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CONTRIBUTION TO THE THEORY OF PLASMA FLUCTUATIONS

A. I. AKHIEZER, I. A. AKHIEZER, and A. G. SITENKO

Physico-Technical Institute, Academy of Sciences, Ukrainian S.S.R.; Khar'kov State University

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Spectral distributions and correlation functions are derived for various fluctuating quantities (electron and ion densities, electron and ion current densities, electric and magnetic fields, particle distribution functions). The possible difference between the electron and ion temperatures is taken into account. The cross sections for the scattering of electromagnetic waves on plasma fluctuations are determined. It is shown that the possibility of propagation of Langmuir oscillations as well as of high-frequency sound waves (in a nonisothermal plasma) leads to the appearance of satellites in the spectrum of the scattered radiation. When a constant magnetic field is applied, the scattered-radiation spectrum contains satellites due to the possibility of propagation of Langmuir, Alfven, and magnetic-sound waves in the plasma.

1. A study of plasma fluctuations can be of interest to plasma physics. Since it yields directly the spectrum of the plasma oscillations, such a study can serve to determine many plasma parameters (density, temperature) and can probably help explain the role of oscillations in transport processes and in the establishment of equilibrium in the plasma. This paper deals with a theoretical determination of the spectral distributions and correlation functions of various fluctuating quantities (including the particle distribution function) and a determination of the cross section for the scattering of electromagnetic waves on fluctuations in a plasma without collisions. We consider here free plasma as well as plasma in a uniform magnetic field, and take into account the possible difference between the electron and ion temperatures.*

2. Fluctuations in a plasma which is in complete statistical equilibrium can be investigated in the general theory (Callen and Welton,[4] Leontovich and Rytov,[5 6] Landau and Lifshitz[7]) by using the known dielectric-constant tensor of the plasma.† In particular, the space-time Fourier components of the correlation functions of the current density $j(\mathbf{r}, t)$ can be determined from

$$\langle j_i j_j\rangle_{\mathbf{k}\omega} \equiv \int \langle j_i(\mathbf{r}, t)\, j_j(\mathbf{r}', t')\rangle e^{-i\mathbf{k}(\mathbf{r}-\mathbf{r}')+i\omega(t-t')}\, d\mathbf{r}\, dt = \frac{\hbar}{1-e^{-\hbar\omega/T}}\,\frac{1}{2i}(\alpha_{ij}-\alpha^*_{ji}), \tag{1}$$

where $\alpha_{ij}(\mathbf{k}, \omega)$ is the tensor relating the Fourier components of the current $j_i(\mathbf{k}, \omega)$ and the potential $\widetilde{A}_i(\mathbf{k}, \omega)$ of the external field:

$$j_i(\mathbf{k}, \omega) = c^{-1}\alpha_{ij}(\mathbf{k}, \omega)\widetilde{A}_j(\mathbf{k}, \omega)$$

(the symbol $\langle\ldots\rangle$ denotes averaging over the fluctuations). The tensor α_{ij}, as follows from Maxwell's material equations, is connected with the dielectric-constant tensor $\epsilon_{ij}(\mathbf{k}, \omega)$ by the relation

$$\alpha_{ij} = \tfrac{1}{2}\omega^2\{\Lambda^0_{ij} - \Lambda^{-1}\Lambda^0_{ik}\lambda_{kl}\Lambda^0_{lj}\}, \tag{2}$$

where

$$\Lambda^0_{ij} = \eta^2(k_ik_jk^{-2}-\delta_{ij})+\delta_{ij}, \quad \Lambda_{ij} = \Lambda^0_{ij}-\delta_{ij}+\varepsilon_{ij};$$
$$\Lambda^{-1}\lambda_{ij} = (\Lambda^{-1})_{ij}, \qquad \Lambda = \det\Lambda_{ij}, \qquad \eta = kc/\omega. \tag{2'}$$

We note that the equation $\Lambda(\mathbf{k}, \omega) = 0$ is the dispersion equation of the system. Consequently the spectral distribution of the fluctuations has sharp maxima near the natural frequencies of the system.*

