PAIR CORRELATION EFFECTS NEAR CLOSED SHELLS

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Pair correlation effects are studied for nuclei that are close to closed shell. Equations for the Green's functions are obtained. It is shown that these equations are valid to within corrections of the order $A^{-1/3}$. A possibility is indicated for testing the results by means of stripping or pick-up reactions.

1. INTRODUCTION

THERE are two main methods commonly used in the theory of superconductivity: the canonical-transformation method, developed in the "Bogolyubov papers",[1] and the Green's-function method proposed by Gor'kov.[2] The canonical-transformation method has been used by Belyaev[3] to study the effects of pair correlation in spherical nuclei. In papers by Migdal and his co-workers[4,5] the Gor'kov method has been applied to the study of the influence of superfluidity on the values of the moments of inertia of nonspherical nuclei.

In all these calculations, however, an essential assumption is that there is a large number of correlated pairs. For spherical nuclei this means a large number of particles in unfilled shells, and for nonspherical nuclei it means that the condition $\rho\Delta \gg 1$ must hold, where ρ is the density of one-particle levels near the Fermi surface and Δ is a parameter that characterizes the energy of the pair correlation. It is interesting to examine the properties of nuclei that are near closed shells, in which the stated assumptions are not satisfied because of the smallness of the number of nucleons above the closed shells. This situation calls for more accurate methods, which make it possible to treat pair-correlation effects in this case also.

The purpose of the present paper is to generalize the Gor'kov method so as to find the Green's functions, the density matrices, and the pair-correlation energies in the case of nuclei that are near closed shells.

We shall assume that we can describe the motion of the nucleons in the nucleus by using the Hamiltonian

$$H = \sum_{\lambda} \varepsilon_{\lambda} a_{\lambda}^{+} a_{\lambda} + \tfrac{1}{2} \sum_{\lambda\lambda_1} g_{\lambda\lambda_1} a_{\lambda}^{+} a_{-\lambda}^{+} a_{-\lambda_1} a_{\lambda_1}, \qquad (1)$$

where λ is a complete set of quantum numbers necessary for the description of the state of a nucleon (the sets λ and $-\lambda$ differ from each other only in the sign of the projection m of the total angular momentum of the nucleon along some arbitrary axis), ϵ_λ are the eigenvalues of the energy of the nucleon in the self-consistent field $U(\mathbf{r})$:

$$H_0\varphi_\lambda(\mathbf{r}) = \varepsilon_\lambda\varphi_\lambda(\mathbf{r}), \qquad H_0 = -(\hbar^2/2m_{\text{eff}})\nabla^2 + U(\mathbf{r});$$

m_{eff} is the effective mass of the nucleon, and $g_{\lambda\lambda_1}$ is the interaction matrix element that leads to pairing. The stated Hamiltonian assumes that the interaction between nucleons that leads to pairing occurs in states with opposite signs of the angular-momentum projection m. The part of the interacaction that does not lead to pairing is taken into account by the introduction of the self-consistent field and the effective mass of the nucleon.

For the further exposition we need Green's functions of the system of N nucleons, defined in the following way:

$$\begin{aligned} G_\lambda^{N+}(\tau) &= -ie^{iE_N\tau}\langle\Phi_N a_\lambda e^{-iH\tau} a_\lambda^{+}\Phi_N\rangle, \ \tau > 0,\\ G_\lambda^{N-}(\tau) &= ie^{-iE_N\tau}\langle\Phi_N a_\lambda^{+} e^{iH\tau} a_\lambda\Phi_N\rangle, \ \tau < 0;\\ G_\lambda^{+}(\tau=0) &- G_\lambda^{-}(\tau=0) = -i. \end{aligned} \qquad (2)$$

Let us also define the functions

$$\begin{aligned} F_\lambda^{N+}(\tau) &= (-1)^{\lambda} e^{iE_N\tau}\langle\Phi_{N+2}\, a_{-\lambda}^{+}\, e^{-iH\tau}\, a_\lambda^{+}\Phi_N\rangle, \ \tau > 0,\\ F_\lambda^{N-}(\tau) &= (-1)^{-\lambda} e^{-iE_N\tau}\langle\Phi_N a_\lambda^{+} e^{iH\tau} a_{-\lambda}^{+}\Phi_{N-2}\rangle, \ \tau < 0;\\ F_\lambda^{N+2-}(0) &= F_\lambda^{N+}(0), \end{aligned} \qquad (3)$$

where Φ_N is the wave function of the ground state of the system of N fermions, E_N is the energy of this state, and $(-1)^\lambda = \text{sign}\,\lambda$.

