# *A MODEL OF LOCAL FIELD THEORY WITH FINITE CHARGE RENORMALIZATION*

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The local field theory proposed by Bialynicki-Birula is treated by a previously developed method.[3,4] The S matrix and renormalization constants are derived. It is proved that the charge renormalization is finite in all orders and does not contain logarithmic singularities. It is shown that the ultraviolet contribution results in a series with a finite sum, and that the Green's function series converges absolutely at low values of the time and has a branch point at $t = 0$, the singularity at zero being integrable.

## INTRODUCTION

In view of the still unsurmounted difficulties in the solution of the exact field equations in quantum field theory, the use of various models for an investigation of the intrinsic consistency of this theory has become quite popular. An investigation of the Lee model[1] has seemingly led to the conclusion that the theory is intrinsically inconsistent. It was shown later, however,[2] that the model was not self consistent because of the simplifications made to obtain an exactly solvable Hamiltonian. In particular, these simplifications violated the important requirement of crossing symmetry.

In the present article we investigate, by a method previously developed,[3,4] the modified Lee model proposed by Bialynicki-Birula,[5] in which the crossing symmetry condition is satisfied. The solutions are obtained in the form of series in the renormalized constant $\Delta m$ ($\Delta m$ is the physical parameter causing the mass difference between two fermion states in the model). The convergence of this series is proved for the ultraviolet region $E \gg \Delta m$. An important property of the model is the finite renormalization, in the case of a point interaction, of the charge in all orders in $\Delta m$, in contrast with the Lee model, where the well known zero-charge problem exists.

The Green's function of the model considered has all the properties of the Green's function of the renormalized theory.[6,7] Specifically, the renormalized Green's function is analytic in the $t$ plane and has a branch point at $t = 0$; when $g^2/\pi^2 < 1$ there exists a Fourier transform of the Green's function, which admits of an expansion about the point $g^2 = 0$.

## 1. THE S MATRIX OF THE MODEL

Bialynicki-Birula[5] has considered a model of local field theory with fixed nucleon, in which the nucleon can be in two states of different mass (we shall call these, arbitrarily, proton and neutron states). The Hamiltonian of the system has the form

$$H = m_0(\psi^+\psi) + \frac{1}{2}\int dx : [\pi^2(x) + (\nabla\varphi(x))^2 + \mu^2\varphi^2(x)]: + g(\psi^+\tau_1\psi)\int dx\, \varphi(x)\,\delta(x) + \Delta m_0(\psi^+\tau_3\psi), \quad (1)$$

where $\psi = v_p c_p + v_n c_n$ is the nucleon field operator, $c_N$ $(N = p, n)$ is the nucleon annihilation operator, $v_N$ is a spinor describing the nucleon,

$$v_p = \begin{pmatrix}1\\0\end{pmatrix}, \qquad v_n = \begin{pmatrix}0\\1\end{pmatrix},$$

$\pi(x)$ and $\varphi(x)$ are the meson field operators, and $\tau_1$ and $\tau_3$ are the matrices of isotopic spin $\frac{1}{2}$.

Noting that when $\Delta m_0 = 0$ we obtain the exactly solvable case of scalar mesons with fixed source, it is possible to consider perturbations in the constant $\Delta m_0$ without limiting the interaction forces between the nucleon and the mesons. By this method, Bialynicki-Birula[5] obtained the interesting result that the renormalization of the charge is finite and contains no logarithmic singularities. From the point of view of the method which we have developed earlier,[3,4]* the Hamiltonian (1) is of interest because the Lappo-Danilevskii series coincides here with the perturbation-theory series in the constant $\Delta m_0$, but the new method, unlike per-

*We denote throughout reference 4 by I, and the corresponding formulas in this paper will be designated (I, 4.2) etc.

turbation theory, makes it possible to obtain the n-th term of the series.

Thus, let us consider the equation for the S matrix in the interaction representation. We seek directly the "adiabatic" $S^\alpha$ matrix, so as to use formulas (I, 4.2) and (I, 4.3). In the interaction representation we have

$$i\partial S^\alpha(t, t_0)/\partial t = H_I(t)e^{-\alpha|t|}S^\alpha(t, t_0), \quad S^\alpha(t, t_0)|_{t=t_0} = 1,$$
$$H_I(t) = g(\psi^+\tau_1\psi)\hat{\varphi}(t) + \Delta m_0(\psi^+\tau_3\psi),$$
$$\hat{\varphi}(t) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega}}(a_{\mathbf{k}}e^{-i\omega t} + a_{\mathbf{k}}^+ e^{i\omega t}). \quad (2)$$

Repeating in its entirety the procedure developed in I, (Secs. 1 – 3), we obtain the following expression for the $S^\alpha(t, t_0)$ matrix:

$$S^\alpha(t, t_0) = 1 - [2(\psi^+\psi) - (\psi^+\psi)^2]$$
$$+ \sum_{q=0}^{\infty} \frac{[-i(\psi^+\tau_3\psi)\Delta m_0]^q}{q!} \int_{t_0}^{t} d\xi_1 \ldots \int_{t_0}^{t} d\xi_q$$
$$\times : \exp\left\{-i(\psi^+\tau_1\psi)g\int_{t_0}^{t} ds \prod_{j=1}^{q} \varepsilon(\xi_j - s)\hat{\varphi}(s)e^{-\alpha|s|}\right\}:$$
$$\times \exp\left\{-\frac{ig^2}{2}\int_{t_0}^{t}\int_{t_0}^{t} ds_1 ds_2 e^{-\alpha(|s_1|+|s_2|)}\right.$$
$$\left.\times \prod_{j=1}^{q} \varepsilon(\xi_j - s_1)\Delta(s_1 - s_2)\varepsilon(\xi_j - s_2)\right\}$$
$$\times \exp\left\{-\alpha\sum_{j=1}^{q}|\xi_j|\right\}. \quad (3)$$

