ASYMPTOTIC BEHAVIOR OF THE SCATTERING AMPLITUDE AT INFINITE ENERGIES

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The behavior of the scattering amplitude at infinite energies is investigated on the basis of the unitarity condition. A relation between the asymptotic form of the forward scattering amplitude and the derivatives of the differential cross sections for elastic scattering with respect to the angle is established.

In the present paper we investigate the asymptotic behavior of the scattering amplitude at infinite energies on the basis of the unitarity condition for all possible channels of a given reaction and the analytic properties of the scattering amplitude with respect to the momentum transfer (reference 1).* We shall investigate the general case and not just special models of interaction, as is usually done.[2,3] We shall in fact use only the most general assumptions of the quantum theory, and, in particular, the "optical" theorem,[4] which is a consequence of the unitarity condition.

The investigation of the asymptotic behavior of the scattering amplitude for $E \to \infty$, based on the most general axioms and assumptions of the quantum theory, is at present of great importance, since the application of the usual dispersion relations (between the real and imaginary parts of the scattering amplitude)[4] and Mandelstam's spectral representations[5] depends directly on the detailed asymptotic behavior of the scattering amplitude.

It is true, as has been shown earlier, that a formulation of the dispersion relations (between the modulus and the phase of the scattering amplitude) and the spectral representations is possible which does not depend directly on the exact asymptote of the scattering amplitude for $E \to \infty$; this formulation depends on information on the possible complex zeroes of the scattering amplitude,† which can be obtained from certain additional reasonable physical assumptions.[6,7]

*I have been informed that a similar problem has been considered independently by M. Braun and L. Prokhorov.

†There exists a definite connection between the position of the possible zeroes and the asymptotic behavior (for $E \to \infty$) of the scattering amplitude, which will be considered in a separate paper on the basis of the theory of entire functions;[9] the latter have already been used in the quantum theory of the decay of physical systems.[7,8]

Nevertheless, the study of the asymptotic behavior of the scattering amplitude for $E \to \infty$ is also of interest because it allows us to make predictions with regard to experimentally measurable quantities [for example, the total cross section $\sigma(E)$] in the region of high and extremely high energies. Below we shall investigate the asymptotic behavior of the forward scattering amplitude $f(E)$. The asymptotic behavior of the scattering amplitude for an arbitrary angle $f(E, \theta)$ can be investigated in an analogous manner (see also footnote on page 346).

1. Let $f(E)$ be the forward scattering amplitude. The analytic properties of $f(E)$ in the complex half-plane $\mathrm{Im}\, E > 0$ have been determined from the most general principles of the quantum theory [locality, relativistic covariance (causality)] and the spectral condition.[4] For the asymptotic behavior of $f(E)$ these general principles only give the result that the forward scattering amplitude $f(E)$ has no essential singularities for $E \to \infty$.[4] The detailed determination of the asymptotic form of $f(E)$ depends on assumptions with regard to the specific character of the interaction (see, for example, references 2 and 3). The asymptotic form of $f(E)$ obtained in this way, therefore, has only significance in terms of a model, especially in that region of the energy where we have no compelling reason to believe in the presently known models of interaction. Using the empirical fact that the total cross section approaches a constant value for $E \to \infty$, one usually assumes that the elastic forward scattering amplitude $f(E)$ has a simple pole at infinity.[4]

2. We shall make decisive use of the "optical" theorem,[4] which is a direct consequence of the unitarity condition for the forward scattering amplitude:

$$\mathrm{Im}\, f(E) = (k/4\pi)\, \sigma(E) = (k/4\pi)\, \sigma_{\mathrm{el}}(E) + (k/4\pi)\, \sigma_{\mathrm{inel}}(E), \quad (1)$$

where $k^2 \equiv E^2 - \mu^2$, μ is the rest mass of the scattering particles, $\sigma(E)$ is the total cross section, $\sigma_{el}(E)$ is the total elastic cross section, and $\sigma_{inel}(E)$ is the total cross section for the inelastic processes which include all possible inelastic channels of the reaction and, in particular, the many-particle channels. We also note the inequalities

$$\operatorname{Im} f(E) \geqslant 0, \qquad \sigma(E) \geqslant 0, \tag{2}$$

which follow from obvious physical assumptions.

To demonstrate the basic idea of our calculation, we consider first the special case, where the elastic scattering involves only s waves.* In this case

$$\sigma_{el}(E) = 4\pi |f(E)|^2 \tag{3}$$

and the unitarity condition (1) takes the form

$$\operatorname{Im} f(E) = (k/4\pi)\sigma_{inel}(E) + k|f(E)|^2. \tag{4}$$

We now use the elementary inequality

$$|f(E)|^2 - [\operatorname{Im} f(E)]^2 = [\operatorname{Re} f(E)]^2 \geqslant 0 \tag{5}$$

We substitute the expression (4) for Im f(E) in (5) and find

$$|f(E)|^2 [1 - k^2 |f(E)|^2 - (k^2/2\pi)\sigma_{inel}(E)] - (k^2/16\pi^2)\sigma^2_{inel}(E) \geqslant 0 \tag{6}$$

Hence

$$|f(E)|^2 [1 - k^2 |f(E)|^2 - (k^2/2\pi)\sigma_{inel}(E)] \geqslant 0. \tag{7}$$

Then

$$1 - k^2 |f(E)^2| - (k^2/2\pi)\sigma_{inel}(E) \geqslant 0, \tag{8}$$

since $|f(E)|$ cannot vanish in a continuous half-infinite interval because of its analyticity.

