ON THE ASYMPTOTIC BEHAVIOR OF GREEN'S FUNCTIONS IN QUANTUM FIELD THEORY

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The formalism of vacuum expectation values of products of field operators[4-6] is used to prove for quantum field theory a principle of the weakening of correlations analogous to the corresponding Bogolyubov principle in quantum statistics. By the extension of these results to the Green's functions, a proof is obtained for the hypothesis of Freese about the asymptotic behavior of Green's functions at large space separations.

THE study of the properties of Green's functions in quantum field theory is an extremely difficult problem. The question arises of approximating them by Green's functions of the lowest orders, whose properties have been most completely studied. This question has been treated in papers by Freese[1] and Falk.[2] Their conclusions are based on the hypothesis that one can neglect the interaction at points separated by an infinite distance.

The present paper is devoted to a rigorous proof and a refinement of the hypothesis of Freese about the behavior of Green's functions. In our deductions we do not make the assumption that interactions at large distances are small. In the statement of the problem we start from an idea of Bogolyubov.[3] In his lectures on statistical physics Bogolyubov has recently formulated a principle of the weakening of the correlations between particles for systems in the state of statistical equilibrium. In our paper this principle is formulated for quantum field theory and we give a proof of it on the basis of the axioms of the theory.

Our conclusions are based on the study of the behavior of vacuum expectation values of products of operators. For this purpose we use the apparatus of functions developed by Wightman, Källen, and Wilhelmsson.[4-6] The behavior of vacuum expectation values for fixed equal time components has been considered by Dell'Antonio and Gulmanelli[7] in a study of the spatial asymptotic condition of Haag.[8,9]

1. THE PRINCIPLE OF WEAKENING OF CORRELATIONS

Let us consider a neutral scalar field which is described by a Hermitian operator $A(x)$. By hypothesis the operator $A(x)$ satisfies the conditions of causality and of translational and Lorentz invariance. It is also assumed that there exists a unique normalized vacuum state $|0\rangle$ and that there are no states with negative energies.

Let us consider a set of points M consisting of $n+1$ vectors x_i, where $x_i = (\mathbf{x}_i, x_i^0)$ and the scalar product is defined in the following way:

$$x_i^2 = (x_i^1)^2 + (x_i^2)^2 + (x_i^3)^2 - (x_i^0)^2 .$$

Let us divide the set of points M into two subsets M_1 and M_2:

$$M = M_1 + M_2,\ \mathbf{x}_\alpha \in M_1,\ \mathbf{x}_\beta \in M_2;$$

$$\alpha = 1, 2, \ldots, i; \quad \beta = i+1, \ldots, n+1.$$

Let the sets M_1 and M_2 be such that each of them taken separately can be inclosed in some sphere of finite radius. Let us now make these spheres undergo a displacement relative to each other. In doing so we shall assume that the configurations of the points of the sets M_1 and M_2 change only inside the respective spheres, and that the time components of the vectors x_i are arbitrarily fixed. The analog of Bogolyubov's principle of the weakening of correlations[3] holds for this case.

The vacuum average

$$f(x_1, \ldots, x_{n+1}) = \langle 0 | A(x_1) \ldots A(x_i) A(x_{i+1}) \ldots A(x_{n+1}) | 0 \rangle \quad (1)$$

separates into a product of vacuum averages

$$\langle 0 | A(x_1) \ldots A(x_i) | 0 \rangle \langle 0 | A(x_{i+1}) \ldots A(x_{n+1}) | 0 \rangle, \quad (2)$$

if the set of points M_1 goes to an infinite distance from the set M_2 in the sense defined above.

For the proof we make use of the results of

Källen, Wightman, and Wilhelmsson.[4-6] We expand the vacuum average (1) in terms of a complete system of intermediate states

$$f(x_1, \ldots, x_{n+1}) = \sum_{|z_k\rangle} \exp\{i\Sigma p^{(z_k)}\xi_k\}$$

$$\times \langle 0 | A_1 | z_1\rangle \langle z_1 | A_2 | z_2\rangle \ldots \langle z_n | A_{n+1} | 0\rangle, \tag{3}$$

where $p^{(z_k)}$ is the eigenvalue of the operator of energy-momentum of the state $|z_k\rangle$, and moreover

$$\xi_k = x_k - x_{k+1}, \ \langle z_{k-1} | A_k(x_k) | z_k\rangle$$

$$= \langle z_{k-1} | A_k | z_k\rangle \exp\{i(p^{(z_k)} - p^{(z_{k-1})})x_k\}. \tag{4}$$