The resultant $S^\alpha$ matrix is defined accurate to a phase-shift factor, and satisfies the equations

$$S^\alpha(\infty, -\infty)|0\rangle = |0\rangle,$$
$$S^\alpha(\infty, -\infty)|N\rangle = e^{i\,\mathrm{const}/\alpha}|N\rangle, \quad (4)$$

where $|0\rangle$ and $|N\rangle$ denote the vacuum and single-nucleon states of the zero-order Hamiltonian. The usual conditions of stability of the vacuum and of the single-particle states require,[8] however, that

$$S(\infty, -\infty)|0\rangle = |0\rangle, \quad S(\infty, -\infty)|N\rangle = |N\rangle. \quad (5)$$

In all further calculations we shall use the adiabatic hypothesis, which enables us, in particular, to eliminate correctly the phase-shift factor, which is indeterminate when $\alpha = 0$ (reference 9), and consequently to satisfy condition (5) (see Appendix A).

We consider now the most important matrix elements of the S matrix. The eigenvalue of the energy of the single-fermion state is[9] (see Appendix B)

$$E_N = \lim_{\alpha\to 0} \frac{\langle N|HS^\alpha(0, -\infty)|N\rangle}{\langle N|S^\alpha(0, -\infty)|N\rangle}$$
$$= m + \delta_N\Delta m \sum_{q=0}^{\infty}(-\delta_N\Delta m)^q \int_0^\infty dx_1 \ldots \int_0^\infty dx_q x_1 \ldots x_q$$
$$\times \frac{\partial^q}{\partial x_1 \ldots \partial x_q}\exp\left\{2g^2\sum_{\mathbf{k}}\frac{1}{\omega^3}\sum_{l=1}^{q}\sum_{m=1}^{l}\right.$$
$$\left.\times(-1)^{l+m}\exp\left(-\omega\sum_{j=m}^{l}x_j\right)\right\},$$
$$\delta_N = \begin{cases} +1 \text{ for the proton } (N = p) \\ -1 \text{ for the neutron } (N = n). \end{cases} \quad (6)$$

Further

$$m = m_0 - \frac{1}{2}g^2\sum_{\mathbf{k}}\omega^{-2}, \quad (6')$$

$$\Delta m = \Delta m_0 \exp\left\{-g^2\sum_{\mathbf{k}}\omega^{-3}\right\}. \quad (6'')$$

The requirement that the observed m and $\Delta m$ be finite causes us to consider the bare quantities $m_0$ and $\Delta m_0$ as infinite. We note that the renormalization of $m_0$ coincides precisely with the renormalization for the case of scalar mesons in a field of a fixed source. After carrying out the renormalizations (6′) and (6″), each term of the series (6) is finite subject to the condition* $g^2/\pi^2 < 1$ (see Appendix B for details).

The renormalization constant of the fermion field $Z_2^N$ is determined, in accordance with its probabilistic sense, by the square of the matrix elements

$$Z_2^N = |\langle N|\mathbf{N}\rangle|^2 = |\langle N|S^\alpha(0, -\infty)|N\rangle|^2$$
$$= Z_2^{sc}\left[\sum_{q=0}^{\infty}(\delta_N\Delta m)^q\int_0^\infty dx_1\ldots\int_0^\infty dx_q x_1\ldots x_q\right.$$
$$\times\frac{\partial^q}{\partial x_1\ldots\partial x_q}\exp\left\{g^2\sum_{\mathbf{k}}\frac{1}{\omega^3}\left[-\sum_{l=1}^{q}(-1)^l\exp\left(-\omega\sum_{j=1}^{l}x_j\right)\right.\right.$$
$$\left.\left.\left.+2\sum_{l=2}^{q}\sum_{m=2}^{l}(-1)^{l+m}\exp\left(-\omega\sum_{j=m}^{l}x_j\right)\right]\right\}\right]^2. \quad (7)$$

Here

$$Z_2^{sc} = \exp\left\{-\frac{1}{2}g^2\sum_{\mathbf{k}}\omega^{-3}\right\}$$

is the value of $Z_2$ for scalar mesons, and $|\mathbf{N}\rangle$ denotes the single-nucleon state of the complete Hamiltonian.

Let us emphasize that the constant $Z_2^N$ in this model is equal to the product of the constant $Z_2^{sc}$ of the scalar neutral theory with fixed source and

*Bialynicki-Birula[5] gives an incorrect condition for the finiteness of the terms of the series $g^2/4\pi^2 < e^{-1}$.

a series of finite terms, so that when $g^2/\pi^2 < 1$ all the terms of the series are finite. Later on this circumstance will enable us to draw some conclusions regarding the constant $Z_1$.

The renormalized coupling constant is defined as usual:

$$\frac{g_r}{g} = \langle p \mid \psi^+\tau_1\psi \mid n\rangle = \lim_{\alpha\to 0} \frac{\langle p \mid S^\alpha(\infty, 0)\,\psi^+\tau_1\psi S^\alpha(0, -\infty) \mid n\rangle}{[\langle p \mid S^\alpha(\infty, -\infty) \mid p\rangle \langle n \mid S^\alpha(\infty, -\infty) \mid n\rangle]^{1/2}}$$
$$= 1 + \sum_{q=1}^{\infty} (\Delta m)^{2q} \int_0^\infty dx_1 \ldots \int_0^\infty dx_{2q-1} x_1 \ldots x_{2q-1} \sum_{j=1}^{q} x_{2j-1} \times \frac{\partial^{2q-1}}{\partial x_1 \ldots \partial x_{2q-1}}$$
$$\times \exp\left\{2g^2 \sum_{\mathbf{k}} \frac{1}{\omega^3} \sum_{l=1}^{2q-1} \sum_{m=1}^{l} (-1)^{l+m} \exp\left(-\omega \sum_{j=m}^{l} x_j\right)\right\}. \quad (8)$$

The situation in this model differs appreciably from that obtaining in the charge theory [see (I, 4.10) ff.], for at the point $g^2 = 0$ all the integrals are bounded, in contrast with expression (I, 4.12). Therefore, when perturbation theory is used here, i.e., when the solution is represented as a series in $g^2$, no logarithmic divergences in the maximum cut-off momentum takes place, such as occur in the local field theory. In this connection the given model does not reflect, in our opinion, some of the fundamental difficulties inherent in the exact equations of mesodynamics.