*The particle-density fluctuations in a free plasma were determined in a paper by Salpeter,[1] with which we became acquainted after completing the present work (the results of[1] coincide with the ones we obtained by a different method). The scattering of electromagnetic waves by Langmuir plasma oscillations was considered earlier.[2] Scattering with small frequency variation in the presence of a magnetic field is treated by Dougherty and Farley.[3]

†Equations for the spatial correlation functions of particle systems with electromagnetic interaction are derived in the papers by Tolmachev, Tyablikov, and Klimontovich.[8-10] The correlation functions of microcurrents were calculated by Shafranov[11] from the laws of motion. The spatial dispersion as it affects the fluctuations was treated by Bass and Kaganov[12] and by Silin.[13]

*Relation (1) can be derived, according to Landau and Lifshitz,[7] from the expression of the rate of change of the plasma energy under the influence of the external field

$$\dot{U} = (2c)^{-1}\,\mathrm{Re}\sum_{\mathbf{k},\omega} i\omega j^*(\mathbf{k}, \omega)\widetilde{A}(\mathbf{k}, \omega).$$

The dielectric tensor constant of a free plasma has the form

$$\varepsilon_{ij}(\mathbf{k}, \omega) = k_i k_j k^{-2} \varepsilon_l(k, \omega) + (\delta_{ij} - k^{-2} k_i k_j)\, \varepsilon_t(k, \omega),$$

where ϵ_l and ϵ_t are the longitudinal and transverse dielectric constants of the plasma

$$\varepsilon_l = 1 + (ak)^{-2}\,[1 - \varphi(z) - \varphi(\mu z) + \tfrac{1}{2} i \sqrt{\pi} z\, (e^{-z^2} + \mu e^{-\mu^2 z^2})],$$

$$\varepsilon_t = 1 - 2\frac{\Omega^2}{\omega^2}\left[\varphi(z) - \frac{i}{2}\sqrt{\pi} z e^{-z^2}\right]; \quad \varphi(z) = z e^{-z^2}\int_0^z e^{t^2} dt \tag{3}$$

($z = \omega/ks$, $\mu^2 = M/m$, $a^2 = (8\pi e^2 n_0)^{-1} T$, $\Omega^2 = 4\pi e^2 n_0/m$, and $s^2 = 2T/m$; n_0 is the equilibrium density of the electrons, while m and M are the electron and ion masses (the plasma is assumed nondegenerate).

Substitution of (3) into (1) and (2) leads to the following expressions for the Fourier components of the correlators of the charge densities and of the transverse current in the plasma

$$\langle \rho^2 \rangle_{\mathbf{k}\omega} = \frac{T}{2\pi}\frac{k^2}{\omega}\frac{\operatorname{Im}\varepsilon_l}{|\varepsilon_l|^2} = \frac{T\,(ak)^3}{4\sqrt{\pi} as} \times \frac{e^{-z^2} + \mu e^{-\mu^2 z^2}}{[1 + (ak)^2 - \varphi(z) - \varphi(\mu z)]^2 + (\pi/4)\, z^2 (e^{-z^2} + \mu e^{-\mu^2 z^2})^2},$$

$$\langle j_t^2 \rangle_{\mathbf{k}\omega} = \frac{T}{2\pi}\,\omega\,(\eta^2 - 1)^2 \frac{\operatorname{Im}\varepsilon_t}{|\eta^2 - \varepsilon_t|^2} = \frac{T\omega}{\sqrt{\pi}}\left(\frac{\omega}{\Omega}\right)^2 \frac{(1-\eta^2)^2 z e^{-z^2}}{[\omega^2 (1-\eta^2)/\Omega^2 - 2\varphi(z)]^2 + \pi z^2 e^{-2z^2}}. \tag{4}$$

