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ON THE THREE-BODY PROBLEM WITH SHORT-RANGE FORCES

G. S. DANILOV

Leningrad Physico-Technical Institute, Academy of Sciences, U.S.S.R.

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It is shown that the scattering amplitude for the scattering of a particle on the bound state of two other particles can be expressed in terms of the parameters of the two-particle problem and the energy of the bound state of the three particles.

SKORNYAKOV and Ter-Martirosyan[1] have derived equations for the determination of the wave function of a system of three identical particles in the limiting case of zero-range forces. These equations allow one to express the scattering amplitude for the scattering of a neutron on a deuteron with the total spin $S = 3/2$ for the system in terms of the parameters of the two-particle problem. Analogous calculations for the case $S = 1/2$, however, have not been successful.[2]

For the total spin $S = 1/2$ the wave function of the system does not vanish when the distances between all particles become zero, as in the case when $S = 3/2$. It will be shown in the present paper that the above-mentioned equations have a non-unique solution in this case.* We shall also see that the wave functions for different energies are proportional to one another in the region where the three particles are sufficiently close to each other. We must therefore choose that solution of the equations of Skornyakov and Ter-Martirosyan which guarantees that the wave functions for different energies are proportional to each other in the above-mentioned region. This requirement allows us to choose a unique solution for an arbitrary energy if the wave function for a single value of the energy is known. For this wave function one can, for example, take the wave function of tritium. The determination of this wave function requires the knowledge of the binding energy of tritium. To solve the three-body problem with short-range forces, we thus require one more parameter in addition to the parameters of the two-body problem, as for example, the binding energy of tritium.

The idea that the non-uniqueness of the solution of the equations of Skornyakov and Ter-Martirosyan can be removed by introducing an experimental parameter is due to V. N. Gribov.

*This question was raised by Skornyakov.[2] He demonstrated the non-uniqueness of the solution of equations of similar type with the help of various examples.

In Sec. 1 of the present paper we explain why the equation of Skornyakov and Ter-Martirosyan for three identical spinless particles has no unique solution. We also determine the behavior of the wave function in the region where the distances between all particles are close to zero. The rigorous proof of the non-uniqueness of the solution of the equation of Skornyakov and Ter-Martirosyan is given in the Appendix. In Sec. 2 we prove that the wave functions for different energies are proportional to one another in the region where the distances between all particles are close to zero. In Sec. 3 we derive for the wave function of three identical spinless particles an equation with a unique solution. In this form the equation of Skornyakov and Ter-Martirosyan can be solved numerically. In Sec. 4 we consider the same problem, but take into account the spin and the isotopic spin of the particles.

1. NON-UNIQUENESS OF THE SOLUTION OF THE EQUATION FOR A SYSTEM OF THREE IDENTICAL SPINLESS PARTICLES AND THE ASYMPTOTIC FORM OF THE WAVE FUNCTION

It was shown in reference 1 that it is sufficient for the solution of the scattering problem to know the wave function in the region $\rho_{23} \leq r_0$, $\rho_1 \gg r_0$, where ρ_{23} and ρ_1 are the Jacobi coordinates and r_0 is the range of the forces. In this region the wave function $\Psi_E(\rho_{23}, \rho_1)$ can be written in the form of a product of the deuteron wave function $\varphi_0(\rho_{23})$ and some function $U_E(\rho_1)$. In the problem of the scattering of a particle with relative momentum $\mathbf{k}_0$ on a bound state of two other particles, the Fourier component $\chi_E(\mathbf{k})$ of the function $U_E(\rho_1)$ is equal to

$$\chi_E(\mathbf{k}) = (2\pi)^3\delta(\mathbf{k} - \mathbf{k}_0) + 4\pi a(\mathbf{k}, \mathbf{k}_0)/(k^2 - k_0^2 - i\delta);$$

$$a(\mathbf{k}, \mathbf{k}_0) = \sum_l (2l + 1)\, a_l(k, k_0)\, P_l(\cos\theta). \tag{1}$$

The square of the momentum k_0^2, the energy of the system E, and the binding energy of the deuteron α^2 are connected through the relation (we assume $\hbar = M = 1$, where M is the mass of the nucleon)

$$E = {}^3/_4 k_0^2 - \alpha^2. \tag{2}$$

The functions $a_l(k, k_0)$ satisfy the equations (b) of the Appendix of reference 1.

We shall be interested in the equation for $a_0(k, k_0)$, since it is this equation which has a non-unique solution. The index 0 of the function a_0 will be omitted in the following.

