as simply, and we obtain

$$2\mathrm{Re}\,(M_1^{RL}M_3^{RL*}) = \frac{(2\pi)^8}{2\omega}\frac{S_{13}^{RL}}{r_1^2r_2^2\alpha_1^2\alpha_3^2}$$
$$= 2\frac{(2\pi)^8}{2\omega r_1^2r_2^2\alpha_1^2\alpha_2^2}\frac{1}{\varepsilon^2\varepsilon_1\varepsilon_2}\Big[p_1p_2\alpha_1e\,(q_1q_2\alpha_3e - q_1\alpha_3q_2e)$$
$$- p_1p_2q_1e\,(q_2\alpha_1\alpha_3e - \alpha_1\alpha_3q_2e) + \frac{1}{2}e^2p_1p_2\,(q_1\alpha_1q_2\alpha_3$$
$$+ q_1\alpha_3q_2\alpha_1 - q_1q_2\alpha_1\alpha_3)\Big],$$

$$2\mathrm{Re}\,(M_1^{RL}M_4^{RL*}) = \frac{(2\pi)^8}{2\omega}\frac{S_{14}^{RL}}{r_1^2r_2^2\alpha_1^2\alpha_4^2} = 2\frac{(2\pi)^8}{2\omega r_1^2r_2^2\alpha_1^2\alpha_4^2}\frac{1}{\varepsilon^2\varepsilon_1\varepsilon_2}(p_2\alpha_4p_1e$$
$$+ p_1p_2\alpha_4e - p_1\alpha_4p_2e)(q_1q_2\alpha_1e + q_1\alpha_1q_2e - q_2\alpha_1q_1e).$$

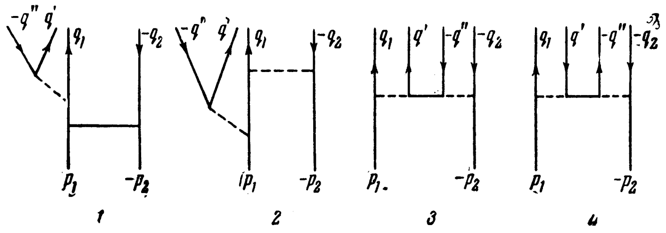
The expressions for S_{24}^{RL} and S_{23}^{RL} are obtained from S_{13}^{RL} and S_{14}^{RL} by interchanging p_1 and $-q_1$ and p_2 and $-q_2$ (and changing the signs of all the α's).

Rules 1 and 2 can be used to obtain expressions for the quadratic combinations of matrix elements when the colliding particles have the same helicities. We need remark only that the expressions for the $S_{1,2,12}^{LL}$ are obtained from the $S_{1,2,12}^{RL}$ by interchanging p_1 and $-q_1$, while those for $S_{3,4,34}^{LL}$ are obtained from the $S_{3,4,34}^{RL}$ by interchanging p_2 and $-q_2$. The expressions for the $S_{13,24,14,23}^{LL}$ are obtained from the $S_{14,23,13,24}^{RL}$ by interchanging p_1 and $-q_1$ and changing the signs of all the α's.

Summing over the polarizations of the emitted photon, we obtain an expression for the differential cross section for radiation in the RL and LL cases. In the RL case, which is the only one for which we will exhibit the expression, one obtains

$$d\sigma_{RL} = \frac{\alpha^3}{2\pi^2}do_2do\frac{d\omega}{\omega}(\varepsilon-\omega)^2\Big\{[2\varepsilon(\varepsilon-\omega)z$$
$$+ \omega(2\varepsilon-\omega y')y - 2\omega(\varepsilon-\omega)y']^{-2}\Big(A\frac{\sin^2\theta'}{(1-\cos\theta')^2}$$
$$+ B\frac{\sin^2\theta}{(1-\cos\theta)^2} - C\frac{\sin\theta'\sin\theta\cos\varphi}{(1-\cos\theta')(1-\cos\theta)}\Big)$$
$$+ \frac{1}{4\varepsilon^2(\varepsilon-\omega)^2z^2}\Big(B\frac{\sin^2\theta'}{(1+\cos\theta')^2} + B\frac{\sin^2\theta}{(1+\cos\theta)^2}$$
$$- D\frac{\sin\theta'\sin\theta\cos\varphi}{(1+\cos\theta')(1+\cos\theta)}\Big) + [2\varepsilon(\varepsilon-\omega)z\,(2\varepsilon(\varepsilon-\omega)z$$
$$+ \omega(2\varepsilon-\omega y')y - 2\omega(\varepsilon-\omega)y')]^{-1}$$
$$\times\Big(\sqrt{ABD} - 2C\cos\varphi\frac{1-\cos\theta'\cos\theta}{\sin\theta'\sin\theta}\Big)\Big\}; \qquad (12)$$

$$A = (2\varepsilon^2 - 2\varepsilon\omega + \omega^2)/\varepsilon(\varepsilon-\omega), \quad B = (2\varepsilon^2 - 2\varepsilon\omega + \omega^2)/\varepsilon^2,$$
$$C = (2\varepsilon-\omega)^2/\varepsilon(\varepsilon-\omega), \qquad D = (2\varepsilon-\omega)^2/\varepsilon^2,$$
$$y' = 1-\cos\theta', \quad y = 1-\cos\theta, \quad z = 1-\cos\theta_2,$$