By the use of the definitions (2), (3) it is easy to obtain the spectral expansions of the Green's functions. In particular, we have for the functions $F_\lambda(\tau)$:

$$\begin{aligned} F_\lambda^{N+}(\tau) &= (-1)^{\lambda}\sum_s\langle\Phi_{N+2}\, a_{-\lambda}^{+}\Phi_{N+1,\,s}\rangle\\ &\times\langle\Phi_{N+1,\,s}\, a_\lambda^{+}\Phi_N\rangle\exp\{-i(E_{N+1,\,s}-E_N)\tau\},\\ F_\lambda^{N-}(\tau) &= (-1)^{-\lambda}\sum\langle\Phi_N a_\lambda^{+}\Phi_{N-1,\,s}\rangle\\ &\times\langle\Phi_{N-1,\,s}\, a_{-\lambda}^{+}\Phi_{N-1}\rangle\exp\{-i(E_N-E_{N-1,\,s})\tau\}. \end{aligned} \qquad (4)$$

Using the Hamiltonian (1), we get the equations for the functions (2) and (3):

$$\left(i\frac{\partial}{\partial\tau}-\varepsilon_\lambda\right)G_\lambda^{N+}(\tau)+i\sum_{\lambda_1}g_{\lambda\lambda_1}e^{iE_N\tau}\langle\Phi_N a_{-\lambda}^+ b_{\lambda_1}e^{-iH\tau}a_\lambda^+\Phi_N\rangle=0,$$

$$\left(i\frac{\partial}{\partial\tau}-\varepsilon_\lambda\right)G_\lambda^{N-}(\tau)-i\sum_{\lambda_1}g_{\lambda\lambda_1}e^{-iE_N\tau}\langle\Phi_N a_\lambda^+ e^{iH\tau}a_{-\lambda}^+ b_{\lambda_1}\Phi_N\rangle=0,$$

$$\left(i\frac{\partial}{\partial\tau}-2\mu_N^++\varepsilon_\lambda\right)F_\lambda^{N+}(\tau)$$
$$+\sum_{\lambda_1}(-1)^{-\lambda}g_{\lambda_1\lambda}e^{iE_N\tau}\langle\Phi_{N+2}b_{\lambda_1}^+a_\lambda e^{-iH\tau}a_\lambda^+\Phi_N\rangle=0,$$

$$\left(i\frac{\partial}{\partial\tau}-2\mu_N^-+\varepsilon_\lambda\right)F_\lambda^{N-}(\tau)$$
$$+\sum_{\lambda_1}(-1)^{\lambda}g_{\lambda_1\lambda}e^{-iE_N\tau}\langle\Phi_N a_\lambda^+ e^{iH\tau}b_{\lambda_1}^+a_\lambda\Phi_{N-2}\rangle=0, \qquad (5)$$

$$b_\lambda=a_{-\lambda}a_\lambda,\quad 2\mu_N^+=E_{N+2}-E_N,\qquad 2\mu_N^-=E_N-E_{N-2}.$$

By definition

$$N=-i\sum_\lambda G_\lambda^{N-}(0).$$

In order to get equations analogous to the Gor'kov equations for the Green's functions, it is necessary to find a connection between the functions $G_\lambda^\pm$ and $F_\lambda^\pm$. Such a connection can be obtained if we can calculate the mean values of products of four Fermi amplitudes that appear in Eq. (5). The main content of the present paper is the calculation of these averages in the approximation $A^{-1/3}$ for nuclei that are near closed shells. It must be emphasized that in the existing theories the parameter on which all the approximations are based is the large number of places in the subshell that is in the course of being filled ($\Omega = 2j + 1 \gg 1$). This limitation does not exist in the method that is developed in the present paper. The possibility of writing the equations for the Green's functions $G_\lambda^\pm$ and of the following recurrence scheme for their solution was pointed out by Migdal.[6]

2. THE GREEN'S FUNCTION FOR TWO NUCLEONS ABOVE A CLOSED SHELL

We begin the accomplishment of the program we have indicated with the treatment of nuclei with two nucleons above a closed shell. We shall assume that there is no pairing in the nucleus that corresponds to the doubly closed shell. Let Φ_0 be the wave function of the doubly closed shell. If we neglect the perturbation of the self-consistent field by the addition of one nucleon (an A^{-1} effect), then the wave function of such a nucleus can be written in the form $\Phi_1 = \overset{+}{a}_{\lambda_0}\Phi_0$, where Φ_0 is the state in which the odd particle is placed.

