CHANGE IN THE MOMENTA OF CHARGES COLLIDING IN A MAGNETIC FIELD

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Scattering of two charged particles in a homogeneous and time-independent magnetic field is examined under the assumption of weak Coulomb interaction. The change in the particle momenta in scattering of the particles is calculated. It is shown that for two like charges the momentum change decreases exponentially with increase of the impact parameter. Because of Larmor resonance, the momentum change of oppositely charged particles decreases with increase in the impact parameter according to a power law. The magnetic field screens the interaction of the charged particles during the scattering. Equivalent spherically symmetric potentials constitute a graphic interpretation of this type of effect of the magnetic field. In the case of like charges, these potentials would be of the Debye type; in the case of opposite charges, they would be of the power type.

THE changes in momenta (and coordinates) of charged particles colliding in a magnetic field were reduced to quadratures in the work of E. M. Lifshitz[1] under the assumption of a weak Coulomb interaction. However, the author limited himself to a certain qualitative analysis of these integrals. The present research is a further development of the work of E. M. Lifshitz and contains a detailed analysis of the change of momenta of charges colliding in a magnetic field.

We shall therefore consider the collision of two charged particles e_k, m_k $(k = 1, 2)$ in a homogeneous and time-independent magnetic field $\mathbf{H}$. Initially, we shall assume that the particles approach each other along the magnetic field without a Coulomb interaction. At some instant of time $(t = 0)$ they lie in the x, y plane, which is perpendicular to the magnetic field. We shall represent the radius vector of the particles at this instant of time in the following fashion: $\mathbf{r}_k = \mathbf{Q}_k + \boldsymbol{\rho}_k$. The quantity $\boldsymbol{\rho}_k = \mathbf{h} \times \mathbf{p}_k^{\perp}/m_k\omega_k$ is the Larmor radius vector of the particle, $\mathbf{p}_k$ is the component of momentum perpendicular to the magnetic field, $\omega_k = e_kH/m_kc$ is the Larmor frequency, h is the unit vector along the magnetic field. The vector $\mathbf{Q}_{\perp} = \mathbf{Q}_1 - \mathbf{Q}_2$ is known as the impact parameter of the collision.

We now take the Coulomb interaction of the particles into consideration. In accordance with the work of E. M. Lifshitz just cited, the increase in the momenta of the particles in the collision can be written in the form

$$\Delta\left(p_{kx} + ip_{ky}^{\perp}\right) = -\int_{-\infty}^{\infty} dt\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\Phi_t e^{-i\omega_k},$$

$$\Delta p_k^{\parallel} = -\int_{-\infty}^{\infty} dt\,\frac{\partial\Phi_t}{\partial z}. \tag{1}$$

The formulas given here were obtained by perturbation theory with respect to the Coulomb interaction potential Φ. We use the notation $\Phi_t = S_{-t}\Phi(|\mathbf{r}_1 - \mathbf{r}_2|)$, where S_t is the operator of motion of the charges in the magnetic field without interaction,[2] $p_k^{\parallel}$ is the component of the momentum of the particles along the magnetic field.

We proceed to the calculation of the integrals (1). First, we consider the special case in which $\rho_1 = \rho_2 = 0$ or $\omega_1 = \omega_2$ and $\boldsymbol{\rho}_1 = \boldsymbol{\rho}_2$. In this case, the quadratures are calculated in finite form. The following formulas are obtained for the increases in the momenta of the particles:

$$\Delta\mathbf{p}_k = \pm\frac{2e_1e_2\mathbf{R}}{|U|R^2}\frac{R}{|\rho_k^{\parallel}|}K_1\left(\frac{R}{|\rho_k^{\parallel}|}\right). \tag{2}$$

Here,

$$U = p_1^{\parallel}/m_1 - p_2^{\parallel}/m_2, \qquad \rho_k^{\parallel} = U/\omega_k, \qquad \mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2,$$

K_1 is a MacDonald function;[3] the plus sign holds for $k = 1$, the minus sign for $k = 2$.