The equation for $a(k, k_0)$ has the following form:

$$a(k,k_0) = \frac{8}{3}(\gamma_{k_0}(k) + \alpha) L_E(k, k_0) + \frac{16}{3\pi}(\gamma_{k_0}(k) + \alpha)\int_0^\infty \frac{a(k', k_0) k'^2}{k'^2 - k_0^2 - i\delta} L_E(k, k')\, dk', \tag{3}$$

where

$$L_E(k, k') = \frac{1}{2kk'} \ln \frac{k^2 + k'^2 + kk' - E}{k^2 + k'^2 - kk' - E},$$

$$\gamma_{k_0}(k) = \sqrt{{}^3/_4 k^2 - E}$$

[$\gamma_{k_0}(k) = -i\,|\gamma_{k_0}(k)|$ if $(3/4)k^2 - E < 0$; for the choice of the branch of the logarithm, see reference 1]. We note that in the problem of the bound state of the three particles

$$\chi_{E_0}(k) = 4\pi a_{E_0}(k)/(k^2 - k_0^2), \tag{4}$$

where k_0^2 is given by formula (2) and E_0 is the energy of the bound state. The function $a_{E_0}(k)$ satisfies Eq. (3) without the free term.

As will be shown in the Appendix, the homogeneous equation corresponding to Eq. (3) has a solution for arbitrary values of E. The solution to this equation is well-defined except for a normalization factor. Therefore, the solution of the inhomogeneous equation (3) (if it exists) contains an arbitrary constant. The existence of a solution to Eq. (3) will also be shown in the Appendix.

Let us determine the asymptotic form of $a(k, k_0)$ for large k. It is easily shown that, up to terms which decrease as k^{-1}, the function $a(k, k_0)$ has the form

$$a(k, k_0) \sim \sum A_i(k_0) k^{s_i}, \tag{5}$$

where the s_i are the roots of the equation

$$1 - 4L(s)/\sqrt{3} = 0, \tag{6}$$

with

$$L(s) = \frac{2\sin(\pi s/6)}{s\cos(\pi s/2)}. \tag{7}$$

The summation in (5) goes over all roots of Eq. (6) which lie in the strip $|\mathrm{Re}\, s| < 1$.

To prove formula (5), we must substitute it in Eq. (3) and use the fact that for large k the integral is essentially determined by the region of large k′.

It can be shown that Eq. (6) has two roots in the strip $|\mathrm{Re}\, s| < 1$:

$$s = \pm i s_0, \quad s_0 \approx 1.$$

Except for terms which decrease as k^{-1}, the asymptotic form of $a(k, k_0)$ is therefore of the following form:*

$$a(k, k_0) \sim A_1(k_0) k^{is_0} + A_2(k_0) k^{-is_0}. \tag{8}$$

We can also compute the next term in the asymptotic expression for $a(k, k_0)$. For this purpose we must separate the integral in Eq. (3) into two parts, one of which includes the region of large k′ and the other, the region of small k′. For large k the integral over the region of small k′ and the free term decrease as const$\cdot k^{-1}$. However, the asymptotic form of $a(k, k_0)$ cannot contain such terms, for their substitution in the integral in Eq. (3) would lead to the appearance of terms in the asymptotic expression which decrease as const$\cdot k^{-1}\ln k$ for large k.† Therefore, the terms proportional to k^{-1} coming from the free term of Eq. (3) and from the integral over the region of small k′, and similar terms coming from the integral over the region of large k′, must sum out to zero. Thus, except for terms which decrease faster than k^{-1}, only the integral over the region of large k′ gives a contribution to the asymptotic form of $a(k, k_0)$. It is easily shown that, with this accuracy, the asymptotic expression for $a(k, k_0)$ is

$$a(k, k_0) = A_1(k_0)\varphi(k, s_0) + A_2(k_0)\varphi(k, -s_0), \tag{9}$$

where

$$\varphi(k, s_0) = k^{is_0}[1 + B(s_0)/k], \tag{10}$$

$$B(s_0) = 2\alpha/\sqrt{3}\,[1 - 4L(is_0 - 1)/\sqrt{3}]. \tag{11}$$

In principle, one of the two arbitrary constants $A_1(k_0)$ and $A_2(k_0)$ is determined by the other when this solution is joined to the solution in the region of small k. Thus it is already clear from the form of the asymptotic expression for $a(k, k_0)$ that the solution of (3) depends on a single arbitrary parameter. The asymptotic expression for $\tilde{a}(k, k_0)$, the

*Since $a(k, k_0)$ does not vanish as $k \to \infty$, the derivation of the equation for $\chi_E(k)$ given in reference 1 is not very convincing. Nevertheless, these equations are correct.[3]

†The absence of terms of the form $k^{-1}\ln k$ in the asymptotic expression for $a(k, k_0)$ can be rigorously shown with the help of the formulas derived in the Appendix.

solution of the homogeneous equation corresponding to (3), is evidently of the same form.