It follows from these expressions that when ka $\gtrsim 1$ the principal role in the fluctuations of ρ and j is played by the low-frequency oscillations. The frequency of the fluctuations increases with decreasing ka and at ka $\ll 1$ the only frequencies remaining in the spectrum are those close to the natural frequencies of the longitudinal (in the spectrum of ρ) and transverse (in the spectrum of j_t) oscillations of the plasma ($\omega \gg ks$):

$$\langle \rho^2 \rangle_{\mathbf{k}\omega} = \tfrac{1}{4} T k^2 \delta(\omega - \omega_p), \qquad \omega_p = \Omega\left(1 + \tfrac{3}{4} k^2 s^2\right),$$

$$\langle j_t^2 \rangle_{\mathbf{k}\omega} = \tfrac{1}{2} T \Omega^4 \omega^{-2} \delta(\omega - \omega_t), \quad \omega_t = \sqrt{k^2 c^2 + \Omega^2}. \tag{5}$$

Integrating $\langle \rho^2 \rangle_{\mathbf{k}\omega}$ with respect to the frequencies (which is most readily done with the aid of the Kramers-Kronig relations, see [7]), we derive the well known formula for the spatial Fourier component of the charge-density fluctuation

$$\langle \rho^2 \rangle_{\mathbf{k}} \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle \rho^2 \rangle_{\mathbf{k}\omega}\, d\omega = \frac{T}{4\pi a^2}\frac{(ak)^2}{1 + (ak)^2}. \tag{6}$$

From this we can then obtain the mean frequency of the plasma space-charge fluctuations

$$\overline{\omega^2} \equiv \left\{\int_{-\infty}^{\infty} \langle \rho^2 \rangle_{\mathbf{k}\omega}\, d\omega\right\}^{-1} \int_{-\infty}^{\infty} \omega^2 \langle \rho^2 \rangle_{\mathbf{k}\omega}\, d\omega = \Omega^2 + \tfrac{1}{2}(ks)^2.$$

Finally, (6) leads to the following expression for the correlation function of the charge density

$$\langle \rho(\mathbf{r}', t)\, \rho(\mathbf{r}'', t) \rangle = 2e^2 n_0 \left[\delta(\mathbf{r}) - \frac{1}{4\pi a^2}\frac{e^{-r/a}}{r}\right], \quad \mathbf{r} = \mathbf{r}' - \mathbf{r}''. \tag{7}$$

Analogously we can determine the correlation function for the density of the total current

$$\langle \mathbf{j}(\mathbf{r}', t)\, \mathbf{j}(\mathbf{r}'', t) \rangle = 3e^2 n_0 (T/m)\, \delta(\mathbf{r}). \tag{8}$$

The spectral distributions of the fluctuations of the electric and magnetic fields **E** and **H** are also expressed in terms of ϵ_l and ϵ_t:

$$\langle \mathbf{E}^2 \rangle_{\mathbf{k}\omega} = \frac{8\pi T}{\omega}\left(\frac{\operatorname{Im}\varepsilon_l}{|\varepsilon_l|^2} + 2\frac{\operatorname{Im}\varepsilon_t}{|\eta^2 - \varepsilon_t|^2}\right),$$

$$\langle \mathbf{H}^2 \rangle_{\mathbf{k}\omega} = \frac{16\pi T}{\omega}\,\eta^2 \frac{\operatorname{Im}\varepsilon_t}{|\eta^2 - \varepsilon_t|^2}. \tag{9}$$

Inserting (3) and integrating with respect to ω, we obtain

$$\langle \mathbf{E}^2 \rangle_{\mathbf{k}} = 8\pi T\left(1 + \frac{1}{2}\frac{1}{1 + a^2 k^2}\right), \qquad \langle \mathbf{H}^2 \rangle_{\mathbf{k}} = 8\pi T.$$

From this we readily determine the correlation functions for the fields

$$\langle \mathbf{E}(\mathbf{r}', t)\, \mathbf{E}(\mathbf{r}'', t) \rangle = 8\pi T\left[\delta(\mathbf{r}) + \frac{1}{8\pi a^2}\frac{e^{-r/a}}{r}\right],$$

$$\langle \mathbf{H}(\mathbf{r}', t)\, \mathbf{H}(\mathbf{r}'', t) \rangle = 8\pi T \delta(\mathbf{r}). \tag{10}$$