The renormalization constant of the vertex part $Z_1$ can be obtained from the well known equation

$$g_r = Z_1^{-1} (Z_2^p Z_2^n)^{1/2} Z_3^{1/2} g.$$

In our case the renormalization constant of the meson field $Z_3$ is equal to unity. Using (7) and (8), we can conclude that the constant $Z_1$ has a structure

$$Z_1 = Z_2^{\mathrm{sc}} \sigma(g^2, \Delta m), \quad (9)$$

where $\sigma(g^2, \Delta m)$ is a series in $\Delta m$, all the terms of which are finite when $g^2/\pi^2 < 1$.

Let us write down the matrix element for the elastic scattering of a meson by a nucleon (see Appendix A):

$$S_{f\leftarrow i} = \lim_{\alpha\to 0} \frac{\langle N \mid a_{\mathbf{p}_f} S^\alpha(\infty, -\infty)\, a^+_{\mathbf{p}_i} \mid N\rangle}{\langle N \mid S^\alpha(\infty, -\infty) \mid N\rangle} = \delta(\mathbf{p}_i - \mathbf{p}_f) - 2\pi i \delta(\omega_i - \omega_f) M_{f\leftarrow i}(\omega_f);$$

$$M_{f\leftarrow i}(\omega_f) = -\frac{2\delta_N g^2}{\omega_f^2} \frac{\Delta m}{\omega_f} \sum_{q=0}^{\infty} (-i\delta_N \Delta m)^q \int_0^\infty dx_1 \ldots \int_0^\infty dx_q \left[q + 1 - \sum_{l=1}^{q} \sum_{m=1}^{l} (-1)^{l+m} \left\{\exp\left(i\omega_f \sum_{j=m}^{l} x_j\right)\right.\right.$$
$$\left.\left. + \exp\left(-i\omega_f \sum_{j=m}^{l} x_j\right)\right\}\right] (-1)^q \int_{x_1}^\infty dy_1 \ldots \int_{x_q}^\infty dy_q \frac{\partial^q}{\partial y_1 \ldots \partial y_q} \exp\left\{2g^2 \sum_{\mathbf{k}} \frac{1}{\omega^3} \sum_{l=1}^{q} \sum_{m=1}^{l} (-1)^{l+m} \exp\left(-i\omega \sum_{j=m}^{l} y_j\right)\right\}. \quad (10)$$

## 2. ON THE CONVERGENCE OF THE SERIES FOR $E_p$, $Z_2^n$, AND $g_r$

The proof of the convergence of the series (6) – (10) is exceedingly complicated, since an estimate of the n-th term calls for extremely fine approximation methods, without which one cannot judge the behavior of the series as a whole. However, the series for $E_p$, $Z_2^n$, and $g_r$ can be summed in the generalized sense[10] (we shall refer to this as the $\epsilon$-sense), the gist of which we shall explain by using the series for $E_p$ as an example.

Let us consider the series

$$\delta m = \sum_{q=0}^{\infty} (-\Delta m)^q \int_0^\infty dx_1 \ldots \times \int_0^\infty dx_q x_1 \ldots x_q \frac{\partial^q}{\partial x_1 \ldots \partial x_q} F_q(x_1, \ldots, x_q),$$

$$F_q(x_1, \ldots, x_q) = \exp\left\{2g^2 \sum_{\mathbf{k}} \frac{1}{\omega^3} \sum_{l=1}^{q} \sum_{m=1}^{l} (-1)^{l+m} \exp\left(-\omega \sum_{j=m}^{l} x_j\right)\right\}. \quad (11)$$

After integrating by parts in each term, we obtain (see formula (B, 8) in Appendix B)

$$\delta m = \sum_{q=0}^{\infty} (\Delta m)^q \int_0^\infty dx_1 \ldots \int_0^\infty dx_q \prod_{j=1}^{q} (1 - \hat{Q}_j) F_q(x_1, \ldots, x_q), \quad (12)$$

where the operator $\hat{Q}_j$ is defined by the equation

$$\hat{Q}_j F_q(\ldots, x_j, \ldots) = F_q(\ldots, \infty, \ldots).$$

We consider a different series

$$\delta m^\varepsilon = \sum_{q=0}^{\infty} (\Delta m)^q \int_0^\infty dx_1 \ldots \int_0^\infty dx_q \exp\left(-\varepsilon \sum_{j=1}^{q} x_j\right) \times \prod_{i=1}^{q} (1 - \hat{Q}_i) F_q(x_1, \ldots, x_q), \quad (13)$$

which becomes equal to (12) when $\epsilon = 0$. It can be shown that when $\epsilon$ satisfies the inequality

$$\frac{\Delta m}{\varepsilon} < 1 - \Delta m \int_0^\infty dx e^{-\varepsilon x} F_1(x), \quad (14)$$

the series (13) is bounded by the quantity (see Appendix B)

$$\delta m^{\varepsilon} < \left\{1 - \Delta m \int_0^\infty dx e^{-\varepsilon x} [F_1(x) - 1]\right\}^{-1}. \quad (15)$$