For sufficiently large $E \gg \mu$ we obtain from (8) with the help of (2)

$$|f(E)| \leqslant E^{-1}, \tag{9}$$

i.e., if only s waves participate in the elastic scattering, the elastic scattering amplitude cannot decrease faster than $|E|^{-1}$. We note that Eq. (8) also leads to the following restriction on the asymptotic form of σ_{inel}:

$$\sigma_{inel}(E) \leqslant 2\pi / |E|^2. \tag{10}$$

The basic idea of the above-mentioned method of obtaining asymptotic restrictions of the type (9) is to make use of the fact that Im f(E) in the basic inequality (5) is itself determined by $|f(E)|$ through the unitarity condition. This leads us to the important conclusion that the contribution of the inelastic processes to the unitarity condition is, in general, not essential for the derivation of restrictions on the asymptotic behavior of f(E).

*This special case of the scattering of scalar mesons was considered in the work of Castillejo, Dalitz, and Dyson[10] and also by Ter-Martirosyan.[11]

3. Let us now turn to the general case. The total elastic scattering cross section $\sigma_{el}(E)$ is, evidently, given by the expression

$$\sigma_{el}(E) = 2\pi \int_0^\pi \sin\theta |f(E, \theta)|^2 d\theta, \tag{11}$$

where $f(E, \theta)$ is the amplitude for elastic scattering into the angle θ. At the first glance it seems impossible to carry over the method described in Sec. 2 to the general case, since Im f(E) is given by an integral of $|f(E, \theta)|^2$, which is insensitive to a change of the integrand at the isolated point $\theta = 0$, especially since the integrand contains $\sin\theta$. However, if we assume certain analytic properties of $f(E, \theta)$ with respect to θ, we can evidently hope that the above-mentioned method for studying the asymptotic behavior of f(E) on the basis of the unitarity condition will remain applicable.

As a preliminary, we note that the basic inequality (5) will hold a fortiori if we discard the positive definite terms in Im f(E). Using (2) and (11), we then obtain, instead of (5), the following basic inequality:*

$$|f(E)|^2 - \frac{k^2}{4}\left[\int_0^{\theta_1} \sin\theta |f(E, \theta)|^2 d\theta\right]^2 \geqslant 0, \qquad \theta_1 < \pi, \tag{12}$$

which is completely independent of the contribution from the inelastic processes to the unitarity condition.

Let us now make the basic assumption that $f(E, \theta)$ is analytic in θ in the neighborhood of $\theta = 0$. This assumption is equivalent to the fact, proved by Lehmann[1] (we admit that Lehmann's proof is valid only under severe restrictions on the masses of the scattering particles), that the scattering amplitude is analytic in the momentum transfer in a region which includes the physical scattering region.

We expand $|f(E, \theta)|^2$ into a series about $\theta = 0$,

$$|f(E, \theta)|^2 = |f(E)|^2 + \sum_{k=1}^{\infty} \frac{\theta^k}{k!} \frac{\partial^k |f(E, 0)|^2}{\partial \theta^k} \tag{13}$$

*According to the mean value theorem, we can use this inequality to carry over some of the results regarding the asymptotic behavior of the forward scattering amplitude f(E) to the asymptotic behavior of the scattering amplitude for an arbitrary angle $f(E, \theta)$.

and substitute (13) in (12):

$$|f(E)|^2 - \frac{k^2}{4}\left[A(\theta_1)|f(E)|^2 + \sum_{k=1}^{\infty}\frac{B_k(\theta_1)}{k!}\frac{\partial^k|f(E,0)|^2}{\partial\theta^k}\right] \geqslant 0, \tag{14}$$

where $A(\theta_1)$ and $B_k(\theta_1)$ are positive functions of θ_1. A stronger inequality is

$$|f(E)|^2 - \frac{k^2}{4}\Big[A^2(\theta_1)|f(E)|^4 + 2A(\theta_1)|f(E)|^2\sum_{k=1}^{\infty}\frac{B_k(\theta_1)}{k!} \times \frac{\partial^k|f(E,0)|^2}{\partial\theta^k} + 2\sum_{k\neq j}\frac{B_kB_j}{k!j!}\frac{\partial^k|f(E,0)|^2}{\partial\theta^k}\frac{\partial^j|f(E,0)|^2}{\partial\theta^j}\Big] \geqslant 0. \tag{15}$$