Let us rewrite the relation (3) in the form

$$f(x_1, \ldots, x_{n+1}) = \langle 0 | A(x_1) \ldots A(x_i) | 0\rangle \langle 0 |$$

$$\times A(x_{i+1}) \ldots A(x_{n+1}) | 0\rangle + f_1, \tag{5}$$

where

$$f_1 = \sum_{|z_1\rangle \ldots |z_i\rangle \ldots |z_n\rangle} \exp\left\{i \sum p^{(z_k)}\xi_k\right\} \langle 0 | A_1 | z_1\rangle$$

$$\times \langle z_1 | A_2 | z_2\rangle \ldots \langle z_n | A_{n+1} | 0\rangle, \tag{6}$$

with the sum taken over all intermediate states with the exception of the states $|z_i\rangle$, the sum over which is taken starting with the one-particle state.

It can be seen from the relation (5) that for our purpose it is enough to prove the equation

$$\lim_{R\to\infty} R^m f_1 = 0, \tag{7}$$

where R is the distance between the spheres that were introduced above and m is an arbitrary positive integer.

Starting from the conditions of our problem, we shall find out certain properties of the function f_1. If we denote the Fourier transform of the function f_1 by $G(p_1, \ldots, p_n)$, then we can write

$$f_1 = \frac{1}{(2\pi)^{3n}} \int \ldots \int dp_1 \ldots dp_n$$

$$\times \exp\left\{i \sum_{k=1}^{n} p_k \xi_k\right\} G(p_1, \ldots, p_n). \tag{8}$$

In virtue of invariance under Lorentz transformations the function $G(p_1, \ldots, p_n)$ is a function of all possible scalar products of the vectors $p_1, \ldots, p_n$. In virtue of the condition that the energy is positive the function $G(p_1, \ldots, p_n)$ is different from zero only when all of the vectors p_k lie in the upper light cone, and furthermore the vector p_i must lie in the upper hyperboloid $p_i^2 \leq -\alpha_{ii}$, $(p_i^0)^2 \geq \alpha_{ii} > 0$, which is also in the corresponding upper light cone. Here α_{ii} is defined by the relation $p_i^2 = \alpha_{ii}$, under the condition that the vector p_i corresponds to a one-particle state.

The fact that the vector p_i lies in the upper hyperboloid is due to the sum in Eq. (6) being taken over states $|z_1\rangle$ beginning with the one-particle state. What we have said can be written in the following way:

$$G(p_1, \ldots, p_n)$$

$$= G(p_1^2, \ldots, p_k p_l, \ldots, p_n^2) \prod_{k=1,\ k\neq i}^{n} \theta(p_k)\, \theta(p_i^0 - \alpha_{ii});$$

$$\theta(p_k) = \frac{1}{2}\left[1 + \frac{p_k^0}{|p_k^0|}\right].$$

We can now write the relation (8) in the form

$$f_1 = i^n \int_{\alpha_{kl}}^{\infty} \ldots \int \prod_{k,\, l=1}^{n} d\alpha_{kl} G(-\alpha_{11}, \ldots, -\alpha_{kl}, \ldots, -\alpha_{nn})$$

$$\times \Delta_{n+1}^{(+)}(\xi_k;\, \alpha_{kl}), \tag{9}$$

where

$$\Delta_{n+1}^{(+)}(\xi_k;\, \alpha_{kl}) = \frac{(-i)^n}{(2\pi)^{3n}} \int \ldots \int dp_1 \ldots dp_n$$

$$\times \exp\left\{i \sum_{k=1}^{n} p_k \xi_k\right\} \prod_{k\leqslant l=1}^{n} \delta(p_k p_l + \alpha_{kl}) \prod_{k=1}^{n} \theta(p_k). \tag{10}$$

The quantities α_{kl} are zero, except that $\alpha_{ii} > 0$.

It can be seen from Eq. (9) that for the proof of the equation (7) it is enough to establish the relation

$$\lim_{R\to\infty} R^m \Delta_{n+1}^{(+)}(\xi_k;\, \alpha_{kl}) = 0. \tag{11}$$

In this connection one also has to justify the passage to the limit under the signs of integration in Eq. (9).

Thus the proof of the principle of the weakening of correlations has been reduced to the study of the asymptotic behavior of the singular functions of Källen and Wilhelmsson.[6] We can study the asymptotic behavior of these functions only in the region where they exist. Therefore it is also necessary to show that the passage to the limit $R \to \infty$ occurs in the region of existence.