As $\epsilon \to 0$, the left half of (15) goes into the original series (12), while the right half has a finite positive limit when

$$\Delta m \int_0^\infty dx\, [F_1(x) - 1] < 1.$$

This procedure signifies that the series (11) is summable in the general sense* ( $\epsilon$-sense) when

$$\Delta m \int_0^\infty dx\, [F_1(x) - 1] < 1$$

and its sum (in the $\epsilon$-sense) is bounded by the quantity

$$\delta m < \left\{1 - \Delta m \int_0^\infty dx\, [F_1(x) - 1]\right\}^{-1}. \quad (16)$$

The series for $Z_2^n$ and $g_r$ are also summed in the $\epsilon$-sense. We cite the result for the sums (in the $\epsilon$-sense)

$$Z_2^n < Z_2^{sc}\left\{1 - \Delta m \int_0^\infty dx\, [F_1(x) - 1]\right\}^{-2}, \quad (17)$$

$$\left|\frac{g_r}{g} - 1\right| < 2(\Delta m)^2 \int_0^\infty dx\, x [F_1(x) - 1]\left\{1 - \left(\Delta m \int_0^\infty dx\, [F_1(x) - 1]\right)^2\right\}^{-2}. \quad (17')$$

Let us discuss the physical meaning of summation in the $\epsilon$-sense. We have seen that the series (13) converges in the usual manner at sufficiently large $\epsilon$ [condition (14)] but the series (13) differs from (12) in that the contribution from the large values of the time (large $x_j$), i.e., from low energies, is smaller. Consequently, the contribution from the ultraviolet region is found to be summable to a finite quantity, which is unexpected from the point of view of local field theory. Thus, the convergence of the series in the $\epsilon$-sense presupposes that the contribution from large times (low energies) is not large enough to violate the summability of expressions (6) — (10).

In this connection, it is interesting to investigate the behavior of the Green's function for small terms, i.e., in the ultraviolet region. The Green's function is defined as

$$G(t) = \langle 0 | T\{\psi(t)\psi^+(0) S(\infty, -\infty)\} | 0\rangle = Z_2^{sc} G_{sc}(t) \sum_{q=0}^{\infty} (-i\tau_3 \Delta m)^q \int_0^t d\xi_1 \int_0^{\xi_1} d\xi_2 \ldots \int_0^{\xi_{q-1}} d\xi_q \exp\left\{-g^2 \sum_{\mathbf{k}} \frac{1}{\omega^3}\right.$$

$$\left.\times\left[\sum_{l=1}^{q} (-1)^l \left(e^{-i\omega(t-\xi_l)} - e^{-i\omega\xi_l}\right) + 2\sum_{l=2}^{q}\sum_{m=1}^{l-1} (-1)^{l+m} e^{-i\omega(\xi_m - \xi_l)}\right]\right\},$$

$$G_{sc}(t) = \theta(t) e^{-imt} \exp\left\{\frac{1}{2} g^2 \sum_{\mathbf{k}} \frac{1}{\omega^3} e^{-i\omega t}\right\}. \quad (18)$$

The function $G_{sc}(t)$ is the Green's function of the scalar neutral theory, and behaves as $t^{-g^2/4\pi^2}$ at small values of the time.[11]

It can be shown (see Appendix D) that the series in $\Delta m$ converges absolutely at small values of t ($\mu t \ll 1$), provided that $g^2/\pi^2 < 1$ and has a branch point at $t = 0$. Thus, the function $G(t)$ has for $t = 0$ a singularity which is integrable because of the condition $g^2/\pi^2 < 1$, which makes the Fourier transform possible.

## CONCLUSION

The proposed example of a model with finite charge renormalization shows another possibility (compared with the Lee model) which can be realized in the rigorous theory. In our opinion, however, this model also fails to reflect the true situation in field theory, since an investigation of more complicated models (see reference 4) leads apparently to a conclusion that the exact solution has a singularity $g^2 = 0$, a Taylor-series expansion about which leads to additional divergences. As already noted above, the model considered here does not have such a property.

In conclusion, the authors express deep gratitude to Prof. D. I. Blokhintsev for continuous interest in the work and for stimulating discussions, and also to L. G. Zastavenko for a discussion of the mathematical problems.

## APPENDIX A

Since the $S^\alpha$ matrix is specified in the form of a series, the matrix elements are presented in the form of a limit of a ratio of two series as $\alpha \to 0$. It is found that if one series is divided by the other and terms of equal powers of $\Delta m$ collected, the phase shift cancels out in the terms obtained in this manner and consequently one can go in each term separately to the limit as $\alpha \to 0$.

*The method proposed is regular, i.e., convergent series such as (11) are summed to their usual sum.[10]

Let us illustrate this method of eliminating the phase shift by an example of the matrix element of the scattering of a meson by a nucleon [see formula (10)]:

$$M_{f\leftarrow i}(\omega_f) = \frac{g^2}{2i\omega_f}\int_{-\infty}^{\infty} d\tau\, e^{-i\omega_f\tau} \lim_{\alpha\to 0} \frac{\sum\limits_{q=1}^{\infty}(-i\delta_N\Delta m_0)^q B_q^\alpha(\tau)}{\sum\limits_{q=0}^{\infty}(-i\delta_N\Delta m_0)^q\, b_q^\alpha}, \qquad (A.1)$$

