STABILITY CONDITIONS ON THE ELECTRON DISTRIBUTION FUNCTION FOR A PLASMA

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General conditions for the stability of the electron distribution function for a plasma with respect to high-frequency plasma oscillations are deduced neglecting collisions. Free plasma and plasma immersed in a constant, uniform electric or magnetic field are considered.

1. It is known that the mean free path of particles in a plasma and the equilibration time increase with rising temperature. Consequently, in investigating a high-temperature plasma (as well as a very rarefied one) one has to deal with particle distribution functions that may differ appreciably from the equilibrium functions. Under these circumstances the question arises as to the stability of such distributions with respect to small oscillations in the field and charge density within the plasma. The present paper is concerned with an analysis of the stability of the electron distribution function with respect to plasma oscillations.

The nature of the field oscillations created when a plasma is disturbed depends, of course, on the unperturbed electron distribution function. For example, as Landau has shown,[1] in the case of a Maxwell distribution the field oscillations will be damped. But if a delta-function like term corresponding to the passage of an electron beam through the plasma is added to the Maxwell distribution, then the field oscillations and the perturbation in the original distribution function increase in time for any velocity of the beam, i.e., the plasma plus beam system is unstable (Akhiezer and Faĭnberg[2]).

More general statements are also true, namely, that any distribution function, which is an arbitrary even function of the velocity with a single maximum at zero is stable, while the super-position of an arbitrary even function and a delta-like term is unstable.[3,4]

2. To establish the general conditions for the stability of the electron distribution function one need only study the behavior of the individual spatial Fourier components of both the potential φ and the perturbation f in the original electron distribution function $F_0(\mathbf{v})$ ($\mathbf{v}$ being the electron velocity). The behavior of these functions as $t \to \infty$ (t being the time) is determined by the singular points of their Laplace transforms φ_p and f_p (here $p = i\omega$ and ω is the complex frequency).

First we shall consider a free plasma, i.e., one not under the influence of external fields. In this case φ_p and f_p are related by[1]

$$f_p(u) = (p + iku)^{-1} \{g(u) + ikem^{-1}\varphi_p f_0'(u)\}, \qquad (1)$$

where u is the electron velocity parallel to the wave vector $\mathbf{k}$, $f_0(u)$ is the initial distribution function for u, i.e., $f_0(u) = \int F_0(\mathbf{v})\,d\mathbf{v}_\perp$ ($\mathbf{v}_\perp$ being the electron velocity perpendicular to $\mathbf{k}$), and g is the initial value for $f(u,t)$.

The poles of φ_p are the roots of the equation[1]

$$\frac{i\omega_0^2}{k}\int\limits_C \frac{f_0'(u)\,du}{p + iku} = 1, \qquad (2)$$

where ω_0 is the plasma frequency and the integration is along a line parallel to the real axis and below all the singularities of the integrand.

If all the roots of this equation (with respect to p) lie in the left half-plane, then $\varphi(t)$ will tend to zero as $t \to \infty$. On the other hand, the distribution function will undergo undamped fluctuations of constant amplitude with a frequency ku, which depend on the electron velocity (this behavior of the distribution function is due to the fact that $f_p(u)$ has a purely imaginary pole $p = -iku$ [see Eq. (1)]. On the other hand, if at least one of the roots of Eq. (2) lies in the right half-plane, then $\varphi(t)$ and $f(u,t)$ will increase indefinitely with time and the initial distribution $f_0(u)$ will be unstable.

Thus, the necessary and sufficient condition for the stability of the distribution function $F_0(\mathbf{v})$ consists in there being no roots to the equation

$$G(s) \equiv \int\limits_{-\infty}^{\infty} \frac{f_0'(u)\,du}{u - s} = \frac{k^2}{\omega_0^2}, \quad s = \frac{ip}{k} \qquad (3)$$

in the upper half-plane s for arbitrary k $(k > 0)$. If no such roots exist, then the function $w = G(s)$, which we shall assume to be finite, maps the upper half plane of s into an area D which does not intersect the positive half-axis of w. This area is bounded by some curve K traced by the point w as s moves along the real axis. Thus this curve is situated in a finite part of the w plane and, because of the principle of maximum modulus, encloses an area D. Therefore, the curve K does not intersect the positive half-axis.

In the case where Eq. (3) does have roots in the upper half plane, then the area D and with it the curve K intersect the positive half-axis.