The wave function of the nucleus with two nucleons above the closed shell already cannot be represented in as simple a form as the wave function of the nucleus just mentioned. For two nucleons above the closed shell there are already important pairing effects, which lead to the "smearing out" of the pair of particles over an entire subshell of the given shell. Owing to this we must write the wave function of the ground state of a nucleus with two nucleons above a closed shell in the form

$$\Phi_2=\sum_{j,m}c_jC_{jm\,j-m}^{00}b_{jm}^+\Phi_0'=\sum_\lambda c_\lambda b_\lambda^+\Phi_0', \qquad (6)$$

where $b_\lambda^\pm = a_\lambda^\pm a_{-\lambda}^\pm$ is the operator for production of a pair, $|c_j|^2$ is the probability of finding the pair in the subshell j, and Φ_0' is the wave function of the closed shell as perturbed by the presence of pairing; the summation is taken over all states λ.

After these comments, let us go on to the derivation of the equations for the Green's function G_λ^{2-}. On the basis of the form of Φ_2 and the commutation rules, we have in the main term of the second equation in (5):

$$\langle\Phi_2 a_\lambda^+ e^{iH\tau}a_{-\lambda}^+ b_{\lambda_1}\Phi_2\rangle=2c_\lambda^*\langle\Phi_0' a_{-\lambda}e^{iH\tau}a_{-\lambda}^+ b_{\lambda_1}\Phi_2\rangle$$
$$+\sum_{\lambda_2}c_{\lambda_2}^*\langle\Phi_0' a_\lambda^+ e^{iH\tau}a_{-\lambda}^+ b_{\lambda_1}\Phi_2\rangle. \qquad (7)$$

Let us consider the first term in the right member of Eq. (7). The effect of the pairing on the closed shell can be treated by perturbation theory. Therefore for $\tau = 0$

$$2c_\lambda^*\sum_{\lambda_1}g_{\lambda\lambda_1}\langle\Phi_0' a_{-\lambda}a_{-\lambda}^+ b_{\lambda_1}\Phi_2\rangle=2c_\lambda^*\sum_{\lambda_1}g_{\lambda\lambda_1}\langle\Phi_0' b_{\lambda_1}\Phi_2\rangle.$$

Here the difference between Φ_0' and Φ_0 is important because of the summation over λ_1.

The dependence of this four-pole on τ is associated with the fixed index λ. Therefore the time dependence of the first term is, in first approximation:

$$2c_\lambda^*\sum_{\lambda_1}g_{\lambda\lambda_1}\langle\Phi_0' a_{-\lambda}e^{iH\tau}a_{-\lambda}^{+1}b_{\lambda_1}\Phi_2\rangle=2c_\lambda^* e^{i(E_0+\varepsilon_\lambda)\tau}\sum_{\lambda_1}g_{\lambda\lambda_1}\langle\Phi_0' b_{\lambda_1}\Phi_2\rangle.$$

The difference between Φ_0' and Φ_0 is due to the transition of nucleons from the filled shell into the unfilled shell. For this reason there occurs in the second term of Eq. (7) an additional oscillating factor of the form $e^{i2\omega_0\tau}$, where ω_0 is the distance between the shells.

By means of the commutation rules we find that for $\tau = 0$ the second term is of the order of magnitude of

$$c_\lambda\langle\Phi_0' b_{\lambda_1}\Phi_2\rangle.$$

Thus for $\tau = 0$ the first and second terms in Eq. (7) are of the same order of magnitude. Solving Eq. (5) for $G_\lambda^{2-}(\tau)$ by using both the first and the second terms, we can convince ourselves that the ratio of the contribution of the second term to that of the first amounts to a quantity

$$\sim (\varepsilon_\lambda - \mu_2)/(\varepsilon_\lambda - \mu_2 + \omega_0),$$

where $2\mu_2 = E_2 - E_0$.

This ratio is small if the states λ are close to the Fermi surface ($\epsilon_\lambda - \mu_2 \ll \omega_0$). Therefore to accuracy $\sim A^{-1/3}$ the equation for $G_\lambda^{2-}(\tau)$ is of the form

$$\left(i\frac{\partial}{\partial\tau} - \varepsilon_\lambda\right) G_\lambda^{2-}(\tau) - i2c_\lambda^* e^{-i(2\mu_2-\varepsilon_\lambda)\tau} \sum_{\lambda_1} g_{\lambda\lambda_1} \langle \Phi_0' \, b_{\lambda_1} \Phi_2 \rangle = 0. \quad (8)$$

It is convenient to put this equation in the form

$$(i\partial/\partial\tau - \varepsilon_\lambda) G_\lambda^{2-}(\tau) - i\Delta_2^* F_\lambda^{2-}(\tau) = 0, \quad (9)$$

$$\Delta_2 = \sum_{\lambda_1} g_{\lambda\lambda_1} (-1)^{\lambda-\lambda_1} F_{\lambda 1}^{2-}(0), \quad (10)$$

where the summation is taken over all λ_1 without exception.