$$B_q^\alpha(\tau) = \frac{1}{q!}\int_{-\infty}^{\infty} d\xi_1 \ldots \int_{-\infty}^{\infty} d\xi_q \times \exp\left\{-\alpha\sum_{j=1}^{q}|\xi_j|\right\}\prod_{l=1}^{q}\varepsilon(\xi_l)\,\varepsilon(\xi_l-\tau) \times \exp\left\{-\frac{ig^2}{2}\iint_{-\infty}^{\infty} ds_1 ds_2\, e^{-\alpha(|s_1|+|s_2|)} \times \prod_{j=1}^{q}\varepsilon(\xi_j-s_1)\,\Delta(s_1-s_2)\,\varepsilon(\xi_j-s_2)\right\}, \qquad (A.2')$$

$$b_q^\alpha = \frac{1}{q!}\int_{-\infty}^{\infty} d\xi_1 \ldots \int_{-\infty}^{\infty} d\xi_q \exp\left\{-\alpha\sum_{j=1}^{q}|\xi_j| - \frac{ig^2}{2}\iint_{-\infty}^{\infty} ds_1 ds_2\, e^{-\alpha(|s_1|+|s_2|)} \times \prod_{j=1}^{q}\varepsilon(\xi_j-s_1)\,\Delta(s_1-s_2)\,\varepsilon(\xi_j-s_2)\right\}. \qquad (A.2'')$$

The integral in the exponential function is

$$-\frac{ig^2}{2}\iint_{-\infty}^{\infty} ds_1 ds_2 \exp\{-\alpha(|s_1|+|s_1|)\} \times \prod_{j=1}^{q}\varepsilon(\xi_j-s_1)\,\Delta(s_1-s_2)\,\varepsilon(\xi_j-s_2)$$
$$= -\frac{g^2}{2i\alpha}\sum_{\mathbf{k}}\frac{1}{\omega^2} - g^2\sum_{\mathbf{k}}\frac{1}{\omega^3}\left\{q + 2\sum_{\nu_1=2}^{q}\sum_{\nu_2=1}^{\nu_1-1}\prod_{j_1\neq\nu_1}\varepsilon(\xi_{\nu_1}-\xi_{j_1}) \times \prod_{j_2\neq\nu_2}\varepsilon(\xi_{\nu_2}-\xi_{j_2})\exp(-i\omega|\xi_{\nu_1}-\xi_{\nu_2}|)\right\}. \qquad (A.3)$$

The first term is the same for all terms of the series and therefore cancels out.

The presence of a phase shift in the expression for the matrix element manifests itself in the fact that the integrals with respect to time (with respect to $\xi_j$) diverge linearly at infinity when $\alpha = 0$. However, as stated above, a series that represents a ratio of series that diverge when $\alpha = 0$ contains integrals that are finite when $\alpha = 0$. This circumstance is due to the fact that the divergent integrals are grouped about each power of $\Delta m$ in such a way, that the infinities cancel out.

The regularization procedure is as follows. We change in the integrals (A.2′) and (A.2″) to integration over the simplex and make a change of variables

$$\xi_1 = \xi, \quad \xi_l = \xi - \sum_{j=1}^{l-1} x_j, \qquad l \geqslant 2. \qquad (A.4)$$

Then

$$B_q^\alpha(\tau) = \int_{-\infty}^{\infty} d\xi \int_0^{\infty} dx_1 \ldots \int_0^{\infty} dx_{q-1} \times \exp\left\{-\alpha\left(|\xi| + \sum_{l=1}^{q-1}\left|\xi - \sum_{j=1}^{l} x_j\right|\right)\right\}$$
$$\times\, \varepsilon(\xi)\,\varepsilon(\xi-\tau)\prod_{l=1}^{q-1}\varepsilon\left(\xi - \sum_{j=1}^{l} x_j\right)\varepsilon\left(\xi - \tau - \sum_{j=1}^{l} x_j\right) \times F_{q-1}(x_1, \ldots, x_{q-1})\,(\Delta m/\Delta m_0)^q$$

$$b_q^\alpha = \int_{-\infty}^{\infty} d\xi \int_0^{\infty} dx_1 \ldots \int_0^{\infty} dx_{q-1} \times \exp\left\{-\alpha\left(|\xi| + \sum_{l=1}^{q-1}\left|\xi - \sum_{j=1}^{l} x_j\right|\right)\right\} \times F_{q-1}(x_1, \ldots, x_{q-1})\,(\Delta m/\Delta m_0)^q \qquad (A.5)$$

where

$$F_{q-1}(x_1, \ldots, x_{q-1}) = \exp\left\{2g^2\sum_{\mathbf{k}}\frac{1}{\omega^3}\sum_{l=1}^{q-1}\sum_{m=1}^{l}(-1)^{l+m}\exp\left(-i\omega\sum_{j=m}^{l}x_j\right)\right\}.$$

The integral of the function $F_{q-1}$ diverges linearly at infinity with respect to any argument $x_j$, but the function $F_{q-1}(\ldots, x_j, \ldots) - F_{q-1}(\ldots, \infty, \ldots)$ is already integrable with respect to $x_j$ at infinity. We note further that

$$F_{q-1}(x_1, \ldots, x_{j-1}, \infty, x_{j+1}, \ldots, x_{q-1}) = F_{j-1}(x_1, \ldots, x_{j-1})\,F_{q-j-1}(x_{j+1}, \ldots, x_{q-1}). \qquad (A.6)$$