This intersection means that for some real value of s the value of

$$w = \int_{-\infty}^{\infty} \frac{f_0'(u)du}{u-s} + \pi i f_0'(s) \tag{4}$$

is positive, i.e.,

$$f_0'(s) = 0, \qquad \int_{-\infty}^{\infty} \frac{f_0'(u)du}{u-s} > 0 .$$

The first of these conditions is fulfilled at the extrema of $f_0(u)$, i.e., at $s = u_1, u_2, \ldots, u_n$. Therefore, for the distribution function to be unstable it is necessary and sufficient that at least one of the following inequalities be satisfied*

$$\int_{-\infty}^{\infty} \frac{f_0'(u)\,du}{u-u_j} > 0, \quad j = 1, 2, \ldots, n . \tag{5}$$

It is easy to see that this instability criterion need be verified only for the minimum points of $f_0(u)$. Actually, when s moves along the real axis from $-\infty$ to $+\infty$, the upper half-plane of s remains on the left. Therefore, when w moves along curve K, the area D also remains on the left. This means that w either does not intersect the positive semi-axis at all, or intersects it at least once while passing from the lower to the upper half plane. The derivative $f_0'(s)$ passes from negative to positive values at the same time, which is possible only in the vicinity of the minimum for $f_0(u)$.

Therefore, the stability criterion for $f_0(u)$ has the form

$$\int_{-\infty}^{\infty} \frac{f_0'(u)\,du}{u-u_j} < 0, \quad f_0'(u_j) = 0, \quad f_0''(u_j) > 0. \tag{6}$$

3. It follows directly from Eq. (6) that a distribution function with only one maximum is stable.†

If the distribution function $f_0(u)$ has two maxima, one of which is fairly pronounced, it will be unstable. (This is the case when an electron beam with a small velocity dispersion passes through a plasma.)

To prove this we separate the distribution function into two terms, $f_0(u) = f_1(u) + f_2(u)$, with $f_1(u)$ coinciding with $f_0(u)$ outside the interval (u_2, u_2') and equal to $f_0(u_2)$ inside this interval [u_2 is the minimum of $f_0(u)$ lying between the maxima u_1 and u_3 and u_2' is the point nearest u_2 at which $f_0(u_2') = f_0(u_2)$]. It is evident that the contribution of $f_2(u)$ to the integral in (5) will be positive and rather large provided that u_2' is close to u_2. In fact, integration by parts yields

$$\int_{-\infty}^{\infty} \frac{f_2'(u)\,du}{u-u_2} > \frac{\nu}{(u_2'-u_2)^2}, \qquad \nu = \int_{u_2}^{u_2'} f_2(u)\,du > 0 .$$

Therefore, the integral given by (5) will be positive at the minimum point $u = u_2$ if $u_2' - u_2$ is sufficiently small. Consequently, $f_0(u)$ is unstable.

4. We now demonstrate that an arbitrary, spherically symmetrical distribution function $F_0(|\mathbf{v}|)$, which never becomes zero is stable.*

Noting that

$$f_0(u) = \int F_0(\mathbf{v})\,d\mathbf{v}_\perp = 2\pi \int_0^{\infty} F_0\left(\sqrt{u^2 + v_\perp^2}\right) v_\perp dv_\perp ,$$

we have

$$f_0'(u) = -2\pi u F_0(|u|) .$$

Therefore, Eq. (3) assumes the form

$$2\pi \int_{-\infty}^{\infty} \frac{uF_0(|u|)}{s-u}\,du = \frac{k^2}{\omega_0^2}, \tag{7}$$

and hence

$$2\pi \int_{-\infty}^{\infty} \frac{uF_0(|u|)}{s-u}\,du - 2\pi^2 is F_0(|s|) = \frac{k^2}{\omega_0^2}. \tag{8}$$

The imaginary part of this expression becomes zero when $s = 0$. Therefore, the stability condition requires that

$$-\int_{-\infty}^{\infty} F_0(|u|)\,du < 0,$$

which is always satisfied.

5. We now estimate the growth rate of the oscillations, i.e., the magnitude of $\mathrm{Im}\,\omega$ for the unstable distribution function, assuming that this quantity is sufficiently small. In this connection it is obvious that the curve K intersects the real half-axis of the plane $w \equiv G(s)$ at two neighboring points k_0^2/ω_0^2

*This criterion has been established independently by Noerdlinger,[5] Penrose,[6] and ourselves.