In Eq. (9) we have introduced the notation

$$F_\lambda^{2-}(\tau) = 2c_\lambda^* (-1)^{-\lambda} \exp\{-i(2\mu_2 - \varepsilon_\lambda)\tau\}.$$

It is not hard to see that $F_\lambda^{2-}(\tau)$ can also be represented in the following form:

$$F_\lambda^{2-}(\tau) = (-1)^{-\lambda} e^{-iE_2\tau} \langle \Phi_2 a_\lambda^+ e^{iH\tau} a_{-\lambda}^+ \Phi_0' \rangle. \quad (11)$$

In order to find the Green's function from Eq. (9), we must determine $F_\lambda^{2-}(\tau)$. For this purpose it is convenient to introduce the function

$$F_\lambda^{0+}(\tau) = (-1)^\lambda e^{iE_0\tau} \langle \Phi_2 a_{-\lambda}^+ e^{-iH\tau} a_\lambda^+ \Phi_0' \rangle, \quad (12)$$

with

$$F_\lambda^{2-}(0) = F_\lambda^{0+}(0), \ \tau = 0.$$

Equations for the determination of $F_\lambda^{2-}(\tau)$ and $F_\lambda^{0+}(\tau)$ follow from the equations (5) by a procedure like that used to get Eq. (9). These equations are

$$(i\partial/\partial\tau - 2\mu_2 + \varepsilon_\lambda) F_\lambda^{2-}(\tau) = 0,$$

$$(i\partial/\partial\tau - 2\mu_2 + \varepsilon_\lambda) F_\lambda^{0+}(\tau) + i\Delta_2 G_\lambda^{0+}(\tau) = 0, \quad (13)$$

where $G_\lambda^{0+}(\tau)$ is the Green's function of the magic nucleus:

$$G_\lambda^{0+}(\tau) = -ie^{-i\varepsilon_\lambda\tau}.$$

By means of the spectral representations (4) we get the following solutions of the system (13):

$$F_\lambda^{0+}(\tau) = -\tfrac{1}{2}\Delta_2(\varepsilon_\lambda - \mu_2)^{-1} \exp\{-i\varepsilon_\lambda\tau\},$$

$$F_\lambda^{2-}(\tau) = -\tfrac{1}{2}\Delta_2(\varepsilon_\lambda - \mu_2)^{-1} \exp\{-i(2\mu_2 - \varepsilon_\lambda)\tau\}. \quad (14)$$

Using Eq. (14), we get the solution of Eq. (9) in the form

$$G_\lambda^{2-}(\tau) = i\,|\Delta_2/2(\varepsilon_\lambda - \mu_2)|^2 \exp\{-i(2\mu_2 - \varepsilon_\lambda)\tau\}. \quad (15)$$

A comparison of the functions $G_\lambda^{2-}(\tau)$ and $F_\lambda^{2-}(\tau)$ with their general expansions (4) shows that, as had been assumed, the excitation energy of the nucleus with one particle above the magic core is determined by the one-particle model, $\Delta E_{12} = \epsilon_\lambda - \epsilon_{\lambda_0}$. Besides this, by means of the Green's functions G_λ^{2-} and F_λ^{2-} we can obtain the following matrix elements:

$$\sum_s |\langle \Phi_2 a_\lambda^+ \Phi_{1s} \rangle|^2 = |\Delta_2/2(\varepsilon_\lambda - \mu_2)|^2,$$

$$\left|\sum_s \langle \Phi_2 a_\lambda^+ \Phi_{1s} \rangle \langle \Phi_{1s} a_{-\lambda}^+ \Phi_0 \rangle\right| = |\Delta_2/2(\varepsilon_\lambda - \mu_2)|, \quad (16)$$

where the summation is taken over all states Φ_{1s} that correspond to a fixed excitation energy $\Delta E_{1s} = \epsilon_\lambda - \epsilon_{\lambda_0}$.

Since

$$\Phi_2 = \sum_\lambda c_\lambda b_\lambda^+ \Phi_0',$$

the formulas (16) can be simplified for states $\Phi_{1s} = a_{\lambda s}^+ \Phi_0$, and the two formulas reduce to one:

$$|\langle \Phi_2 b_\lambda^+ \Phi_0 \rangle| = |\Delta_2/2(\varepsilon_\lambda - \mu_2)|. \quad (17)$$

Thus the functions G_λ^{2-} and F_λ^{2-} found above do not contradict each other.