It follows from the foregoing that in order to regularize the function $B_q^\alpha(\tau)$ it is necessary to subtract from the function $F_{q-1}$ its value at infinity in each of its arguments, i.e., we substitute instead of $F_{q-1}$ the function

$$\prod_{l=1}^{q-1}(1-\hat{Q}_j)\,F_{q-1}(x_1, \ldots, x_{q-1}), \qquad (A.7)$$

where the operator $\hat{Q}_j$ is defined by the equation $\hat{Q}_j F_{q-1}(\ldots, x_j, \ldots) = F_{q-1}(\ldots, \infty, \ldots)$. Such a substitution of (A.7) for the function $F_{q-1}$ is realized precisely by dividing one series by the other and grouping the terms about equal powers of $\Delta m$. The same applies also to the regularization of the integral with respect to $\xi$, (A.5). Finally, the scattering matrix element with the phase shift eliminated is written in the form

$$M_{f\leftarrow i}(\omega_f) = -\frac{g^2}{2i\omega_f}\int_{-\infty}^{\infty} d\tau e^{-i\omega_f\tau} \sum_{q=1}^{\infty}(-i\delta_N\Delta m)^q \int_0^{\infty} dx_1 \ldots \int_0^{\infty} dx_q$$
$$\times \prod_{j=1}^{q}(1-\hat{Q}_j) F_q(x_1,\ldots,x_q) \int_{-\infty}^{\infty} d\xi \, [\varepsilon(\xi)\varepsilon(\xi-\tau)$$
$$\times \prod_{l=1}^{q}\varepsilon\Big(\xi-\sum_{j=1}^{l} x_j\Big)\varepsilon\Big(\xi-\tau-\sum_{j=1}^{l} x_j\Big)-1\Big]. \quad (A.8)$$

From (A.8) we obtain (10) directly if we recognize that the operator $(1-\hat{Q}_j)$ can be represented in the form

$$1-\hat{Q}_j = -\int_{x_j}^{\infty} dy_j \frac{\partial}{\partial y_j}. \quad (A.9)$$

The phase shift for any other matrix element is eliminated analogously.

## APPENDIX B

According to formula (6), the eigenvalue of the one-fermion state is defined as

$$E_N = \lim_{\alpha\to 0}\frac{\langle N\,|\,HS^\alpha(0,-\infty)\,|\,N\rangle}{\langle N\,|\,S^\alpha(0,-\infty)\,|\,N\rangle} = m_0 + \delta_N\Delta m_0 + \delta E_N, \quad (B.1)$$

$$\delta E_N = \lim_{\alpha\to 0}\frac{\langle N\,|\,g(\psi^+\tau_1\psi)\,\hat{\varphi}(0)\,S^\alpha(0,-\infty)\,|\,N\rangle}{\langle N\,|\,S^\alpha(0,-\infty)\,|\,N\rangle}. \quad (B.1')$$

The matrix element in the numerator is

$$M_1^\alpha = \langle N\,|\,g(\psi^+\tau_1\psi)\,\hat{\varphi}(0)\,S^\alpha(0,-\infty)\,|\,N\rangle$$
$$= -\sum_{q=1}^{\infty}(-i\delta_N\Delta m_0)^q \int_{-\infty}^{0} d\xi_1 \ldots \int_{-\infty}^{\xi_{q-1}} d\xi_q$$
$$\times \exp\Big(\alpha\sum_{j=1}^{q}\xi_j\Big)\, ig^2 \int_{-\infty}^{0} ds e^{\alpha s} i\Delta(s)$$
$$\times \prod_{j=1}^{q}\varepsilon(s-\xi_j)\exp\Big\{-\frac{ig^2}{2}\iint_{-\infty}^{0} ds_1 ds_2 e^{\alpha(s_1+s_2)}$$
$$\times \prod_{j=1}^{q}\varepsilon(s_1-\xi_j)\,\Delta(s_1-s_2)\,\varepsilon(s_2-\xi_j)\Big\}. \quad (B.2)$$

The matrix element in the denominator of (B.1′) is obtained analogously

$$M_2^\alpha = \langle N\,|\,S^\alpha(0,-\infty)\,|\,N\rangle = \sum_{q=0}^{\infty}(-i\delta_N\Delta m_0)^q r_q,$$
$$r_q = \int_{-\infty}^{0} d\xi_1 \ldots \int_{-\infty}^{\xi_{q-1}} d\xi_q \exp\Big(\alpha\sum_{j=1}^{q}\xi_j\Big)$$
$$\times \exp\Big\{-\frac{ig^2}{2}\iint_{-\infty}^{0} ds_1 ds_2 e^{\alpha(s_1+s_2)}$$
$$\times \prod_{j=1}^{q}\varepsilon(s_1-\xi_j)\,\Delta(s_1-s_2)\,\varepsilon(s_2-\xi_j)\Big\}. \quad (B.3)$$

The integral in the exponential function is (when $\xi_1 > \xi_2 > \ldots > \xi$)

$$I_q(\xi_1,\ldots,\xi_q)$$
$$= -\frac{ig^2}{2}\iint_{-\infty}^{0} ds_1 ds_2 e^{\alpha(s_1+s_2)}\prod_{j=1}^{q}\varepsilon(s_1-\xi_j)\,\Delta(s_1-s_2)$$
$$\times \varepsilon(s_2-\xi_j) = -\frac{g^2}{4}\sum_{\mathbf{k}}\frac{1}{\omega^2}\Big(\frac{1}{i\alpha}+\frac{1}{\omega}\Big) - g^2\sum_{\mathbf{k}}\frac{1}{\omega^3}$$
$$\times\Big[q+\sum_{l=1}^{q}(-1)^l e^{i\omega\xi_l} + 2\sum_{l=2}^{q}\sum_{m=1}^{l-1}(-1)^{l+m}e^{-i\omega(\xi_m-\xi_l)}\Big]. \quad (B.4)$$