θ', θ, θ_2 are the angles between **k** and $\mathbf{q}_2$, **k** and $-\mathbf{p}$, and $\mathbf{q}_2$ and $-\mathbf{p}$, while φ is the angle between the $(-\mathbf{p}, \mathbf{k})$ and $(\mathbf{q}_2, \mathbf{k})$ planes.

The maxima of $d\sigma_{RL}$ occur when θ_2 and θ are about equal to zero, and when θ_2 is about equal to 0 and θ about equal to π. At these angles $d\sigma_{RL} = d\sigma_{LL}$. Integrating (12) over small θ_2 and θ, with a lower limit in θ_2 equal to $[2m^2/\epsilon(\epsilon-\omega)]^{1/2}$ and that for θ of order of magnitude $(m\omega/\epsilon(\epsilon-\omega))^{1/2}$ (where m is the electron mass), we obtain, with logarithmic accuracy, the formula first obtained by Garibyan.[3]

Integrating (12) in the region $\theta_2 \sim 0$ and $\theta \sim \pi$ with lower limits $(2m^2/\epsilon^2)^{1/2}$ and about $[m\omega/\epsilon(\epsilon-\omega)]^{1/2}$, respectively, we arrive at

$$d\sigma_{RL} \approx 2r_0^2\alpha\frac{d\omega}{\omega}\Big(1-\frac{\omega}{\varepsilon}\Big)\ln\frac{\varepsilon(\varepsilon-\omega)}{m\omega}, \quad r_0 = \frac{\alpha}{m}. \qquad (13)$$

The radiation described by (13) is due to the recoil electron.

6. Finally, let us consider pair production in fast electron-positron collisions in the c.m.s. system.

Let p_1, p_2, q_1, q_2 be the electron and positron momenta before and after collision, and let q' and q'' be the momenta of the created pair. The region of angles which makes the largest contribution to the cross sections is determined by the minima in the denominators of the matrix elements, and corresponds to small angles of emission for both the created and scattered particles. The figure shows those diagrams which contribute most strongly to the cross section in the region of small angles about the direction of p_1.

Because the phase volumes for q_1 and q' are equal in our region of angles, the diagrams differing by exchange of q_1 and q' give the same contribution as do the direct diagrams pictured here, and thus we can take them into account merely by multiplying the cross section we get by 2.

The contribution from the cross terms between these diagrams and the exchange diagrams, as well as between diagrams 1, 2 and 3, 4 may be neglected as the maxima of their matrix elements do not overlap. One may also neglect pair production by virtual annihilation of the colliding particles, because of the large denominators of the virtual photon lines. Finally, the diagrams in which the pair is created by the positron give no contribution to our region of angles.

The matrix elements corresponding to diagrams 1 and 2 for the collision of an R and an L particle and the creation of a pair of L particles are

$$M_1 = -i\frac{(2\pi)^4}{r^2r_2^2\alpha_1^2}\bar u_R(q_1)\overset{+}{\sigma}_\mu\bar\alpha_1\overset{+}{\sigma}_\nu u_R(p_1)\bar v_L(-p_2)\sigma_\nu v_L$$

$$\times(-q_2)\bar u_L(q')\sigma_\mu v_L(-q''),$$

$$M_2 = -i\frac{(2\pi)^4}{r^2r_2^2\alpha_2^2}\bar u_R(q_1)\overset{+}{\sigma}_\mu\bar\alpha_2\overset{+}{\sigma}_\nu u_R(p_1)\bar v_L(-p_2)\sigma_\mu v_L$$

$$\times(-q_2)\bar u_L(q')\sigma_\nu v_L(-q''),$$

where $r = q' + q''$, $r_2 = p_2 - q_2$, $\alpha_1 = q_1 + q' + q''$, and $\alpha_2 = p_1 - q' - q''$.