†This stability condition was obtained by Auer.[7]

*Here the distribution function can have an arbitrary number of maxima and minima. The stability of a spherically symmetrical distribution function with a single maximum has been demonstrated by Penrose.[6]

and k_1^2/ω_0^2 and that $G(s)$ can be expanded in a Taylor series in the vicinity of $s = s_0$ [where $G(s_0) = k_0^2/\omega_0^2$ and $f_0'(s_0) = 0$]. Thus we obtain, correct to quadratic terms,

$$(s - s_0)^2 + \frac{2G'(s_0)}{G''(s_0)}(s - s_0) - \frac{2(k^2 - k_0^2)}{\omega_0^2 G''(s_0)} = 0,$$

and consequently the maximum value of $\mathrm{Im}\,\omega = k\,\mathrm{Im}\,s$ is

$$\max \mathrm{Im}\,\omega = \max_{k_0 \leqslant k \leqslant k_1} k\,\mathrm{Im}\left\{-\frac{G'(s_0)}{G''(s_0)} \pm \left(\left[\frac{G'(s_0)}{G''(s_0)}\right]^2 + \frac{2(k^2 - k_0^2)}{\omega_0^2 G''(s_0)}\right)^{1/2}\right\}. \quad (9)$$

In deriving the instability criterion for the distribution function we have disregarded collisions among particles. Obviously these collisions hinder the development of the instability. Hence, the e-folding frequency of the oscillations, $\mathrm{Im}\,\omega$, must exceed a minimum value equal to the effective collision frequency in order for the instability actually to occur.

6. One can raise the question as to a general form for all the stable $f_0(u)$ distribution functions. Taken together, these functions allow the following representation. If $g(\xi)$ is the Fourier component of $f_0'(u)$, i.e.,

$$f_0'(u) = \int_{-\infty}^{\infty} g(\xi)\, e^{i\xi u}\, d\xi,$$

and if the distribution function is stable, then $g(\xi)$ can be represented in the form

$$g(\xi) = -\int_0^{\xi} \psi(\xi - \xi')\,\psi(\xi')\,d\xi', \quad (10)$$

where

$$\psi(\xi) = \int_{-\infty}^{\infty} e^{-i\xi\lambda}\, d\sigma(\lambda)$$

and $\sigma(\lambda)$ is an arbitrary, continuous, non-decreasing, bounded function.* For every such function there corresponds a definite stable distribution function.

7. We now proceed to examine the conditions for the stability of the distribution function for electrons in a plasma immersed in constant, uniform magnetic field **H**. Here we confine our investigation to plasma waves for which the electric field is longitudinal. The dispersion equation for these waves has the form[8]

$$\varepsilon_{11}\sin^2\theta + \varepsilon_{33}\cos^2\theta + 2\varepsilon_{13}\cos\theta\sin\theta = 0, \quad (11)$$

*The representation given by (10) was found by B. Ya. Levin (see reference 3).

where $\epsilon_{ik}(\omega, k)$ is the dielectric permittivity tensor for a plasma in a magnetic field and θ is the angle between **k** and **H**. These quantities are complicated functionals of the distribution function $F_0(\mathbf{v})$ (see Kitsenko and Stepanov[9]).

We assume that the plasma waves are long enough and the magnetic field H is strong enough for the inequality $k\bar{v}_\perp/\omega_H \ll 1$ to hold, here $\bar{v}_\perp$ is the mean value of the electron velocity perpendicular to **H** and $\omega_H = eH/mc$ is the electron gyrofrequency. In this case (11) assumes the form

$$1 - \frac{\omega_0^2\cos^2\theta}{\chi}\int_{-\infty}^{\infty}\frac{f_0'(u)du}{\chi u - \omega} + \frac{\omega_0^2\sin^2\theta}{2\omega_H} \times \int_{-\infty}^{\infty}\left(\frac{1}{\chi u - \omega + \omega_H} - \frac{1}{\chi u - \omega - \omega_H}\right) f_0(u)\,du = 0, \quad (12)$$

where $f_0(u)$ is the distribution of electron velocities parallel to **H** and $\chi = |k\cos\theta|$ is the component of **k** parallel to **H**.

Integrating Eq. (12) by parts one obtains

$$G_H(s) \equiv \int_{-\infty}^{\infty}\left(\frac{\cos^2\theta}{u - s} + \frac{\sin^2\theta}{2s_H}\ln\frac{u - s + s_H}{u - s - s_H}\right) f_0'(u)\,du = \frac{\chi^2}{\omega_0^2}, \quad (13)$$

where

$$s_H = |\omega_H|/\chi.$$

The necessary and sufficient condition for the stability of the $f_0(u)$ distribution function is that the roots of Eq. (13) do not lie in the upper half-plane. Otherwise, as was shown in Sec. 2, the function $w \equiv G(s)$ maps the real axis of s into a curve K that intersects the positive half-axis (here the passage along curve K from the lower to the upper half-plane corresponds to an increase in s).