The asymptotic form of $a(k, k_0)$ for large k determines the behavior of the function $U_E(\rho_1)$ for small ρ_1. Then we have for $\rho_{23} \sim r_0$ and $r_0 \ll \rho_1 \ll (|E|^{-1/2}, \alpha^{-1}, k_0^{-1})$

$$\Psi_E(\rho_{23}, \rho_1) \sim \varphi_0(\rho_{23})$$

$$\times\left[d_1 \frac{\sin\xi}{\rho_1}(1 + d_2\rho_1) + d_3\frac{\cos\xi}{\rho_1}(1 + d_4\rho_1)\right], \quad (12)$$

where $\xi = s_0 \ln(\rho_1\sqrt{|E|})$ and $\varphi_0(\rho_{23})$ is the deuteron wave function. The constants d_1, d_2, d_3, and d_4 are functions of A_1 and A_2.

In the following we shall require the leading term in the asymptotic expression for the wave function in the region

$$r_0 \ll \rho_{23} \ll (|E|^{-1/2}, \alpha^{-1}, k_0^{-1}),$$
$$r_0 \ll \rho_1 \ll (|E|^{-1/2}, \alpha^{-1}, k_0^{-1}).$$

To compute this term, we must substitute the main term of (12) in the integral of the equation which determines the wave function of the system in terms of the Green's function and use the fact that the principal contribution to the integral comes from the region where all three particles are close to one another. The calculation shows that in this region the wave function is proportional to the expression

$$\Psi_E(\rho_{23}, \rho_1) \sim [d_1(k_0)\sin\tilde{\xi} + d_3(k_0)\cos\tilde{\xi}]\, Q(\rho_{23}, \rho_1), \quad (13)$$

where

$$\tilde{\xi} = s_0 \ln(\sqrt{|E|}R), \qquad R = \tfrac{2}{3}(\rho_{12}^2 + \rho_{23}^2 + \rho_{13}^2) = \rho_{23}^2 + \tfrac{4}{3}\rho_1^2,$$

$$Q(\rho_{23}, \rho_1) = \frac{1}{\rho_1\rho_{23}} f\left(\frac{\rho_{23}}{\rho_1}\right) + \frac{1}{\rho_2\rho_{13}} f\left(\frac{\rho_{13}}{\rho_2}\right) + \frac{1}{\rho_3\rho_{12}} f\left(\frac{\rho_{12}}{\rho_3}\right),$$

where ρ_{12}, ρ_{23}, and ρ_{13} are the distances between the particles and ρ_1, ρ_2, and ρ_3 denote the distance of a given particle from the center of mass of the two others. The function $f(x)$ is equal to*

$$f(x) = \mathrm{sh}\frac{s_0}{2}[\pi - \vartheta(x)], \quad \vartheta(x) = \arccos\frac{4/3 - x^2}{4/3 + x^2}.$$

2. CHOICE OF SOLUTION

In order to select the correct solution from the set of solutions to Eq. (3), we may avail ourselves of the assertion, which will be proved below, that the wave functions for different energies are proportional to each other if ρ_1 and ρ_{23} are small. In particular, the wave functions of the continuous spectrum $\Psi_E(\rho_{23}, \rho_1)$ must be proportional to the wave function of tritium $\Psi_{E_0}(\rho_{23}, \rho_1)$:

*sh = sinh.

$$\Psi_E(\rho_{23}, \rho_1) = C(E)\,\Psi_{E_0}(\rho_{23}, \rho_1). \quad (14)$$

Condition (14) determines uniquely the arbitrary constant in the solution to Eq. (3). Indeed, it follows from (14) that for $k \to \infty$

$$a(k, k_0) = C(E)\, a_{E_0}(k). \quad (15)$$

For a given function $a_{E_0}(k)$, Eq. (15) can be satisfied only with a definite value of the arbitrary constant entering in the solution to Eq. (3). If the binding energy of tritium E_0 is known, we can find the function $a_{E_0}(k)$ from the solution of the homogeneous equation corresponding to Eq. (3) for $E = E_0$.