3. Using the well-known expression for the dielectric constant of a degenerate electron gas [14] we can use (1) to determine the spectral distribution of the fluctuations in an electron gas with $T \ll m v_0^2/2$ (v_0 — boundary velocity). In particular, we have for $\langle \rho^2 \rangle_{\mathbf{k}\omega}$

$$\langle \rho^2 \rangle_{\mathbf{k}\omega} = \frac{3}{4}\frac{\hbar \zeta k^2}{1 - e^{-\hbar\omega/T}}\left\{\frac{1}{2} z\theta(1 - |z|) \times \left[\left(\zeta + 1 - \frac{z}{2}\ln\frac{1+z}{1-z}\right)^2 + \left(\frac{\pi z}{2}\right)^2\right]^{-1} + \delta\left(\zeta + 1 - \frac{z}{2}\ln\left|\frac{z+1}{z-1}\right|\right)\operatorname{sign} z\right\},$$

$$\theta(z) = \begin{cases} 0, & z < 0 \\ 1, & z > 0 \end{cases}, \tag{11}$$

where $z = \omega/kv_0$ and $\zeta = (kv_0/\Omega)^2/2$.

4. The fluctuations in a plasma situated in a constant homogeneous magnetic field $\mathbf{H}_0$ are also determined by the general formula (1). (The components of the tensor ϵ_{ij} of a plasma in a magnetic field are known [15,16].)

If we disregard the thermal motion of the plasma particles ($\omega \gg ks$), then ϵ_{ij} has the form[17]

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_1 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix},$$

$$\varepsilon_1 = 1 - \frac{\Omega^2}{\omega^2 - \omega_H^2}, \quad \varepsilon_2 = \frac{\omega_H}{\omega} \frac{\Omega^2}{\omega^2 - \omega_H^2},$$

$$\varepsilon_3 = 1 - \frac{\Omega^2}{\omega^2}, \quad \omega_H = \frac{eH_0}{mc}. \tag{12}$$

In this case relation (1) becomes ($T \gg \hbar\omega$)

$$\langle j_i j_j \rangle_{\mathbf{k}\omega} = \tfrac{1}{4} T\omega \Lambda_{ik}^0 \lambda_{kl} \Lambda_{lj}^0 \delta\left(A(\eta^2 - \eta_1^2)(\eta^2 - \eta_2^2)\right),$$

$$A = \varepsilon_1 \sin^2\vartheta + \varepsilon_3 \cos^2\vartheta, \tag{13}$$

where $\eta_{1,2}$ are the refractive indices of the ordinary and extraordinary waves, and λ_{ij} is determined by formula (2′) (ϑ is the angle between $\mathbf{H}_0$ and $\mathbf{k}$).

Equating the argument of the δ-function to zero, we obtain the dispersion equation for the high-frequency plasma oscillations. The equation $A = 0$ determines here the frequencies of the Langmuir oscillations of the plasma in a magnetic field

$$\omega_\pm^2 = \tfrac{1}{2}(\Omega^2 + \omega_H^2) \pm \tfrac{1}{2}\left[(\Omega^2 + \omega_H^2)^2 - 4\Omega^2\omega_H^2 \cos^2\vartheta\right]^{1/2}.$$

With the aid of (13) we can find the spectral distribution of the charge-density fluctuations in a plasma at high frequencies ($\omega \gg ks$):

$$\langle \rho^2 \rangle_{\mathbf{k}\omega} = \frac{T}{4}\left(\frac{k\omega}{\Omega}\right)^2 \frac{(\omega^2 - \omega_H^2)^2}{\omega^4 \cos^2\vartheta + (\omega^2 - \omega_H^2)^2 \sin^2\vartheta} \{\delta(\omega - \omega_+) + \delta(\omega + \omega_+) + \delta(\omega - \omega_-) + \delta(\omega + \omega_-)\}. \tag{14}$$

5. We now proceed to determine separately the electron and ion density fluctuations, and also the fluctuations of the plasma-particle distribution functions, without assuming the electron and ion temperatures T^e and T^i to be equal. Since the energy exchange between the electrons and ions is much slower than exchange between like particles, we can regard a nonisothermal plasma as a quasi-equilibrium system and apply the general methods of fluctuation theory to the fluctuations produced in this plasma.