2. PROOF OF THE RELATION (11)

Källen and Wilhelmsson[6] have obtained the following representation for the function $\Delta_{n+1}^{(+)}(\xi; \alpha_{kl})$:

$$\Delta_{n+1}^{(+)}(\xi_k;\, \alpha_{kl}) = \frac{(-i)^{n-4}}{(2\pi)^{3(n-4)}} (-D)^{(n-4)^2/2} \prod_{k_1\leqslant k_2=5}^{n} \delta\Big(D\alpha_{k_1k_2}$$

$$- \sum_{\lambda,\, \lambda'=1}^{4} \alpha_{k_1\lambda}\alpha_{\lambda\lambda'}\alpha_{\lambda' k_2}\Big)\, \Delta_5^{(+)}(y_\varkappa;\, \alpha_{\varkappa\varkappa'}) \prod_{k=5}^{n} \theta(\alpha_{1k}),$$

$$y_\varkappa = x_\varkappa + \sum_{\lambda=1}^{4} \sum_{k=5}^{n} D^{-1}\Delta_{\varkappa\lambda}\alpha_{\lambda k} x_k,$$

$$D = \mathrm{Det}\, \| \alpha_{ik} \| < 0 \quad (1 \leqslant i, k \leqslant 4),$$

$$D_0 = \mathrm{Det}\, \| \alpha_{ik} \| \quad (1 \leqslant i, k \leqslant 3). \tag{12}$$

The function $\Delta_5^{(+)}$ has the form

$$\Delta_5^{(+)} (\xi ; a_{kl}) = \frac{1}{(2\pi)^{12}} \frac{1}{\sqrt{-D}} \theta$$
$$\times (-D)\, \theta (D_0)\, \theta (a_{12}^2 - a_{11}a_{22}) \prod_{k=2}^{4} \theta (a_{1k})\, I; \quad (13)$$

$$I = I^{(1)} + I^{(2)}, \qquad I^{(1,2)} = \frac{(2\pi)^3}{2} \int_{-\infty}^{\infty} \frac{t\,dt H_0^{(1)}(t)}{\sqrt{\Sigma_{1,2}(t)}}; \quad (14)$$

$$\Sigma_{1,2}(t) = [(t^2 - Q)^2 - P \pm \sqrt{T}]^2 - t^2 S \mp 8\sqrt{T} t^2 (t^2 - Q),$$

$$Q = I_1, \quad P = 2 (I_1^2 - I_2), \quad S = \tfrac{32}{3} (I_1^3 - 3I_1I_2 + 2I_3),$$

$$T = \tfrac{8}{3} (I_1^4 - 6I_1^2 I_2 + 3I_2^2 + 8I_3I_1 - 6I_4),$$

$$I_k = \mathrm{Sp}\, [(AX)^k]; \quad k = 1, 2, 3, 4. \quad (15)$$

Here A and X denote the matrices $\| a_{kl} \|$ and $\| \xi_k \cdot \xi_l \|$.

In the passage to the limit $R \to \infty$ the distance between points that belong to the same set, M_1 or M_2, remains less than some constant, which is fixed by the radius of the corresponding sphere. The distances that increase will be those between pairs of points, one of which belongs to the set M_1 and the other to M_2. The function $\Delta_{n+1}^{(+)}(\xi_k; a_{kl})$ involves only one such pair of points: $\xi_i = x_i - x_{i+1}$. In the passage to the limit the difference between the coordinates of the points increases in proportion to the quantity R. Consequently, we can introduce a new quantity ξ_i', by setting $\xi_i = (R\xi_i', \xi_i^0)$. By the theorem of Hall and Wightman[10] the function $\Delta_{n+1}^{(+)}(\xi_k; a_{kl})$, and consequently also the quantities Q, P, S, T, are functions of the scalar products $\xi_k \cdot \xi_l$.

For the further argument it is convenient to introduce the quantities

$$q = Q / R^2, \quad p = P/R^4, \quad s = S/R^6, \quad \tau = T/R^8, \quad t = Rt' \quad (16)$$

The relations (14) and (15) then take the forms

$$I^{(1,2)} = \frac{(2\pi)^3}{2} \frac{1}{R^2} \int_{-\infty}^{\infty} \frac{t'\,dt' H_0^{(1)}(Rt')}{\sqrt{\Sigma_{1,2}(t')}}, \quad (17)$$

$$\Sigma_{1,2}(t') = [(t'^2 - q)^2 - p \pm \sqrt{\tau}]^2 - t'^2 s \mp 8\sqrt{\tau} t'^2 (t'^2 - q). \quad (18)$$

Direct calculation of the integrals (17) is extremely complicated.