The quantities μ_2 and Δ_2 that appear in the expression (15) for the Green's function are determined from Eq. (10) and the normalization condition for the Green's function. In Eq. (10) the sum over filled states can be included as a renormalization of the matrix element $g_{\lambda\lambda_1}$. If besides this we assume the renormalized matrix element to be constant and equal to $\bar{g}$, we then get the following equation for the determination of μ_2 and Δ_2:

$$1 = \tfrac{1}{2}|\bar{g}| \sum_\lambda (\varepsilon_\lambda - \mu_2)^{-1}, \qquad 2 = \sum_\lambda |\Delta_2/2(\varepsilon_\lambda - \mu_2)|^2, \quad (18)$$

where the summation is taken over the states in the unfilled shell.

We can obtain an estimate of the quantities μ_2 and Δ_2 by considering the limiting case of a single level (Sec. 4). In this case

$$2(\varepsilon_\lambda - \mu_2) = |g|\Omega, \qquad \Delta_2 = |g|\sqrt{2\Omega},$$

where Ω is the degree of degeneracy of the level ($\Omega \sim A^{1/3}$). Thus near a closed shell the density matrix

$$\rho_\lambda^{(2)} = |\Delta_2/2(\varepsilon_\lambda - \mu_2)|^2$$

is of the order $A^{-1/3}$.

3. THE GREEN'S FUNCTIONS OF ODD NEARLY MAGIC NUCLEI

By using the results of the preceding section, it is not hard to find the equation for the Green's functions of odd nearly magic nuclei. Let us begin the discussion with the case of one nucleon above the closed shell. The wave function of the ground state is $\Phi_1 = a^+_{\lambda_0}\Phi_0$. It follows from Eq. (5) that in this case $G^{1-}_\lambda(\tau)$ satisfies the free equation

$$(i\partial/\partial\tau - \varepsilon_\lambda) G^{1-}_\lambda(\tau) = 0, \tag{19}$$

and the equation for $G^{1+}_\lambda(\tau)$ is of the form

$$(i\partial/\partial\tau - \varepsilon_\lambda) G^{1+}_\lambda(\tau) = -i\delta_{\lambda-\lambda_0} \sum_{\lambda_1} g_{\lambda_0\lambda_1} e^{iE_1\tau} \langle \Phi_0 b_{\lambda_1} e^{-iH\tau} b^+_{\lambda_0}\Phi_0\rangle. \tag{20}$$

As was to be expected, pairing effects appear in the function $G^{1+}_\lambda(\tau)$ only for $\lambda = -\lambda_0$, since the odd particle is in the state λ_0.

It is not hard to see that the right member of Eq. (20) cannot be calculated by the method of summation over the intermediate states Φ_{2S} by leaving only the term in the sum that corresponds to the ground state Φ_2. In fact, we have for example $\langle\Phi_0 b_\lambda b^+_\lambda \Phi_0\rangle = 1$, whereas $\langle\Phi_0 b_\lambda \Phi_2\rangle\langle\Phi_2 b^+_\lambda\Phi_0\rangle \sim A^{-1/3}$ (cf. Sec. 2).

Let us try to find the function $G^{1+}_\lambda(\tau)$ by using the known solution for $G^{2-}_\lambda(\tau)$. Obviously, for $\lambda \neq \pm\lambda_0$,

$$G^{1+}_\lambda(\tau) = -ie^{-i\varepsilon_\lambda\tau}.$$

Furthermore,

$$G^{1+}_{\lambda_0}(\tau) \equiv 0, \qquad G^{1+}_{-\lambda_0}(0) = -i.$$

Therefore, in accordance with the spectral representation, the function $G^{1+}_\lambda(\tau)$ must have the following form:

$$G^{1+}_\lambda(\tau) = i(\delta_{\lambda\lambda_0} - 1 + \delta_{\lambda-\lambda_0}B)\exp\{-i\varepsilon_\lambda\tau\} - i\delta_{\lambda-\lambda_0}B\exp\{-i(2\mu_2 - \varepsilon_\lambda)\tau\}. \tag{21}$$

The first exponential corresponds to the free motion of particles in the excited nucleus with two particles above the core and the second to the ground, paired, state of this nucleus.

The coefficient B is determined by a comparison of the spectral representations for G^{2-}_λ and G^{1+}_λ with the expressions (15) and (21); we get:

$$B = |\langle\Phi_1 a_{-\lambda}\Phi_2\rangle|^2 = |\Delta_2/2(\varepsilon_\lambda - \mu_2)|^2. \tag{22}$$

Thus

$$G^{1+}_\lambda(\tau) = i(\delta_{\lambda\lambda_0} - 1 + \delta_{\lambda-\lambda_0}|\Delta_2/2(\varepsilon_\lambda - \mu_2)|^2)\exp\{-i\varepsilon_\lambda\tau\} - i\delta_{\lambda-\lambda_0}|\Delta_2/2(\varepsilon_\lambda - \mu_2)|^2\exp\{-i(2\mu_2 - \varepsilon_\lambda)\tau\}. \tag{23}$$