Using (3) and (5) we obtain the following expressions for the quadratic combinations of these matrix elements:

$$|M_1|^2 = i\frac{(2\pi)^8}{r^4r_2^4\alpha_1^4}\frac{2}{\varepsilon^2\varepsilon_1\varepsilon_2\varepsilon'\varepsilon''}p_1q_2q_1q''(2p_2\alpha_1q'\alpha_1 - \alpha_1^2p_2q'),$$

$$|M_2|^2 = \frac{(2\pi)^8}{r^4r_2^4\alpha_2^4}\frac{2}{\varepsilon^2\varepsilon_1\varepsilon_2\varepsilon'\varepsilon''}q_1p_2p_1q''(2q_2\alpha_2q'\alpha_2 - \alpha_2^2q_2q'), \quad (14)$$

$$2\mathrm{Re}(M_1M_2^*) = \frac{(2\pi)^8}{r^4r_2^4\alpha_1^2\alpha_2^2}\frac{1}{\varepsilon^2\varepsilon_1\varepsilon_2\varepsilon'\varepsilon''}$$

$$\times\{(p_2q_1q''\alpha_2 + p_2\alpha_2q_1q'' - p_2q''q_1\alpha_2)(p_1q'q_2\alpha_1$$

$$+ p_1q_2q'\alpha_2 - q_2q'p_1\alpha_1) + (q_1q_2q''\alpha_2 + q_1q''q_2\alpha_2$$

$$- q_2q''q_1\alpha_2)(p_1q'p_2\alpha_1 + p_1p_2q'\alpha_1 - p_2q'p_1\alpha_1)$$

$$- p_2q_2[q''\alpha_2(p_1q'q_1\alpha_1 + p_1q_1q'\alpha_1 - q_1q'p_1\alpha_1)$$

$$+ q_1q''(p_1q'\alpha_1\alpha_2 + p_1\alpha_2q'\alpha_1 - p_1\alpha_1q'\alpha_2)$$

$$- q_1\alpha_2(p_1q'q''\alpha_1 + p_1q''q'\alpha_1 - q'q''p_1\alpha_1)]\}.$$

The corresponding expressions for production of a pair of R-particles are obtained from (14), according to rule 1, by interchanging q' and q''. The quadratic combinations for equal helicities of the incoming particles are obtained from the above by interchanging p_2 and $-q_2$.

Writing the expression for the differential cross section in the usual way, integrating over the variables of the scattered positron (corresponding to eliminating the δ function of the momentum variables) and the angle variables of the created positron, with a lower limit equal to $(2m^2/\epsilon^2)^{1/2}$, we obtain the cross section for small angles of scattering and production in the form

$$d\sigma_{RL} = 4r_0^2\frac{\alpha^2}{\pi^2}d\psi\theta_1d\theta_1\theta'd\theta'\frac{\varepsilon^2}{\varepsilon_1^2}\frac{d\varepsilon'd\varepsilon''}{(\varepsilon'+\varepsilon'')^2}$$

$$\times\frac{\sin^2\psi}{\theta_1^2 + \theta'^2 - 2\theta_1\theta'\cos\psi}\frac{\theta_1^2}{(\theta_1^2 + \eta^2\theta'^2 + 2\eta\theta_1\theta'\cos\psi)^2},$$

where $\eta = (\epsilon' + \epsilon'')/\epsilon_1$, and $\epsilon_1 = \epsilon - \epsilon' - \epsilon''$. Integrating over ψ, θ_1, and θ' with lower limits of order m/ϵ, we find, for $\epsilon - \epsilon' - \epsilon'' > \epsilon' + \epsilon''$ (remembering the exchange diagrams), that

$$d\sigma_{RL} \approx 4r_0^2\frac{\alpha^2}{\pi}\frac{d\varepsilon'd\varepsilon''}{(\varepsilon'+\varepsilon'')^2}\ln\frac{\varepsilon}{m}\ln\frac{\varepsilon(\varepsilon'+\varepsilon'')}{m(\varepsilon-\varepsilon'-\varepsilon'')}. \quad (15)$$

In our case of small angles, $d\sigma_{RL} = d\sigma_{LL}$. To find the total cross section let us integrate (15) over ϵ' and ϵ''. with the condition $\epsilon - \epsilon' - \epsilon'' > \epsilon' + \epsilon''$ and lower limits of order m. We obtain

$$\sigma_{RL} \approx 2r_0^2\frac{\alpha^2}{\pi}\ln^3\frac{\varepsilon}{m}.$$

Let us now evaluate the contribution from diagrams 3 and 4. The matrix elements in the RR case of the colliding particles and creation of an L pair are

$$M_3 = -i\frac{(2\pi)^4}{r_1^2r_2^2\alpha_3^2}\bar u_R(q_1)\overset{+}{\sigma}_\mu u_R(p_1)\bar v_R(-p_2)\overset{+}{\sigma}_\nu v_R$$

$$\times(-q_2)\bar u_L(q')\bar\sigma_\mu\overset{+}{\alpha}_3\bar\sigma_\nu v_L(-q''),$$

$$M_4 = -i\frac{(2\pi)^4}{r_1^2r_2^2\alpha_4^2}\bar u_R(q_1)\overset{+}{\sigma}_\mu u_R(p_1)\bar v_R(-p_2)\overset{+}{\sigma}_\nu v_R$$