Let us prove Eq. (14). The wave functions $\Psi_E(\rho_{23}, \rho_1)$ and $\Psi_{E_0}(\rho_{23}, \rho_1)$ are orthogonal, i.e.,

$$\int \Psi_E(\rho_{23}, \rho_1)\,\Psi_{E_0}(\rho_{23}, \rho_1)\, d^3\rho_{23}\, d^3\rho_1 = 0. \quad (16)$$

This integral can be written in the form of a sum of an integral over the region $R < R_0$, where $R^2 = \rho_{23}^2 + (4/3)\rho_1^2$, and an integral over the remaining portion of space. We shall assume here that R_0 and the range of the forces r_0 both go to zero in such a way that $r_0 \ll R_0$. Then the integral over the region $R < R_0$ will go to zero. The other integral can be transformed into an integral over the surface $R = R_0$ with the help of the Schrödinger equation:

$$(E - E_0)\int \Psi_E(\rho_{23}, \rho_1)\,\Psi_{E_0}(\rho_{23}, \rho_1)\, d^3\rho_{23}\, d^3\rho_1$$
$$= R^5 \int d\Omega \Big[\Psi_{E_0}(\rho_{23}, \rho_1)\frac{\partial}{\partial R}\Psi_E(\rho_{23}, \rho_1)$$
$$- \Psi_E(\rho_{23}, \rho_1)\frac{\partial}{\partial R}\Psi_{E_0}(\rho_{23}, \rho_1)\Big]\Big|_{R\to 0}, \quad (17)$$

the integration in (17) goes over the angles in the six-dimensional space formed by the vectors $(\rho_{23}, \sqrt{4/3}\,\rho_1)$. Substituting the expression (13) for the wave functions in (17), we find that the right hand side of (17) does not depend on R for small R and is proportional to the expression

$$[d_1(k_0)d_3(E_0) - d_3(k_0)\,d_1(E_0)]\int d\Omega\, Q^2(\mathbf{n}_{23}, \mathbf{n}_1), \quad (18)$$

where $\mathbf{n}_{23}$ and $\mathbf{n}_1$ satisfy the relation $n_{23}^2 + (4/3)n_1^2 = 1$.

We see* that the right-hand side of (17) goes to zero for $R \to 0$ only if

$$d_1(k_0)\, d_3(E_1) - d_3(k_0)\, d_1(E_0) = 0, \quad (19)$$

*If one attempts to verify the orthogonality condition for the wave function with the help of formula (12) of reference 1, it may appear that all solutions of the equation of Skornyakov and Ter-Martirosyan are orthogonal to each other. However, this result is the consequence of an illegitimate interchange of the orders of integration. The correct procedure leads to a formula which agrees with formula (18) of our paper.

i.e., if the wave functions $\Psi_E(\rho_{23}, \rho_1)$ and $\Psi_{E_0}(\rho_{23}, \rho_1)$ are proportional to each other.

3. WAVE EQUATION FOR A SYSTEM OF THREE PARTICLES WITH A UNIQUE SOLUTION

We shall show now how one obtains an integral equation by which $a(k, k_0)$ is determined uniquely. Let us introduce the function

$$\omega(k, k_0) = a(k, k_0) - C(E)\, a_{E_0}(k), \tag{20}$$

which, according to our earlier discussion, has the property

$$k\omega(k, k_0) \to 0 \quad \text{for } k \to \infty. \tag{21}$$

The function $\omega(k, k_0)$ satisfies the equation

$$\frac{3}{8}\frac{\omega(k, k_0)}{\gamma_{k_0}(k)+\alpha} = f_E(k) + \frac{2}{\pi}\int_0^\infty \frac{L_E(k, k')\, k'^2}{k'^2 - k_0^2 - i\delta}\,\omega(k', k_0)\, dk', \tag{22}$$

where

$$f_E(k) = -\frac{3}{8}\frac{C(E)\, a_{E_0}(k)}{\gamma_k(k)+\alpha} + L_E(k, k_0) + \frac{2}{\pi}\int_0^\infty \frac{L_E(k, k')\, k'^2}{k'^2 - k_0^2 - i\delta}\, a_{E_0}(k')\, dk'. \tag{23}$$

It has been shown in Sec. 1 that Eq. (22) has an infinite number of solutions. We must find that solution which satisfies condition (21). It will be evident from the following discussion that the condition that Eq. (22) have such a solution determines the constant $C(E)$ uniquely.