A comparison of this expression with the general expansion of G^{1+}_λ leads to the following relation ($\lambda \neq \lambda_0$):

$$\sum_s |\langle\Phi_1 a_\lambda \Phi_{2s}\rangle|^2 = 1 - |\Delta_2/2(\varepsilon_\lambda - \mu_2)|^2\delta_{\lambda-\lambda_0}, \tag{24}$$

where the summation is taken over all excited states Φ_{2S} that correspond to a prescribed excitation energy $\Delta E_{2S} = \epsilon_\lambda - \epsilon_{\lambda_0} + 2(\epsilon_{\lambda_0} - \mu_2)$. Also it is not hard to see that the function $G^{1+}_\lambda(\tau)$ defined by Eq. (23) satisfies the equation

$$(i\partial/\partial\tau - \varepsilon_\lambda) G^{1+}_\lambda(\tau) + i\delta_{\lambda-\lambda_0}\Delta_2 F^{2-}_\lambda(\tau) = 0. \tag{25}$$

Let us consider a nucleus with three nucleons above the closed shell. In this case the wave function of the ground state is of the form

$$\Phi_3 = a^+_{\lambda_0}\Phi'_2, \tag{26}$$

where

$$\Phi'_2 = \sum_\lambda c'_\lambda b^+_\lambda \Phi'_0$$

is a state with angular momentum zero, which in general does not coincide with the wave function Φ_2 of the ground state, owing to the perturbing effect of the third particle on the pairing. Obviously in the ground state Φ_3 the one-particle state with angular-momentum component $-\lambda_0$ is a free state.[5] Therefore the Green's function for this state is

$$G^{3-}_{-\lambda_0}(\tau) \equiv 0. \tag{27}$$

It is also not hard to see that to accuracy $\sim A^{-1/3}$ by taking off the odd particle in the state λ_0 [this process is described by the function $G^{3-}_{\lambda_0}(\tau)$] we get a nucleus with two particles above the closed core in the ground state. It is not hard to make the calculation for this case, if in the equations of motion we estimate the terms that arise owing to pairing by breaking the expression up into a sum over intermediate states and keeping only the ground states. By using the spectral representation we thus get:

$$G^{3-}_{\lambda_0}(\tau) = i\exp\{-i(E_3 - E_2)\tau\}. \tag{28}$$

For $\lambda \neq \pm\lambda_0$ the Green's function $G^{3-}_\lambda(\tau)$ is obviously determined like the function $G^{2-}_\lambda(\tau)$ for the two particles above the closed core. In fact, by using the wave function Φ_3 of the ground state and the hypothesis that there is only one paired state for this nucleus, we find from Eq. (5);

$$(i\partial/\partial\tau - \varepsilon_\lambda) G^{3-}_\lambda(\tau) - i\Delta_3 F^{3-}_\lambda(\tau) = 0,$$
$$(i\partial/\partial\tau - 2\mu_3 + \varepsilon_\lambda) F^{3-}_\lambda(\tau) = 0,$$
$$(i\partial/\partial\tau - 2\mu_3 + \varepsilon_\lambda) F^{1+}_\lambda(\lambda) + i\Delta_3 G^{1+}_\lambda(\tau) = 0, \tag{29}$$

where

$$\Delta_3 = \sum_{\lambda_1} g_{\lambda\lambda_1}(-1)^{\lambda-\lambda_1} F^{3-}_{\lambda_1}(0). \tag{30}$$

In these equations the functions $F_\lambda(\tau)$ are defined by the general formulas (3), and the respective spectral representations (4) hold for them. It follows from these definitions that

$$F^{1+}_\lambda(0) = F^{3-}_\lambda(0) = 0 \text{ for } \lambda = \pm\lambda_0. \tag{31}$$

Therefore two terms ($\lambda_1 = \pm\lambda_0$) are absent from the sum (30).