The first term in (B.4) is the same for all the terms of the series, both in the numerator and in the denominator, and therefore cancels out. Subtracting in (B.2) the integral with respect to s, we obtain

$$M_1^\alpha = -\frac{g^2}{2}\sum_{\mathbf{k}}\omega^{-2}M_2^\alpha$$
$$-\sum_{q=1}^{\infty}(-i\delta_N\Delta m_0)^q\int_{-\infty}^{0} d\xi_1 \ldots \int_{-\infty}^{\xi_{q-1}} d\xi_q \exp\Big(\alpha\sum_{j=1}^{q}\xi_j\Big)$$
$$\times i\Big(\frac{\partial}{\partial\xi_1}+\ldots+\frac{\partial}{\partial\xi_q}\Big)\exp\{I_q(\xi_1,\ldots,\xi_q)\}. \quad (B.5)$$

Let us consider now the q-th term of the series (B.5)

$$\int_{-\infty}^{0} d\xi_1 \ldots \int_{-\infty}^{\xi_{q-1}} d\xi_q \exp\Big(\alpha\sum_{j=1}^{q}\xi_j\Big)\, i\Big(\frac{\partial}{\partial\xi_1}+\ldots+\frac{\partial}{\partial\xi_q}\Big)$$
$$\times \exp\{I_q(\xi_1,\ldots,\xi_q)\} = ir_{q-1} - i\alpha q r_q. \quad (B.6)$$

Further, substituting (B.6) in (B.5) we obtain for $E_N$, according to (A.1), the expression

$$E_N = m_0 - \frac{g^2}{2}\sum_{\mathbf{k}}\frac{1}{\omega^2} + \lim_{\alpha\to 0} i\alpha\,\frac{\sum_{q=1}^{\infty}(-i\delta_N\Delta m_0)^q q r_q}{\sum_{q=0}^{\infty}(-i\delta_N\Delta m_0)^q r_q}. \quad (B.7)$$

Performing in (B.7) a term by term division of one series into another, we eliminate thereby the phase-shift factor in each power of Δm. This makes it possible to make the transition to the limit in α in each power with respect to Δm (see Appendix A).

As a result we obtain

$$E_N = m_0 - \frac{1}{2}g^2\sum_{\mathbf{k}}\frac{1}{\omega^2} + \sum_{q=1}^{\infty}(-i\delta_N\Delta m)^q$$
$$\times \int_0^{\infty} dx_1 \ldots \int_0^{\infty} dx_{q-1}\prod_{l=1}^{q-1}(1-\hat{Q}_l)\,F_{q-1}(x_1,\ldots,x_{q-1}), \quad (B.8)$$

where $F_{q-1}$ is given by (A.5). We obtain (6) directly from (B.8), if we take (A.9) into account.

## APPENDIX C

Let us consider formula (13). We used the relations

$$\prod_{j=1}^{q}(1-\hat{Q}_j)=\sum_{l=0}^{q}(-1)^l\sum_{s_1=l}^{q}\cdots\sum_{s_i=l-i+1}^{s_{i-1}-1}\cdots\sum_{s_l=1}^{s_{l-1}-1}\hat{Q}_{s_1}\cdots\hat{Q}_{s_i}\cdots\hat{Q}_{s_l}, \quad (C.1)$$

$$\int_0^\infty dx_1\ldots\int_0^\infty dx_q e^{-\varepsilon(x_1+\ldots+x_q)}\times\hat{Q}_{s_1}\ldots\hat{Q}_{s_i}\ldots\hat{Q}_{s_l}F_q(x_1,\ldots,x_q)$$
$$=\left(\frac{1}{\varepsilon}\right)^l I^\varepsilon_{s_l-1}I^\varepsilon_{s_{l-1}-s_l-1}\ldots I^\varepsilon_{s_i-s_{i+1}-1}\ldots I^\varepsilon_{s_1-s_2-1}I^\varepsilon_{q-s_1}, \quad (C.2)$$

$$I^\varepsilon_s=\int_0^\infty dx_1\ldots\int_0^\infty dx_s e^{-\varepsilon(x_1+\ldots+x_s)}F_s(x_1,\ldots,x_s). \quad (C.3)$$

Substituting these relations in (13), we obtain after interchanging the order of summation in the sums with respect to q, $l$, and $s_i$, and interchanging the summation indices (the derivations are simple but somewhat cumbersome and will therefore not be given here)

$$\delta m^\varepsilon=\sum_{q=0}^{\infty}(\Delta m)^q\bar{I}^\varepsilon_q=\sum_{l=0}^{\infty}\frac{(-\Delta m)^l}{\varepsilon^l}\left\{\sum_{q=0}^{\infty}(\Delta m)^q I^\varepsilon_q\right\}^{l+1},$$

$$\bar{I}^\varepsilon_q=\int_0^\infty dx_1\ldots\int_0^\infty dx_q e^{-\varepsilon(x_1+\ldots+x_q)}\times\prod_{j=1}^{q}(1-\hat{Q}_j)F_q(x_1,\ldots,x_q). \quad (C.4)$$

Let us carry out in (C.4) formally the summation

$$\delta m^\varepsilon=\left[\frac{\Delta m}{\varepsilon}+\left\{\sum_{q=0}^{\infty}(\Delta m)^q I^\varepsilon_q\right\}^{-1}\right]^{-1}. \quad (C.5)$$

Since $I^\epsilon_q \le (I^\epsilon_1)^q$, we get

$$\delta m^\varepsilon\leqslant[\Delta m/\varepsilon+1-\Delta m I^\varepsilon_1]^{-1}=[1-\Delta m\bar{I}^\varepsilon_1]^{-1}. \quad (C.6)$$

The summation in (C.4) will be correct if $\epsilon$ satisfies the conditions

$$\Delta m I^\varepsilon_1<1, \qquad \Delta m/\varepsilon<1-\Delta m I^\varepsilon_1, \quad (C.7)$$

of which the second is the stronger.