It follows from (22) that for $k \to \infty$

$$\omega(k, k_0) \sim \frac{4}{\sqrt{3}k}\Big\{1 + \frac{2}{\pi}\int_0^\infty \frac{\omega(k', k_0)\, k'^2}{k'^2 - k_0^2 - i\delta}\, dk' + \frac{2}{\pi}C(E) \times \int_0^\infty a_{E_0}(k')\Big[\frac{k_0^2}{k'^2 - k_0^2 - i\delta} - \frac{4(E_0+\alpha^2)/3}{k'^2 - 4(E_0+\alpha^2)/3}\Big]dk'. \tag{24}$$

On the other hand, the asymptotic form of $\omega(k, k_0)$ does not contain terms which decrease as k^{-1}. The right-hand side of (24) must therefore be equal to zero. It follows from (22) and (24) that $\omega(k, k_0)$ satisfies the equation

$$\tfrac{3}{8}\,\omega(k, k_0)/(\gamma_{k_0}(k)+\alpha) = \varphi_E(k) + \frac{2}{\pi}\int_0^\infty \frac{\omega(k', k_0)\, k'^2}{k'^2 - k_0^2 - i\delta}[L_E(k, k') - L_E(k, k_0)]\, dk', \tag{25}$$

where

$$\varphi_E(k) = \beta(k_0)\, L_E(k, k_0) - \frac{3}{8}\frac{C(E)\, a_{E_0}(k)}{\gamma_{k_0}(k)+\alpha} + \frac{2}{\pi}\int_0^\infty \frac{a_{E_0}(k')\, k'^2}{k'^2 - k_0^2 - i\delta} L_E(k, k')\, dk';$$

$$\beta(k_0) = -\frac{2C(E)}{\pi}\int_0^\infty a_{E_0}(k)\Big[\frac{k_0^2}{k^2 - k_0^2 - i\delta} - \frac{4(E_0+\alpha^2)/3}{k^2 - 4(E_0+\alpha^2)/3}\Big]dk.$$

Equation (25) can be used for the determination of $\omega(k, k_0)$, since it does no longer contain any nondecreasing solutions.

Let us now determine the constant $C(E)$. For this purpose we multiply both sides of (25) by $\tilde{a}(k, k_0)\, k^2 (k^2 - k_0^2 - i\delta)^{-1}$, where $\tilde{a}(k, k_0)$ is the solution of the homogeneous equation corresponding to Eq. (3), and integrate over k from zero to infinity:

$$\frac{3}{8}\int_0^\infty \frac{\omega(k, k_0)\,\tilde{a}(k, k_0)\, k^2\, dk}{[\gamma_{k_0}(k)+\alpha](k^2 - k_0^2 - i\delta)} = \int_0^\infty \frac{f_E(k)\,\tilde{a}(k, k_0)\, k^2}{k^2 - k_0^2 - i\delta}\, dk + \frac{2}{\pi}\int_0^\infty \frac{\tilde{a}(k, k_0)\, k^2\, dk}{k^2 - k_0^2 - i\delta}\int_0^\infty \frac{\omega(k', k_0)\, L_E(k, k')\, k'^2\, dk'}{k'^2 - k_0^2 - i\delta}. \tag{26}$$

Since $\omega(k, k_0)$ vanishes for large k while $\tilde{a}(k, k_0)$ is bounded, we can change the order of integration in the second term on the right-hand side of (26), which evidently leads to

$$\int_0^\infty \frac{f_E(k)\,\tilde{a}(k, k_0)\, k^2}{k^2 - k_0^2 - i\delta}\, dk = 0. \tag{27}$$

From (22) and (27) we obtain the equation for the determination of $C(E)$:

$$C(E)\int_0^\infty \Big[\frac{3}{8}\frac{a_{E_0}(k)}{\gamma_{k_0}(k)+\alpha} - \frac{2}{\pi}\int_0^\infty \frac{L_E(k, k')\, k'^2}{k'^2 - k_0^2 - i\delta}\, a_{E_0}(k')\, dk'\Big] \times \frac{\tilde{a}(k, k_0)\, k^2}{k^2 - k_0^2 - i\delta}\, dk = \frac{3\pi}{16}\frac{\tilde{a}(k_0, k_0)}{2\alpha}. \tag{28}$$