$$\times(-q_2)\bar u_L(q')\bar\sigma_\nu\overset{+}{\alpha}_4\sigma_\mu v_L(-\bar q''),$$

where $r_1 = p_1 - q_1$, $r_2 = p_2 - q_2$, $\alpha_3 = -(p_1 - q_1 - q')$, and $\alpha_4 = p_1 - q_1 - q''$. Using (3) and (5) we arrive at

$$|M_3|^2 = \frac{(2\pi)^8}{r_1^4r_2^4\alpha_3^4}\frac{2}{\varepsilon^2\varepsilon_1\varepsilon_2\varepsilon'\varepsilon''}q_2q''p_1q'(2q_1\alpha_3p_2\alpha_3 - \alpha_3^2q_1p_2),$$

$$|M_4|^2 = \frac{(2\pi)^8}{r_1^4r_2^4\alpha_4^4}\frac{2}{\varepsilon^2\varepsilon_1\varepsilon_2\varepsilon'\varepsilon''}q_1q''p_2q'(2q_2\alpha_4p_1\alpha_4 - \alpha_4^2p_1q_2),$$

$$2\mathrm{Re}(M_3M_4^*) = \frac{(2\pi)^8}{r_1^4r_2^4\alpha_3^2\alpha_4^2}\frac{1}{\varepsilon^2\varepsilon_1\varepsilon_2\varepsilon'\varepsilon''}\{(p_2q'p_1\alpha_4 + p_1q'p_2\alpha_4$$

$$- p_1p_2q'\alpha_4)(q_1q''q_2\alpha_3 + q_2q''q_1\alpha_3 - q_1q_2y''\alpha_3) + (q_2q'p_1\alpha_4$$

$$+ p_1q'q_2\alpha_4 - p_1q_2q'\alpha_4)(q_1q''p_2\alpha_3 + p_2q''q_1\alpha_3 - q_1p_2q''\alpha_3)$$

$$+ p_2q_2[p_1\alpha_4(q_1q''q'\alpha_3 + q'q''q_1\alpha_3 - q_1q'q''\alpha_3)$$

$$- p_1q'(q_1q''\alpha_3\alpha_4 + q_1\alpha_3q''\alpha_4 - q_1\alpha_4q''\alpha_3)$$

$$+ q'\alpha_4(q_1q''p_1\alpha_3 + p_1q''q_1\alpha_3 - p_1q_1q''\alpha_3)]\}.$$

If we now use rule 2 we can find the corresponding expressions for creation of an R pair. The expressions for the RL case of colliding particles are obtained from the above by interchanging p_2 and $-q_2$.

Writing out the expression for the differential cross section and integrating over the variables of the scattered particles [one integration eliminates the δ function of the momentum variables, and the second is taken over angles with lower limit equal to $2m^2/\epsilon\epsilon_1)^{1/2}$], the expression we obtain for small angles of the created pair is

$$d\sigma_{RR} = 8r_0^2\frac{\alpha^2}{\pi^2}d\varphi\theta'd\theta'\theta''d\theta''\frac{\varepsilon_1}{\varepsilon}\frac{\varepsilon'^2\varepsilon''^2d\varepsilon'd\varepsilon''}{(\varepsilon'+\varepsilon'')^2}$$

$$\times\frac{\sin^2\varphi}{(\varepsilon'^2\theta'^2 + \varepsilon''^2\theta''^2 + 2\varepsilon'\varepsilon''\theta'\theta''\cos\varphi)^2}.$$

In our region of angles $d\sigma_{RR} = d\sigma_{RL}$. Integrating over and θ' and θ'' with lower limits of order of magnitude m/ϵ, we find for $\epsilon' > \epsilon''$ (recalling the exchange diagrams), that

$$d\sigma_{RR} \approx 8r_0^2 \frac{\alpha^2}{\pi} \frac{\varepsilon_1}{\varepsilon} \frac{d\varepsilon' d\varepsilon''}{(\varepsilon'+\varepsilon'')^2} \ln \frac{\varepsilon}{m} \ln \frac{\varepsilon'^2}{\varepsilon'^2 - \varepsilon''^2}.$$

To find the total cross section we integrate (16) over ϵ' and ϵ'' from m to ϵ. We finally arrive at

$$\sigma_{RR} \approx 2r_0^2 \frac{\alpha^2}{\pi} \ln^2 \frac{\varepsilon}{m}.$$

For θ' and θ'' close to π, the total cross section is of order of magnitude $r_0^2\alpha^2\pi^{-1} \ln(\epsilon/m)$, whereas when $\theta' = 0$, and θ'' is near π, it is of order of magnitude $r_0^2\alpha^2/\pi$.

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