To determine the fluctuations of the particle distribution functions we must introduce into the kinetic equations that define these functions the random forces $y^{e,i}(\mathbf{v}, \mathbf{r}, t)$ (the indices e and i will henceforth denote electrons and ions):*

$$\left\{\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} + \frac{e}{m}\mathbf{E}\frac{\partial}{\partial \mathbf{v}}\right\} F^e(\mathbf{v}, \mathbf{r}, t) = -\frac{1}{\tau^e} f^e(\mathbf{v}, \mathbf{r}, t) + y^e(\mathbf{v}, \mathbf{r}, t),$$

$$\left\{\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} - \frac{e}{M}\mathbf{E}\frac{\partial}{\partial \mathbf{v}}\right\} F^i(\mathbf{v}, \mathbf{r}, t) = -\frac{1}{\tau^i} f^i(\mathbf{v}, \mathbf{r}, t) + y^i(\mathbf{v}, \mathbf{r}, t), \tag{15}$$

where $F_0^{e,i}$ are the Maxwellian distribution functions for the electrons and ions, $f^{e,i} = F^{e,i} - F_0^{e,i}$ are the deviations of the distribution functions from Maxwellian, and $\tau^{e,i}$ are the relaxation times, which will tend to infinity in the final results. For simplicity we consider first only longitudinal plasma oscillations; in this case

$$\operatorname{div}\mathbf{E} = 4\pi e\{\delta n^e - \delta n^i\} = 4\pi e\int (F^e - F^i)\, d\mathbf{v}, \quad \operatorname{rot}\mathbf{E} = 0. \tag{16*}$$

Taking the time derivative of the entropy of the electron and ion system (separately for specified values of the electron and ion energies and numbers), we obtain

$$\dot{S} = -\int d\mathbf{r}\, d\mathbf{v}\, (\dot{x}^e X^e + \dot{x}^i X^i),$$

$$\dot{x}^{e,i} = -\frac{1}{\tau^{e,i}} f^{e,i} + y^{e,i}, \qquad X^{e,i} = \frac{f^{e,i}}{F_0^{e,i}}.$$

Using further the method developed by Landau and Lifshitz[19] and by Abrikosov and Khalatnikov[18] we obtain the following expression for the mean values of the products of the random forces

$$\langle y^a(\mathbf{v}, \mathbf{r}, t)\, y^b(\mathbf{v}', \mathbf{r}', t')\rangle = \delta_{ab}\, 2(\tau^a)^{-1} F_0^a(\mathbf{v})\, \delta(\mathbf{v} - \mathbf{v}')\, \delta(\mathbf{r} - \mathbf{r}')\, \delta(t - t'), \tag{17}$$

where the indices a and b label the type of particle ($a, b \equiv e, i$). We now must use (15) and (16) to express the distribution functions and the various physical quantities defined by them in terms of the random forces, and average these forces with the aid of (17). For the Fourier components of the fluctuations of the electron and ion densities we obtain

$$\delta n^e(\mathbf{k}, \omega) = i\frac{k}{\omega}\frac{1}{\varepsilon(k, \omega)}\{Y_{k\omega}^e(1 + 4\pi\varkappa^i) + Y_{k\omega}^i 4\pi\varkappa^e\},$$

$$\delta n^i(\mathbf{k}, \omega) = i\frac{k}{\omega}\frac{1}{\varepsilon(k, \omega)}\{Y_{k\omega}^e 4\pi\varkappa^i + Y_{k\omega}^i(1 + 4\pi\varkappa^e)\};$$

$$Y_{k\omega}^a = \int \frac{\mathbf{k}\mathbf{v}}{k}\left(\omega - \mathbf{k}\mathbf{v} + \frac{i}{\tau^a}\right)^{-1} y^a(\mathbf{v}, \mathbf{k}, \omega)\, d\mathbf{v}, \tag{18}$$

where $\epsilon = 1 + 4\pi(\kappa^e + \kappa^i)$, where $\kappa^{e,i}$ are the electric susceptibilities of the electrons and ions:

$$\varkappa^a(k, \omega) = -\frac{e^2}{k^2}\frac{1}{T^a}\int F_0^a(v)\frac{\mathbf{k}\mathbf{v}}{\omega - \mathbf{k}\mathbf{v} + i0}\, d\mathbf{v}. \tag{19}$$

*This method is a generalization of the one employed by Abrikosov and Khalatnikov [18] to find the distribution-function fluctuations in an equilibrium Fermi system.