From formula (28) we can obtain a relation between $C(E)$ and the coefficients $\tilde{A}_1(k_0)$, $\tilde{A}_2(k_0)$, $\tilde{A}_1(E_0)$, and $\tilde{A}_2(E_0)$ in the asymptotic form (9) for $\tilde{a}(k, k_0)$ and $a_{E_0}(k)$:

$$\frac{3\pi}{16}\frac{\tilde{a}(k_0, k_0)}{2\alpha} = 2C(E)\frac{d}{ds_0}\frac{\operatorname{sh}(\pi s_0/6)}{s_0 \operatorname{ch}(\pi s_0/2)}[A_1(E_0)\tilde{A}_2(k_0) - A_2(E_0)\tilde{A}_1(k_0)]. \tag{29*}$$

Formulas (25) and (28) therefore give the solution to our problem.

4. EQUATIONS INCLUDING SPIN AND ISOSPIN OF THE PARTICLES

The preceding discussion is immediately generalized to the case of particles with spin and isotopic spin. We shall give only the results of the investigation of the equations of Skornyakov and Ter-Martirosyan, since all calculations are trivial generalizations of the corresponding calculations for a system of spinless particles.

To determine the wave function in the case of

*ch = cosh; sh = sinh.

real nucleons, we must, according to reference 1, solve the following system of equations for the two quantities $a_1(k, k_0)$ and $a_2(k, k_0)$:

$$\frac{[\gamma_{k_0}(k)\,\delta_{ij} - R_{ij}]\,a_j(k, k_0)}{k^2 - k_0^2 - i\delta} = L_E(k, k_0)\,\eta_i + \frac{2}{\pi}\int_0^\infty \frac{L_E(k, k')}{k'^2 - k_0^2 - i\delta} M_{ij} a_j((k, k_0)\,dk', \quad (30)$$

where

$$\{R_{ij}\} = \begin{pmatrix} \alpha_t & 0 \\ 0 & \alpha_s \end{pmatrix}, \qquad k_0^2 = \tfrac{4}{3}(E + \alpha_t^2),$$

where E is the energy of the system, α_t^2 is the binding energy of the deuteron, and α_s^2 is the energy of the virtual singlet state of the deuteron;

$$\{M_{ij}\} = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}, \qquad \{\eta_i\} = \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix}.$$

Summation over repeated indices is understood in formula (30) and all following formulas.

The wave function of the system in the bound state of the three particles is determined by the quantities $b_1(k)$ and $b_2(k)$, which satisfy the equations (31) without the free term for $E = E_0$, where E_0 is the energy of the bound state of the three particles.

Just as in the case of spinless particles, we can show that the equations (30) have a non-unique solution. For large k the asymptotic forms of $a_j(k, k_0)$ and $b_j(k, k_0)$ and the asymptotic form of the solution $\tilde{a}_j(k, k_0)$ of the homogeneous equations corresponding to the equations (30) are determined by a formula which is analogous to (9) with the same s_0, and we have, up to terms which vanish as $k \to \infty$,

$$a_1(k, k_0) = a_2(k, k_0),$$

$$b_1(k, k_0) = b_2(k, k_0), \qquad \tilde{a}_1(k, k_0) = \tilde{a}_2(k, k_0). \quad (31)$$

As before, it is necessary for the orthogonality of the wave functions for different energies that $a_j(k, k_0)$ satisfy for large k the condition

$$a_j(k, k_0) = C(E)\,b_j(k, k_0). \quad (32)$$

By the method discussed in Sec. 3 we can transform the system (30) to a system which has a unique solution. For this purpose let us introduce the function

$$\omega_j(k, k_0) = a_j(k, k_0) - C(E)\,b_j(k), \quad (33)$$

which has the property that

$$k\omega_j(k, k_0) \to 0 \quad \text{for } k \to \infty. \quad (34)$$

The function $\omega_j(k, k_0)$ satisfies the equation

$$\frac{[\gamma_{k_0}(k)\,\delta_{ij} - R_{ij}]\,a_j(k, k_0)}{k^2 - k_0^2 - i\delta} = \varphi_E^{(i)}(k) + \frac{2}{\pi}\int_0^\infty \frac{k'^2}{k'^2 - k_0^2 - i\delta}\,[L_E(k, k') - L_E(k, k_0)]\,M_{ij}\omega_j(k', k_0)\,dk', \quad (35)$$