It is clear that for every finite E we can choose such a small $\theta_1(E) > 0$ that we can write down the following inequality instead of (15):

$$|f(E)|^2 - \frac{k^2}{4}\Big[A^2(\theta_1)|f(E)|^4 + 2A(\theta_1)B(\theta_1)|f(E)|^2\frac{\partial|f(E,0)|^2}{\partial\theta}\Big] \geqslant 0. \tag{16}$$

The inequalities (15) and (16) are completely rigorous for finite E, for the results of Lehmann[1] imply the analyticity of the scattering amplitude in the momentum transfer for finite energies $E < \infty$. However, the possible asymptotic behavior of $f(E, \theta)$ for $E \to \infty$ does not follow from the Lehmann theorem.[1] But if we make certain assumptions with respect to the behavior of $\partial^k|f(E, 0)|^2/\partial\theta^k$ for $E \to \infty$, we can use (15) to find the required asymptotic behavior of $f(E, 0)$.

Let us give the result in the form of a theorem: If the derivatives of the differential elastic cross section with respect to the angle at $\theta = 0$ are finite for $E \to \infty$ as functions of the energy, the elastic forward scattering amplitude $f(E)$ cannot increase for $E \to \infty$; moreover,

$$|f(E)| < A/|E| + B, \qquad A, B > 0. \tag{17}$$

The proof follows from the basic inequality (15) by going to the limit.

Other restrictions on the possible asymptotic behavior of $f(E)$ can be obtained in an analogous way from the inequality (15), by making different assumptions with regard to the behavior of $\partial^k|f(E, 0)|^2/\partial\theta^k$ as $E \to \infty$. In fact, the basic inequality (15) allows us, by explicit use of the unitarity condition, to replace the assumption with regard to the asymptotic behavior of $f(E)$ by an equivalent assumption with regard to the behavior of the derivatives $\partial^k|f(E, 0)|^2/\partial\theta^k$, which have a direct physical meaning.

It clearly follows from the above-mentioned theorem that the usual assumption that $f(E)$ has a simple pole for $E \to \infty$ can only be correct if the derivatives of the differential elastic cross section with respect to the angle have a discontinuity at $\theta = 0$ for $E \to \infty$. This gives us a new experimental possibility to verify the hypothesis with regard to the asymptotic behavior of the forward scattering amplitude $f(E)$ at high and extremely high energies.

We also note that, together with the above-mentioned estimate of the asymptotic form of $f(E)$, we can also derive a lower limit for the asymptotic form of $f(E)$, by using "the criterion of physical realizability,"[12] which is also based on the most general assumptions of the quantum theory:

$$|f(E)| \geqslant A\exp\{-\gamma|E|^q\}, \qquad A > 0, \quad \gamma > 0, \quad q < 1. \tag{18}$$

Thus we see that very general restrictions on the possible behavior of the scattering amplitude at infinite energies can be derived from the most general assumptions of the quantum theory, including the unitarity condition. To determine the detailed behavior of the asymptotic scattering amplitude at infinite energies, we must, of course, supplement the general assumptions of the quantum theory (necessary for the derivation of the analytic properties and the spectral representations of the scattering amplitude) with specific assumptions about the interactions in the chosen model. In this connection the following problem is of interest: what are the minimal characteristics of the interaction model necessary for the determination of the detailed behavior of the scattering amplitude at infinite energies?

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[1] H. Lehmann, Nuovo cimento **16**, 579 (1958).

[2] K. Symanzik, Nuovo cimento **5**, 659 (1957).

[3] R. Arnowitt and G. Feldman, Phys. Rev. **108**, 144 (1957).

[4] N. N. Bogolyubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields, Interscience, New York (1959). Bogolyubov, Medvedev, and Polivanov, Вопросы теории дисперсионных соотношений (Problems of the Theory of Dispersion Relations) Gostekhizdat (1958).

[5] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

[6] L. A. Khalfin, JETP **36**, 1088 (1959), Soviet Phys. JETP **9**, 774 (1959).

[7] L. A. Khalfin, Dissertation, Institute of Physics, Academy of Sciences (1960).

[8] L. A. Khalfin, Doklady Akad. Nauk SSSR **132**, 1051 (1960), Soviet Phys.-Doklady **5**, 515 (1960).

[9] B. Ya. Levin, Распределение корней целых функций (Distribution of the Zeroes of Entire Functions), Gostekhizdat (1956).

[10] Castillejo, Dalitz, and Dyson, Phys. Rev. **101**, 453 (1956).

[11] K. A. Ter-Martirosyan, JETP **37**, 1005 (1959), Soviet Phys. JETP **10**, 714 (1960).

[12] L. A. Khalfin, Doklady Akad. Nauk SSSR **111**, 345 (1956), Soviet Phys.-Doklady **1**, 671 (1957).

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