The solution of the equations (29) obtained by using the spectral representation is ($\lambda \neq \pm\lambda_0$)

$$F_\lambda^{1+}(\tau) = -\tfrac{1}{2}\Delta_3(\varepsilon_\lambda - \mu_3)^{-1}\exp\{-i\varepsilon_\lambda\tau\},$$
$$F_\lambda^{3-}(\tau) = -\tfrac{1}{2}\Delta_3(\varepsilon_\lambda - \mu_3)^{-1}\exp\{-i(2\mu_3 - \varepsilon_\lambda)\tau\},$$
$$G_\lambda^{3-}(\tau) = i\,|\Delta_3/2\,(\varepsilon_\lambda - \mu_3)|^2\exp\{-i(2\mu_3 - \varepsilon_\lambda)\tau\}. \quad (32)$$

The quantities μ_3 and Δ_3 which characterize the pairing in this nucleus are determined from the normalization conditions and Eq. (30). Recalling the relations (28) and (31), we have:

$$1 = |\bar{g}|\sum_\lambda{}'(1 - \delta_{\lambda\lambda_0} - \delta_{\lambda_-\lambda_0})/2(\varepsilon_\lambda - \mu_3),$$
$$2 = \sum_\lambda{}'|\Delta_3/2(\varepsilon_\lambda - \mu_3)|^2(1 - \delta_{\lambda\lambda_0} - \delta_{\lambda-\lambda_0}), \quad (33)$$

where $\bar{g}$ is the renormalized matrix element (Sec. 2).

By means of Eqs. (33) and (18) we can calculate the differences $\mu_3 - \mu_2$ and $\Delta_3 - \Delta_2$. In first approximation we find:

$$\mu_3 - \mu_2 = \Delta_2^2/4(\varepsilon_{\lambda_0} - \mu_2),$$
$$\Delta_3 - \Delta_2 = -\tfrac{1}{4}\Omega_0[\Delta_2/2(\varepsilon_{\lambda_0} - \mu_2)]^4\Delta_2.$$

It follows from this that the pairing energy in an odd nucleus with three particles above a closed core is smaller than the pairing energy in the preceding even nucleus.

4. THE LIMITING CASE OF A SINGLE DEGENERATE LEVEL

In the limiting case of a single degenerate level the derivation of the equations for the Green's function is decidedly simplified. This simplification arises from the fact that the wave functions of the ground and excited states are known in this case.[7] The resulting system of equations for the Green's functions is, for an arbitrary number N of particles in the given degenerate level:

$$(i\partial/\partial\tau - \varepsilon_\lambda - 2g)\,G_\lambda^{N+}(\tau) - i\Delta_N^{+*}F_\lambda^{N+}(\tau) = 0,$$
$$(i\partial/\partial\tau - 2\mu_N^+ + \varepsilon_\lambda)\,F_\lambda^{N+}(\tau) + i\Delta_N^+G_\lambda^{N+}(\tau) = 0,$$
$$(i\partial/\partial\tau - \varepsilon_\lambda)\,G_\lambda^{N-}(\tau) - i\Delta_N^{-*}F_\lambda^{N-}(\tau) = 0,$$
$$(i\partial/\partial\tau - 2\mu_N^- + \varepsilon_\lambda + 2g)\,F_\lambda^{N-}(\tau) + i\Delta_N^-G_\lambda^{N-}(\tau) = 0. \quad (34)$$

The initial conditions are

$$G_\lambda^{N+}(0) = G_\lambda^{N-}(0) - i, \qquad \mu_{N+2}^- = \mu_N^+, \quad F_\lambda^{N+2-}(0) = F_\lambda^{N+}(0),$$
$$N = -i\sum_\lambda G_\lambda^{N-}(0), \qquad \Delta_N^\pm = \sum_{\lambda_1} g_{\lambda\lambda_1}(-1)^{\lambda-\lambda_1}F_{\lambda_1}^{N\pm}(0),$$
$$g = g_{\lambda\lambda}.$$

The equations (34) are obtained by breaking up the four-poles in Eq. (5) into sums over intermediate states. Because of the properties of the wave functions of the ground states, these sums contain only one term, which corresponds to the ground state.

By means of Eq. (15) one can proceed by recurrence to find the Green's functions for all the nuclei that correspond to the filling up of the given level. The recurrence scheme is treated independently for the even and odd nuclei. For the beginning of the recurrence scheme one needs to know the explicit form of the functions $F_\lambda^{2-}(0)$ and $F_\lambda^{3-}(0)$, which are calculated by means of Eq. (3). Using the initial conditions and the spectral representation we get

$$G_\lambda^{2+}(\tau) = i(2 - \Omega)\,\Omega^{-1}\exp\{-i(\varepsilon_\lambda - 2g)\tau\},$$
$$G_\lambda^{2-}(\tau) = i2\Omega^{-1}\exp\{-i(\varepsilon_\lambda + g\Omega)\tau\}. \quad (35)$$

In this case the energy of the ground state of the nucleus with two nucleons in the unfilled level ϵ_λ turns out to be

$$E_2 = 2\varepsilon_\lambda + g\Omega. \quad (36)$$

For the first odd nucleus the pairing will manifest itself only in the state $-\lambda_0$ which is coupled with the state λ_0 in which the odd particle is placed. Therefore $G_\lambda^{1-}(\tau)$ corresponds to the free motion of the particle

$$G_\lambda^{1-}(\tau) = i\delta_{\lambda\lambda_0}\exp\{-i\varepsilon_\lambda\tau\}.$$