where

$$\varphi_E^{(i)}(k) = \beta_i(k_0)\,L_E(k, k_0) - \frac{[\gamma_{k_0}(k)\,\delta_{ij} - R_{ij}]\,b_j(k)}{k^2 - k_0^2 - i\delta} + \frac{2}{\pi} C(E)\int_0^\infty \frac{L_E(k, k')\,k'^2}{k'^2 - k_0^2 - i\delta} M_{ij} b_j(k')\,dk',$$

$$\beta_i(k_0) = -\frac{2}{\pi} C(E) \times \int_0^\infty \left[\frac{k_0^2}{k^2 - k_0^2 - i\delta} - \frac{{}^4/_3(E_0 + \alpha_t^2)}{k^2 - {}^4/_3(E_0 + \alpha_t^2)}\right] M_{ij} b_j(k)\,dk. \quad (36)$$

The constant C(E) is determined by the equation

$$\frac{\tilde{a}_1(k, k_0)}{2\alpha_t} = C(E)\int_0^\infty \frac{\tilde{a}_j(k, k_0)\,k^2}{k^2 - k_0^2 - i\delta}\left[\frac{(\gamma_{k_0}(k)\,\delta_{ji} - R_{ji})\,b_i(k)}{k^2 - k_0^2 - i\delta} - \frac{2}{\pi}\int_0^\infty \frac{L_E(k, k')\,k'^2}{k'^2 - k_0^2 - i\delta} M_{ji} b_i(k')\,dk'\right] dk. \quad (37)$$

I express my deep gratitude to V. N. Gribov for suggesting this topic and his constant attention, and also to A. A. Ansel'm and Yu. V. Petrov for their interest in this work.

APPENDIX

In order to demonstrate the non-uniqueness of the solution to Eq. (3), we rewrite it in the following form:

$$a(k, k_0) = \Phi(k) + \int_0^\lambda K(k, k')\,a(k', k_0)\,dk' + \int_\lambda^\infty K(k, k')\,a(k', k_0)\,dk', \quad (A.1)$$

where $K(k, k')$ and $\Phi(k')$ are the kernel and the free term of (3), and λ is an arbitrary parameter. We shall assume in the following that

$$\lambda \gg (k_0, \sqrt{|E|}, \alpha).$$

To prove the non-uniqueness of the solution to (A.1), it is sufficient to show that the solution to (A.1) for $k > \lambda$ can be expressed in terms of the value of $a(k, k_0)$ for $k < \lambda$:

$$a(k, k_0) = \int_0^\lambda K_1(k, k')\,a(k', k_0)\,dk' + \Phi_1(k) + C\varphi(k, k_0), \quad (A.2)$$

where the kernel $K_1(k, k')$ and the function $\Phi_1(k)$ can be computed in principle, C is an arbitrary constant, and $\varphi(k, k_0)$ is some solution of the homogeneous equation:

$$\varphi(k, k_0) = \int_\lambda^\infty K(k, k')\,\varphi(k', k_0)\,dk'. \tag{A.3}$$

Indeed, substituting (A.2) in the second integral on the right-hand side of (A.1), we obtain an integral equation for $a(k, k_0)$ for $k < \lambda$. The limits of integration in this equation are finite, and the equation is therefore known to have a solution. We conclude that there exists a solution to (A.1) which depends on a single arbitrary parameter, if Eq. (A.2) is satisfied.

Let us prove formula (A.2). We rewrite (A.1) in the following form:

$$a(k, k_0) = \Phi_2(k) + \int_\lambda^\infty \Delta(k, k')\,a(k', k_0)\,dk' + \int_\lambda^\infty K_0(k, k')\,a(k', k_0)\,dk', \tag{A.4}$$

where $\Phi_2(k)$ is the sum of the two first terms on the right-hand side of (A.1),

$$\Delta(k, k') = K(k, k') - K_0(k, k'), \tag{A.5}$$

$$K_0(k, k') = \frac{4}{\pi\sqrt{3}}\frac{1}{k'}\ln\frac{k^2 + k'^2 + kk'}{k^2 + k'^2 - kk'}. \tag{A.6}$$

For large k and k′ the kernel $\Delta(k, k')$ goes to zero faster than $K(k, k')$.

For convenience, let us replace the variables k and k′ by $y = \lambda^{-1}k$ and $y' = \lambda^{-1}k'$. Then (A.4) takes the form

$$a(y) = \Phi_\lambda(y) + \int_1^\infty \Delta_\lambda(y, y')\,a(y')\,dy' + \frac{4}{\pi\sqrt{3}}\int_1^\infty \frac{1}{y'}\ln\frac{y^2 + y'^2 + yy'}{y^2 + y'^2 - yy'}\,a(y')\,dy'. \tag{A.7}$$

The functions $\Phi_\lambda(y)$ and $\Delta_\lambda(y, y')$ go to zero as $\lambda \to \infty$.