*rot **E** = curl **E**.

Averaging of Y leads, according to (17) and (19), to the expressions $e^2\langle Y^a Y^b\rangle_{k\omega} = 2\delta_{ab}\,\omega T^a\,\mathrm{Im}\,\kappa^a(k,\omega)$. Using next Eq. (18) we obtain ultimately*

$$e^2\langle|\delta n^e|^2\rangle_{k\omega} = \frac{2k^2}{\omega|\varepsilon|^2}\{T^e|1+4\pi\varkappa^i|^2\,\mathrm{Im}\,\varkappa^e + T^i|4\pi\varkappa^e|^2\,\mathrm{Im}\,\varkappa^i\},$$

$$e^2\langle|\delta n^i|^2\rangle_{k\omega} = \frac{2k^2}{\omega|\varepsilon|^2}\{T^e|4\pi\varkappa^i|^2\,\mathrm{Im}\,\varkappa^e + T^i|1+4\pi\varkappa^e|^2\,\mathrm{Im}\,\varkappa^i\},$$

$$e^2\langle\delta n^e\delta n^i\rangle_{k\omega} = e^2\langle\delta n^i\delta n^e\rangle^*_{k\omega} = \frac{2k^2}{\omega|\varepsilon|^2}\{T^e(1+4\pi\varkappa^i)(4\pi\varkappa^i)^*\,\mathrm{Im}\,\varkappa^e + T^i(1+4\pi\varkappa^e)^*(4\pi\varkappa^e)\,\mathrm{Im}\,\varkappa^i\}. \quad (20)$$

From these formulas we can readily obtain the spectral distribution of the charge-density fluctuations:

$$\langle\rho^2\rangle_{k\omega} \equiv e^2\langle|\delta n^e-\delta n^i|^2\rangle_{k\omega} = \frac{2k^2}{\omega|\varepsilon|^2}\,\mathrm{Im}\,\{T^e\varkappa^e + T^i\varkappa^i\}. \quad (21)$$

When $T^e = T^i$, this expression goes into the first expression in (4).

We note that in the region of the small phase velocities ($\omega/k \ll s^e$ and s^i) we have

$$\langle\rho^2\rangle_{k\omega} = \frac{2\sqrt{\pi}}{k}e^2 n_0\left(\frac{1}{s^e}+\frac{1}{s^i}\right)\left(1+\frac{1}{a^2k^2}\right)^{-1},$$

($a^{-1} = \sqrt{k_e^2+k_i^2}$, $k^2_{e,i} = 4\pi e^2 n_0/T^{e,i}$). If $s^e \gg \omega/k \gg s^i$ and $T^e \gg T^i$, then

$$\langle\rho^2\rangle_{k\omega} = \frac{1}{4}BT^e\frac{k^4}{k_e^2+k^2}\{\delta(\omega-\omega_s(k)) + \delta(\omega+\omega_s(k))\};\quad \omega_s(k) = \frac{\Omega_i}{\sqrt{k_e^2+k^2}}k,$$

where $\Omega_i^2 = 4\pi e^2 n_0/M$;

$$B = \left[1+\frac{s_e}{s_i}\frac{T^i}{T^e}\exp\left\{-\left(\frac{\Omega_i}{s_i}\right)^2\frac{1}{k_e^2+k^2}\right\}\right]\Big/\left[1+\frac{s_e}{s_i}\times\exp\left\{-\left(\frac{\Omega_i}{s_i}\right)^2\frac{1}{k_e^2+k^2}\right\}\right].$$

(In particular, if $\ln(T^e/T^i) \ll (\Omega_i/s_i)^2(k^2+k^2)^{-1}$, then $B = 1$.) We see that in a highly non-isothermal plasma with $T^e \gg T^i$ the correlation function of the charge density has a sharp maximum at $\omega = \pm\omega_s(k)$, corresponding to the possibility that specific sound oscillations propagate in the plasma.† At large phase velocities ($\omega/k \gg s^e, s^i$) the principal role in the charge-density fluctuations is played by electrons, and $\langle\rho^2\rangle_{k\omega}$ is determined by the formula (5) with $T = T^e$.