At the same time $G_\lambda^{1+}(\tau)$ becomes unfree for $\lambda = -\lambda_0$:

$$G_\lambda^{1+}(\tau) = i(\delta_{\lambda\lambda_0} - 1 + \delta_{\lambda-\lambda_0}2\Omega^{-1})\exp\{-i\varepsilon_\lambda\tau\}$$
$$- i\delta_{\lambda_-\lambda_0}2\Omega^{-1}\exp\{-i(\varepsilon_\lambda + g\Omega)\tau\}. \quad (37)$$

In the case of a nucleus with three nucleons in the given level the wave function of the ground state can be written in the form

$$\Phi_3 = \overset{+}{a}_{\lambda_0}\Phi_2', \quad (38)$$

where λ_0 is the state in which there is an odd particle, and Φ_2' differs from the wave function Φ_2 of the second nucleus by a factor which is obtained from the normalization requirement for Φ_3:

$$\Phi_2' = \sqrt{\Omega/(\Omega - 2)}\,\Phi_2.$$

Using the wave function Φ_3, one can easily obtain expressions for the Green's functions of the third nucleus:

$$G_{\lambda_0}^{3-}(\tau) = i\Big[\frac{2}{\Omega}\exp\{-i[\varepsilon_\lambda + g(\Omega - 2)]\tau\}$$
$$+\frac{\Omega - 2}{\Omega}\exp\{-i(\varepsilon_\lambda - 2g)\tau\}\Big], \qquad G_{-\lambda_0}^{3-}(\tau) \equiv 0,$$
$$G_\lambda^{3-}(\tau) = i\frac{2}{\Omega - 2}\exp\{-i[\varepsilon_\lambda + g(\Omega - 2)]\tau\}, \quad \lambda \neq \pm\lambda_0;$$
$$G_{-\lambda_0}^{3+}(\tau) = -i\Big[\frac{\Omega - 4}{\Omega}\exp\{-i(\varepsilon_\lambda - 2g)\tau\}$$
$$+\frac{4}{\Omega}\exp\{-i[\varepsilon_\lambda + g(\Omega - 2)]\tau\}\Big], \qquad G_{\lambda_0}^{3+}(\tau) \equiv 0,$$
$$G_\lambda^{3+}(\tau) = i\frac{4 - \Omega}{\Omega - 2}\exp\{-i(\varepsilon_\lambda - 2g)\tau\}, \quad \lambda \neq \pm\lambda_0. \quad (39)$$

By using the results for the second and third nuclei and the recurrence scheme which has been indicated, one can obtain the parameters of a system with an arbitrary number N of particles. In particular, the energy of the ground state of a system with an even number of particles is

$$E_N = N\varepsilon_\lambda + \tfrac{1}{2} Ng(\Omega - N + 2). \tag{40}$$

This expression agrees completely with the quantity obtained by Mottelson.[7] At the same time, the method of quasi-particles gives for this case an expression that agrees with Eq. (40) only in the limit $\Omega \gg 1$.

CONCLUSION

A direct experimental test of the results obtained here can be made by means of stripping or pickup reactions. In fact, a knowledge of the density matrix gives direct information about the nuclear matrix element that determines the cross section of the stripping reaction. For example, for the case of a nucleus with two nucleons above a closed shell

$$\rho_\lambda^{(2)} = \sum_s |\langle \Phi_2 a_\lambda^+ \Phi_{1s}\rangle|^2 = |\Delta_2 / 2(\varepsilon_\lambda - \mu_2)|^2,$$

where all the states s refer to a given level ϵ_λ. If the independent-particle model were valid, then, for example, a pickup reaction could occur only from states corresponding to the single level ϵ_{λ_0}.

In the presence of pairing the pickup reaction can also occur from other levels ϵ_λ. For constant Δ_2 the ratio of the squares of the corresponding matrix elements is given by

$$|(\varepsilon_{\lambda_0} - \mu_2)/(\varepsilon_{\lambda_1} - \mu_2)|^2.$$

Since $\epsilon_{\lambda_1} - \epsilon_{\lambda_0}$ is a quantity of the order of $\epsilon_{\lambda_0} - \mu_2$, this ratio is of the order of unity. Therefore in principle the pickup reaction can occur for all states inside the shell that is being filled up. Incidentally, such reactions can be used to test the hypothesis of the constancy of Δ. The considerations that have been given are of course valid for all spherical even and odd nuclei. If, however, the number of nucleons above the closed shell is large, other effects become important, which can mask the pair-correlation effects.

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