In solving (A.7) we shall regard the first two terms of this equation as being responsible for the non-uniqueness. For the solution we make use of the Mellin transform. Let us introduce the function

$$\tilde{a}(s) = \int_0^\infty y^{s-1} a_1(y)\,dy, \tag{A.8}$$

where $a_1(y) = 0$ for $y < 1$ and $a_1(y) = a(y)$ for $y > 1$. Then

$$a_1(y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} y^{-s}\tilde{a}(s)\,ds. \tag{A.9}$$

The contour in formula (A.9) is supposed to pass possible poles of the function $\tilde{a}(s)$ on the left.

To obtain an equation for $\tilde{a}(s)$, we multiply both sides of (A.7) by y^{s-1} and integrate from one to infinity. Then

$$\tilde{a}(s) = \tilde{\Phi}_\lambda(s) + \frac{1}{2\pi i}\frac{4}{\sqrt{3}}\int_{-i\infty}^{i\infty} \frac{L(s')\,\tilde{a}(s')}{s' - s}\,ds', \tag{A.10}$$

where $L(s)$ is determined by formula (7) and $\Phi_\lambda(s)$ is the sum of the first two terms on the right-hand side of (A.7) integrated in the indicated manner. Equation (A.10) is valid for $\operatorname{Re} s \to -0$. The function $\tilde{a}(s)$ must go to zero as $|\operatorname{Im} s| \to \infty$. Otherwise the integral (A.9) has no meaning.

The solution to (A.10) can be readily obtained by known methods. It turns out that the homogeneous equation corresponding to (A.10) has a decreasing solution. It can be conveniently written in the form

$$\tilde{a}_0(s) = \frac{s - d}{s^2 + s_0^2}\exp\left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\ln\varkappa(s')}{s - s'}\,ds'\right),$$
$$\varkappa(s) = \frac{s^2 - d^2}{s^2 + s_0^2}\left[1 - \frac{4}{\sqrt{3}}L(s)\right], \tag{A.11}$$

where the quantity s_0 is defined in Sec. 1 and d is an arbitrary parameter satisfying the condition $|\operatorname{Re} d| > 1$. It can be shown that in this case the expression (A.11) is independent of d.

The function $\kappa(s)$ is analytic in the strip $|\operatorname{Re} s| < 1$. The branch of the logarithm in (A.11) is chosen in such a way that $\log\kappa(s) \to 0$ in this strip when $|\operatorname{Im} s| \to \infty$. The general solution of (A.10) is written in the form of a sum of some particular solution of the inhomogeneous equation and the function $\tilde{a}_0(s)$ multiplied by an arbitrary constant:

$$\tilde{a}(s) = C\tilde{a}_0(s) + \frac{\tilde{a}_0(s)}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\tilde{\Phi}_\lambda(s')\,ds'}{(s' - s)(s' - d)\,\tilde{a}_0(s')\,[1 - 4L(s')/\sqrt{3}]} \tag{A.12}$$

It is easily verified that a change in the parameter d leads to a redefinition of the constant C.

Let us apply the inverse Mellin transformation to (A.12). Instead of (A.7), we obtain then

$$a(y) = u(y) + \Phi'_\lambda(y) + \int_1^\infty \Delta'_\lambda(y, y')\,a(y')\,dy', \tag{A.13}$$

where $u(y)$ is the solution of (A.7) with $\Delta_\lambda(y, y') = 0$, and $\Phi'_\lambda(y)$ and $\Delta'_\lambda(y, y')$ are certain known functions with the property

$$\Phi'_\lambda(y) \to 0, \qquad \int_1^\infty |\Delta'_\lambda(y, y')|\,dy' \to 0 \tag{A.14}$$

for $\lambda \to \infty$. Since, moreover, the functions $u(y)$

and $\Phi'_{\lambda}(y)$ are bounded, Eq. (A.13) can, in principle, be solved by the method of iterative approximations. This leads to formula (A.2). Since $u(y)$ contains an arbitrary constant, the solution of Eq. (A.13) will, obviously, also contain an arbitrary constant.

The asymptotic form of $a(k\,k_0)$ can be obtained from Eq. (A.13). The resulting expression is in agreement with formula (9).

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