In accord with (2), the increases in the momenta for $R \gg |\rho_k^{\parallel}|$ are exponentially small quantities:

$$\Delta\mathbf{p}_k = \pm\frac{2e_1e_2\mathbf{R}}{|U|R^2}\left(\frac{\pi R}{2|\rho_k^{\parallel}|}\right)^{1/2}\exp\left(-\frac{R}{|\rho_k^{\parallel}|}\right). \tag{3}$$

We return to the calculation of the integrals (1) in the general case. The increases in the momenta of the particles are conveniently written in the form

$$\Delta\mathbf{p}_k = \Delta p_k^{\parallel}\mathbf{h} + \Delta p_k^{\perp}(\mathbf{p}_k^{\perp}/p_k^{\perp}) + \Delta' p_k^{\perp}[\mathbf{h}\mathbf{p}_k^{\perp}]/p_k, \quad (4)^*$$

where $\Delta p_k^{\perp}$ is the increase in the absolute value of the perpendicular component of the momentum, $\Delta' p_k^{\perp}$ is the corresponding change in the direction. We now introduce the phases α_k of the vectors $\mathbf{p}_k^{\perp}$ at the time $t = 0$, and expand the quadratures (1) in a double Fourier series in these phases. The Fourier expansion leads to the following formulas:

$$\Delta p_1^{\parallel} = \frac{2e_1e_2}{|U|}\sum_{n_1,\,n_2=-\infty}^{\infty}\widetilde{k}_{n_1n_2}A_{n_1n_2}\sin(n_1\bar{\alpha}_1 + n_2\bar{\alpha}_2),$$

$$\Delta p_2^{\parallel} = -\Delta p_1^{\parallel},$$

$$\Delta p_k^{\perp} = -\frac{2e_1e_2}{|U|\rho_k}\sum_{n_1,\,n_2=-\infty}^{\infty} n_k A_{n_1n_2}\sin(n_1\bar{\alpha}_1 + n_2\bar{\alpha}_2). \quad (5)$$

We used the notation

$$\bar{\alpha}_1 = \alpha_1 - (\pi/2), \qquad \bar{\alpha}_2 = (\alpha_1 \pm \pi) - (\pi/2),$$

$$\widetilde{k}_{n_1n_2} = (n_1\omega_1 + n_2\omega_2)/U.$$

The phases α_k are measured from the x axis, along which the vector $Q_\perp$ is directed. The amplitudes $A_{n_1n_2}$ are equal to

$$A_{n_1n_2} = \int_0^{\infty}\frac{k_\perp dk_\perp}{k_\perp^2 + \widetilde{k}_{n_1n_2}^2} J_{n_1}(k_\perp\rho_1)\, J_{n_2}(k_\perp\rho_2)\, J_{n_1+n_2}(k_\perp Q_\perp), \quad (6)$$

where J_n is a Bessel function.

We shall limit ourselves below to a consideration of the case in which the distance between the centers of the Larmor circles of the colliding particles at the time $t = 0$ is larger than the sum of their Larmor radii; $Q_\perp > |\rho_1| + |\rho_2|$. In this case (see Watson[3])

$$A_{n_1n_2} = I_{n_1}(|\widetilde{k}_{n_1n_2}|\,\rho_1)\, I_{n_2}(|\widetilde{k}_{n_1n_2}|\,\rho_2)\, K_{n_1+n_2}(|\widetilde{k}_{n_1n_2}|\,Q_\perp), \quad (7)$$

where I_n is a Bessel function of purely imaginary argument, K_n is a MacDonald function.