If s is real, then the real and imaginary parts of $G_H(s)$ are

$$\mathrm{Re}\,G_H(s) = \cos^2\theta \int_{-\infty}^{\infty}\frac{f_0'(u)\,du}{u - s} + \frac{\sin^2\theta}{2s_H}\int_{-\infty}^{\infty} f_0'(u)\ln\left|\frac{u - s + s_H}{u - s - s_H}\right| du,$$

$$\mathrm{Im}\,G_H(s) = \pi\cos^2\theta f_0'(s) + \pi\frac{\sin^2\theta}{2s_H}\int_{s-s_H}^{s+s_H} f_0'(u)\,du.$$

The distribution function will be stable if the real part of $G_H(s)$ is negative for all the values of s for which $\mathrm{Im}\,G_H(s) = 0$, i.e.,

$$f_0'(u_j)\cos^2\theta + (2s_H)^{-1}[f_0(u_j + s_H) - f_0(u_j - s_H)]\sin^2\theta = 0,$$

$$\cos^2\theta \int_{-\infty}^{\infty}\frac{f_0'(u)\,du}{u - u_j} + \frac{\sin^2\theta}{2s_H}\int_{-\infty}^{\infty} f_0'(u)\ln\left|\frac{u - s + s_H}{u - s - s_H}\right| du < 0. \quad (14)$$

We note that when this criterion is used, as was explained in Sec. 2, one need take into account only the roots of Eq. (14) for which

$$\int_{u_j}^{s} \operatorname{Im} G_H(s)\, ds = \frac{\pi \sin^2\theta}{2s_H} \int_{s-s_H}^{s+s_H} f_0(u)\, du + \pi \cos^2\theta\, [f_0(s) - f_0(u_j)]$$

has a minimum.

When $\theta \to 0$ and $s_H \to 0$ or $s_H \to 0$, the instability condition (14) coincides with the instability condition in the absence of a magnetic field. Therefore, generally speaking, a magnetic field restricts the class of stable functions.

8. We now show that an even distribution function $f_0(u)$ with a single maximum (at $u = 0$) is stable. In this case Eq. (14) has a single root $u_1 = 0$. Actually, for $u_1 > 0$ the inequality

$$f_0'(u_1) < 0, \quad |u_1 + s_H| > |u_1 - s_H|,$$

is satisfied so that $f_0(u_1 + s_H) < f_0(u_1 - s_H)$, and Eq. (14) can have no solution for $u_1 > 0$. Because $f_0(u)$ is even, this statement holds true even when $u_1 < 0$. The stability condition now assumes the form

$$\cos^2\theta \int_{-\infty}^{\infty} \frac{f_0''(u)}{u}\, du + \frac{\sin^2\theta}{2s_H} \int_{-\infty}^{\infty} f_0'(u) \ln\left|\frac{u + s_H}{u - s_H}\right| du < 0.$$

Obviously this inequality is satisfied, since the integrands are negative for arbitrary u.

9. In conclusion we derive the stability condition for a plasma located in a constant, uniform electric field $\mathbf{E}_0$. Just as in the case of the magnetic field, we confine ourselves to longitudinal plasma oscillations and assume further that $\mathbf{E}_0$ is sufficiently weak.

Beginning with the equations

$$\frac{\partial F}{\partial t} + \mathbf{v}\,\mathrm{grad}\, F + \frac{e}{m}(\mathbf{E}_0 + \mathbf{E}) \frac{\partial F}{\partial \mathbf{v}} + J\{E\} = 0, \tag{15}$$

$$\mathrm{div}\, \mathbf{E} = 4\pi e n_0 \left(\int F\, d\mathbf{v} - 1\right), \tag{16}$$

where n_0 is the ion density, $J\{E\}$ the collision integral, and $\mathbf{E}$ the electric field due to plasma oscillations, one can derive[10] the following dispersion equation for high-frequency plasma oscillations for a sufficiently weak field $\mathbf{E}_0$

$$\int_{-\infty}^{\infty} \frac{f_0'(u)\, du}{u - s} + \frac{ieE_0}{mk} \cos\theta \int_{-\infty}^{\infty} \frac{du}{u - s} \frac{d}{du}\left[\frac{f_0'(u)}{u - s}\right] = \frac{k^2}{\omega_0^2}, \tag{17}$$

where $f_0(u)$ is the initial electron distribution function and θ is the angle between $\mathbf{k}$ and $\mathbf{E}_0$. In deriving this equation we have neglected the collision integral, which, however, is implicitly contained in f_0, whose form depends on $J\{f_0\}$.

Integrating the second term in Eq. (17) by parts, separating the real and imaginary terms and making use of the method explained in Sec. 2, we obtain the following condition for the stability of the distribution function $f_0(u)$

$$\int_{-\infty}^{\infty} \frac{f_0'(u)\, du}{u - u_j} - \frac{\pi e E_0}{4mk} \cos\theta\, f_0'''(u_j) < 0, \tag{18}$$

where u_j are the roots of the equation

$$f_0'(u_j) + \frac{eE_0}{2\pi mk} \cos\theta \int_{-\infty}^{\infty} \frac{f_0'''(u)}{u - u_j}\, du = 0; \quad e < 0.$$

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153