Let us first find out the properties of the functions q, p, s, τ, starting from the conditions of our problem. When expressed in terms of the matrix elements $(AX)_{ij}$ the functions I_k have the form

$$I_k (AX)_{ii}^k + R_k, \quad k = 1, 2, 3, 4, \quad (19)$$

where R_k contains all the other terms, and

$$(AX)_{ii} = a_{1i}\xi_i \cdot \xi_1 + a_{2i}\xi_i \cdot \xi_2 + \ldots + a_{ii}\xi_i^2 + \ldots + a_{ni}\xi_i \cdot \xi_n. \quad (20)$$

For the further argument it is convenient to write this relation in the form

$$(AX)_{ii} = a_{ii}\xi_i^2 + r_1, \quad (21)$$

where the function r_1 contains all the remaining terms.

Calculating the functions Q, P, S, T by means of Eq. (19) and subsituting the expressions for these functions in Eq. (16), we find

$$q = a_{ii}\xi_i^2 + R_1'/R, \quad (22)$$

$$p = R_2'/R, \qquad s = R_3/R, \qquad \tau = R_4'/R. \quad (23)$$

It follows from the relations (23) that in the limit $R \to \infty$ the functions p, s, τ go to zero. According to Källen and Wilhelmsson[6] in this case the function $\Delta_{n+1}^{(+)}$ reduces to the function

$$\Delta_2^{(+)} (\xi_i; a_{ii}) = -\frac{a_{ii}}{8\pi} H_1^{(1)} (\sqrt{a_{ii} z}) / \sqrt{a_{ii} z}, \quad z = -\xi_i^2. \quad (24)$$

This fact can be interpreted in the following way. In the limiting case $R \to \infty$ the correlation between the sets M_1 and M_2 can be interpreted as a correlation between the two points x_i and x_{i+1}. The correlation between them will then naturally be described by the correlation function $\Delta_2^{(+)} \equiv \Delta_2^{(+)}(\xi_i; a_{ii})$, which depends only on the differences of the coordinates of these two points. The function $\Delta_2^{(+)}$ is the boundary value of a certain analytic function which is regular in the entire complex plane of the variable z with the exception of a cut along the positive real axis. By the conditions of our problem $a_{ii} > 0$, and for sufficiently large R the vector ξ_i is spacelike. This means that we are studying our function on the negative real axis, which is in the region of existence of the function $\Delta_2^{(+)}$. This gives us the right to calculate this function at the point $R = \infty$.

Taking into account the behavior of the Hankel function $H_1^{(1)}$ in the upper half-plane, we find the relation

$$\lim_{R\to\infty} R^m \Delta_2^{(+)} (\xi_i; a_{ii}) = 0.$$

Thus we have proved the relation (11).

It is interesting to extend our arguments to other cases, for example to the case in which the set M is divided into three or four subsets. In these cases the study of the function $\Delta_{n+1}^{(+)}$ reduces to the study of the functions $\Delta_3^{(+)}$ and $\Delta_4^{(+)}$, and the proof can be carried through in analogy with the arguments of Dell'Antonio and Gulmanelli.[7]

The case in which the set M is divided into five or more subsets requires special treatment.

To justify the passage to the limit under the integral signs in Eq. (8) one must show that the passage to the limit in Eq. (9) is uniform in all the a_{kl}. It is obvious that we can always take $(\xi_i')^2$ and R so large that all of the arguments carried out above will hold for all a_{kl}.

With this we have completely proved the relation (2).

3. THE ASYMPTOTIC BEHAVIOR OF THE GREEN'S FUNCTIONS

Let us consider the function

$$\langle 0 | T(x_1, \ldots, x_{n+1}) | 0 \rangle = \Sigma \Theta(x_1 - x_2) \ldots \Theta(x_n - x_{n+1}) \times \langle 0 | A(x_1) \ldots A(x_{n+1}) | 0 \rangle, \quad (25)$$

where the sum is taken over all possible permutations of the indices 1, 2, ..., n + 1. Let us investigate the behavior of this function when the distance between the spheres introduced earlier becomes infinite. We regard the time components as arbitrarily fixed. For our purpose let us apply the principle of the weakening of correlations to each term in the right member of Eq. (25). It is not hard to see that all of the terms take the form (2), because for sufficiently large distance between the spheres the vectors of the sets M_1 and M_2 become spacelike, and consequently in virtue of the condition of local commutativity the operators $A(x_1), \ldots, A(x_i)$ will commute with the operators $A(x_{i+1}), \ldots, A(x_{n+1})$. In the limit the function (25) takes the form

$$\langle 0 | T(x_1, x_2, \ldots, x_i) | 0 \rangle \langle 0 | T(x_{i+1}, \ldots, x_{n+1}) 0 \rangle. \quad (26)$$

With this, we have proved the hypothesis of Freese[1] about the asymptotic behavior of the function (25).

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