We now return to analysis of the amplitude $A_{n_1n_2}$, which is represented by Eq. (7). Taking into account the behavior of the MacDonald functions at large and small values of the argument, we can conclude that the amplitudes are maxima in order of magnitude when $n_1\omega_1 + n_2\omega_2 = 0$ (we assume the ratio ω_1/ω_2 to be rational). The corresponding harmonics in the expansions (5) will be called resonant.† The amplitudes of the resonant harmonics are equal to

$$A_{n_1n_2} = \frac{(|n_1| + |n_2| - 1)!}{2\,|n_1|!\,|n_2|!}\left(\frac{\rho_1}{Q_\perp}\right)^{|n_1|}\left(\frac{\rho_2}{Q_\perp}\right)^{|n_2|}, \quad (8)$$

if the signs of the charges of the corresponding particles are different, and are equal to 0 if the signs of the particles are the same.

We now go on to the amplitudes of the non-resonant harmonics. For their estimate, it is advantageous to make use of the asymptotic function $K_n(nx)$ for large values of the index n:

$$K_n(nx) \sim (\pi/2n)^{1/2}(x^2+1)^{-1/4}[x^{-1} + (x^{-2}+1)^{1/2}]^n$$
$$\times\exp[-n(x^2+1)^{1/2}]\,[1 + a(x)/n + \ldots],$$
$$|a(x)| \leqslant \tfrac{1}{12}. \quad (9)$$

This formula can be obtained by the application of the Laplace method for the asymptotic estimate of integrals[4] to the integral representation of the MacDonald function:[3]

$$K_n(nx) = \frac{1}{2}\int_{-\infty}^{\infty}\exp\{-n(x\,\mathrm{ch}\,t - t)\}\,dt. \quad (10)^*$$

Without going into details, we write down the result of the investigation: if $Q_\perp \gg |\rho_1| + |\rho_2|$ and $Q_\perp \gg |\rho_k^{\parallel}|$, then the amplitudes of the non-resonant harmonics are exponentially small [see Eq. (14) below]. We note that, in accord with (5), the resonant harmonics do not make a contribution to the increase in the components of the momenta of the particles parallel to the magnetic field.

We now sum up. We come to the conclusion that the increases in the components of the momenta of particles perpendicular to the magnetic field, for large values of the impact collision parameter $Q_\perp$, are exponentially small if the signs of the charges are identical, and fall off with increase in the impact parameter according to a power law if the signs of the charges are different. Changes in the components of momenta parallel to the magnetic field are exponentially small.

It was noted above that the amplitudes of the resonant harmonics differ from 0 only in the case of differently charged particles. The latter circumstance has the following physical meaning: if a time periodic force with frequency ν acts on a charge which is rotating in a magnetic field with its own Larmor frequency ω, then resonance takes place for $\nu + \omega = 0$ (and not for $|\nu| = |\omega|$). It is of interest to consider in particular the case in which the specific charges of the colliding particles are identical: $e_1/m_1 = e_2/m_2$. In this case, the law of conservation of momentum is satisfied, which can be written in the form $\Delta\mathbf{p}_1 = \Delta\mathbf{p}$, $\Delta\mathbf{p}_2 = -\Delta\mathbf{p}$, where $\mathbf{p} = (m_2\mathbf{p}_1 - m_1\mathbf{p}_2)/(m_1 + m_2)$ = momentum of the relative motion of the particles. For the latter, a formula is valid which is analogous to (4) in which

*$[\mathbf{h}\mathbf{p}_k^{\perp}] = \mathbf{h} \times \mathbf{p}_k^{\perp}$.

†The idea of the possibility of resonance in the scattering of charges in a magnetic field belongs to A. I. Morozov.