## APPENDIX D

Let us consider the q-th term of the series (1′) for the Green's function

$$J_q(t)=(-i\Delta m)^q\int_0^t d\xi_1\ldots\int_0^{\xi_{q-1}}d\xi_q\exp\left\{-g^2\sum_k\frac{1}{\omega^3}\left[\sum_{l=1}^{q}(-1)^l\left(e^{-i\omega(t-\xi_l)}-e^{-i\omega\xi_l}\right)+2\sum_{l=2}^{q}\sum_{m=1}^{l-1}(-1)^{l+m}e^{-i\omega(\xi_m-\xi_l)}\right]\right\}. \quad (D.1)$$

Since we are interested in the behavior of the Green's function for small times ($\mu t \ll 1$), the functions in the exponential function can be replaced by their asymptotic expansions for small arguments

$$g^2\sum_k\omega^{-3}e^{-i\omega\xi}=-G^2\ln(i\mu\xi)+G^2(\ln 2-C-1)+O(\mu\xi), \quad (D.2)$$

where C is Euler's constant and $G^2 = g^2/2\pi^2$.

Making further a change of variables of integration in (D.1), $\xi_j = \mu t x_j$ and substituting the asymptotic expansion (D.2) therein, we obtain

$$J_q(t)=\left(-\frac{\Delta m}{\mu}\right)^q(i\mu t)^{q-2G^2[q/2]}\exp\{2G^2(\ln 2-C-1)[q/2]\}I_q,$$

$$I_q=\int_0^1 dx_1\ldots\int_0^{x_{q-1}}dx_q\prod_{r=1}^{q}[(1-x_r)/x_r]^{(-1)^r G^2}\times\prod_{l=2}^{q}\prod_{m=1}^{l-1}(x_m-x_l)^{(-1)^{l+m}2G^2},$$

$$\left[\frac{q}{2}\right]=\begin{cases}q/2 & \text{if q is even}\\(q-1)/2, & \text{if q is odd}\end{cases} \quad (D.3)$$

Let us consider first the numbers $I_q$ for even values q = 2n:

$$I_{2n}=\int_0^1 dx_1\ldots\int_0^{x_{2n-1}}dx_{2n}\prod_{r=1}^{n}\left[\frac{(1-x_{2r})x_{2r-1}}{x_{2r}(1-x_{2r-1})}\right]^{G^2}\times\prod_{l=1}^{n}(x_{2l-1}-x_{2l})^{-2G^2}\prod_{m=1}^{l-1}\left[\frac{(x_{2m}-x_{2l})(x_{2m-1}-x_{2l-1})}{(x_{2m}-x_{2l-1})(x_{2m-1}-x_{2l})}\right]^{2G^2} \quad (D.4)$$

We make the following change of variables in the integrals:

$$x_l=\prod_{j=1}^{l}z_j. \quad (D.5)$$

We then obtain

$$I_{2n}=\int_0^1 dz_1\ldots\int_0^1 dz_{2n}\prod_{s=1}^{2n}z_s^{(2n+1-s)(1-G^2)-1}\times\prod_{l=1}^{n}\left(\frac{1-\prod_{j=1}^{2l}z_j}{1-\prod_{j=1}^{2l-1}z_j}\right)^{G^2}(1-z_{2l})^{-2G^2}$$
$$\times\prod_{m=1}^{l-1}\left[\frac{\left(1-\prod_{j=2m+1}^{2l}z_j\right)\left(1-\prod_{j=2m}^{2l-1}z_j\right)}{\left(1-\prod_{j=2m+1}^{2l-1}z_j\right)\left(1-\prod_{j=2m}^{2l}z_j\right)}\right]^{2G^2}. \quad (D.6)$$

It is easy to see that the integrals converge on the upper limit provided

$$2G^2 < 1, \qquad g^2/\pi^2 < 1. \qquad \text{(D.7)}$$

To obtain an estimated upper limit for the number $I_{2n}$, we use the inequality

$$(1-x)/(1-x\beta) \leqslant 1, \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant \beta \leqslant 1. \qquad \text{(D.8)}$$

An estimate yields

$$I_{2n} \leqslant \int_0^1 \frac{dz_1 z_1^{2n(1-G^2)-1}}{(1-z_1)^{G^2}} \prod_{l=2}^{2n} \int_0^1 dz_l \frac{z_l^{(2n+1-l)(1-G^2)-1}}{(1-z_l)^{2G^2}}. \qquad \text{(D.9)}$$

The calculation of the integrals leads to the result

$$I_{2n} \leqslant \frac{[(1-G^2)\Gamma(1-2G^2)]^{2n}}{\Gamma((1-2G^2)/(1-G^2))} \Gamma\left(\frac{(2n+1)(1-G^2)-1}{1-G^2}\right) \times \left[\frac{\Gamma(1-G^2)}{\Gamma((2n+1)(1-G^2))}\right]^2. \qquad \text{(D.10)}$$

As n tends to infinity, the inequality assumes the form $[(2n+1)(1-2G^2) \gg 1]$

$$I_{2n} \leqslant \frac{[\Gamma(1-2G^2)]^{2n}}{\Gamma((2n+1)(1-2G^2))} \frac{(1-G^2)^{2n}[\Gamma(1-G^2)]^2}{\Gamma((1-2G^2)/(1-G^2))} \times \left[\frac{(1-2G^2)^{1-2G^2}}{(1-G^2)^{2(1-G^2)}}\right]^{2n+1}. \qquad \text{(D.11)}$$

The presence of the factor $\Gamma((2n+1)(1-2G^2))$ in the denominator of the right half of the inequality ensures the absolute convergence of the series (D.1) for small t, i.e., $\mu t \ll 1$, for arbitrary $\Delta m$, subject to the condition $g^2/\pi^2 < 1$ on the bare coupling constant. An estimate of the odd numbers $I_{2n+1}$ is obtained analogously and yields the same results.


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136