*ch = cosh.

$$\Delta p^{\parallel} = \frac{4e_1e_2}{|U|\rho^{\parallel}} \sum_{n_1,n_2=-\infty}^{\infty} \left(\frac{\rho}{|\rho|}\right)^n a_n(\xi,\eta)\sin n\bar{\alpha},$$

$$\Delta p^{\perp} = -\rho^{\parallel}\Delta p^{\parallel}/\rho,$$

$$\Delta' p^{\perp} = -\frac{2e_1e_2}{|U||\rho^{\parallel}|}\Big[-\frac{\rho}{|\rho|}\frac{\theta(\xi<\eta)}{\eta} + \sum_{n=1}^{\infty}\left(\frac{\rho}{|\rho|}\right)^{n-1} c_n(\xi,\eta)\cos n\bar{\alpha}\Big]. \quad (11)$$

Here

$$\rho^{\parallel} = p_k^{\parallel}, \qquad \rho = |(p_1^{\perp}/m_1) - (p_2^{\perp}/m_2)|/\omega_k,$$

$$\xi = Q_{\perp}/|\rho^{\parallel}|, \qquad \eta = |\rho|/|\rho^{\parallel}|, \qquad \bar{\alpha} = \alpha - (\pi/2),$$

where α is the phase of the vector $\mathbf{p}^{\perp}$. The function $\theta(\xi < \eta) = 1$ for $\xi < \eta$, and equal to 0 for $\xi > \eta$. The amplitudes a_n and ξ_n are equal to the following:

$$a_n(\xi,\eta) = nI_n(n\eta)K_n(n\xi), \qquad \xi > \eta,$$

$$c_n(\xi,\eta) = 2nI_n'(n\eta)K_n(n\xi), \qquad \xi > \eta. \quad (12)$$

As investigation shows, for $\xi \gg \eta$ and $\xi \gg 1$, the series of amplitudes a_n and ξ_n tend to zero as $n \to \infty$ no more slowly than the geometric progression with base $\eta e^{-\xi}$. Therefore, limiting ourselves in (11) to the first harmonics, we get

$$\Delta p_k^{\parallel} \sim \pm P^{\parallel}\xi^{-1/2}e^{-\xi}, \qquad \Delta p_k^{\perp} \sim P_k^{\perp}\xi^{-1/2}e^{-\xi}, \quad (14)$$

where $P^{\parallel}$, $P_k^{\perp}$ are certain functions of α, $\alpha_k\eta$. Estimates analogous to (14) are given also in the cited paper by E. M. Lifshitz,[1] and in the work of Belyaev.[2]

Up to this point we have considered changes of momenta of particles in collisions. However, no less interest attaches to the consideration of the scattering angles of the particles. By the angle of the scattering of a particle χ_k we shall mean the change of angle between the direction of its momentum and the direction of its magnetic field. We shall regard as equivalent the spherically symmetric potential $\Pi_k(r)$ $(k = 1, 2)$ which, without participation of magnetic field with impact parameter $Q_{\perp}$, scatters the k-th particle through an angle χ_k. The equivalent potential for the given angle of scattering χ_k satisfies an integral equation of the form (see reference 5)

$$\chi_k = -\frac{2m_kQ_{\perp}}{p_k^2}\int_{Q_{\perp}}^{\infty}\Pi_k'(r)\frac{dr}{(r^2-Q_{\perp}^2)^{1/2}}. \quad (15)$$

If we consider the solution of this equation[6] for large values of r, we find that

$$\Pi_k(r) \sim (p_k^2|\rho^{\parallel}|/2m_k r)\exp(-r/|\rho^{\parallel}|) \quad (16)$$

for particles with equal specific charges (a similar formula holds for identically charged particles),

$$\Pi_k(r) \sim (p_k^2/2m_k)(\rho_1/r)^{M_1}(\rho_2/r)^{M_2} \quad (17)$$

for differently charged particles. In the latter formula, M_1 and M_2 are relatively prime natural numbers in which $|\omega_1|/|\omega_2| = M_2/M_1$. The appearance of the Debye potential (16) as an equivalent potential is a graphic physical interpretation of the screening action of the magnetic field[2] in the scattering of two charged particles. However, as follows from (17), the equivalent potential in the scattering of differently charged particles has the form of a power potential.

In conclusion, I take this opportunity to express my deep gratitude to A. I. Morozov for his numerous discussions and valuable observations in the completion of the present work.

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