*Salpeter[1] derived the expression for $\langle|\delta n^e|^2\rangle_{k\omega}$ by a different method.

†These oscillations were investigated by Tonks and Langmuir[20] and by Gordeev.[21]

Let us express now the Fourier component of the distribution-function fluctuation in terms of the random forces

$$f^e(\mathbf{v}) = \frac{4\pi e^2}{k^2}\frac{1}{T^e}\frac{\mathbf{kv}F_0^e(v)}{\omega-\mathbf{kv}+i/\tau^e}(\delta n^e-\delta n^i) + \frac{i}{\omega-\mathbf{kv}+i/\tau^e}y^e(\mathbf{v}),$$

$$f^i(\mathbf{v}) = -\frac{4\pi e^2}{k^2}\frac{1}{T^i}\frac{\mathbf{kv}F_0^i(v)}{\omega-\mathbf{kv}+i/\tau^i}(\delta n^e-\delta n^i) + \frac{i}{\omega-\mathbf{kv}+i/\tau^i}y^i(\mathbf{v}).$$

Using (17), we obtain the distribution-function correlators*

$$\langle f^a(\mathbf{v})f^b(\mathbf{v}')\rangle_{k\omega} = 2\pi\delta_{ab}F_0^a(v)\,\delta(\mathbf{v}-\mathbf{v}')\,\delta(\omega-\mathbf{kv}) \pm 2\pi\cdot 4\pi e^2k^{-2}F_0^a(v)F_0^b(v')S^{ab}(\mathbf{v},\mathbf{v}'), \quad (22)$$

where the upper (or lower) sign is taken in the case of like (unlike) particles and

$$S^{ab} = \frac{1}{T^a}\frac{\mathbf{kv}}{\omega-\mathbf{kv}+i0}\frac{1}{\varepsilon}\delta(\omega-\mathbf{kv}') + \frac{1}{T^b}\frac{\mathbf{kv}'}{\omega-\mathbf{kv}'-i0}\frac{1}{\varepsilon^*}\delta(\omega-\mathbf{kv}) + \frac{4}{\omega}\frac{1}{T^aT^b}\frac{\mathbf{kv}}{\omega-\mathbf{kv}+i0}\frac{\mathbf{kv}'}{\omega-\mathbf{kv}'-i0}\frac{\mathrm{Im}\,(T^e\varkappa^e+T^i\varkappa^i)}{|\varepsilon|^2}.$$

6. We now generalize the obtained results to include a plasma in a constant homogeneous magnetic field $\mathbf{H}_0$. In this case the kinetic equations can be written (in Fourier components) as

$$-iG^a(\mathbf{v},\mathbf{k},\omega)f^a(\mathbf{v},\mathbf{k},\omega) \mp e\mathbf{E}\frac{\mathbf{v}}{T^a}F_0^a(v) = y^a(\mathbf{v},\mathbf{k},\omega),$$

$$G^e = \omega-\mathbf{kv}+i\frac{e[\mathbf{vH}_0]}{mc}\frac{\partial}{\partial\mathbf{v}},\qquad G^i = \omega-\mathbf{kv}-i\frac{e[\mathbf{vH}_0]}{Mc}\frac{\partial}{\partial\mathbf{v}}. \quad (23)†$$

We can use (23) to relate the fluctuations of the electron and ion currents with the random forces

$$j_i^a = \pm ieY_i^a - i\omega\varkappa_{ij}E_j,\qquad Y_i^a = \int v_i(G^a)^{-1}y^a\,d\mathbf{v}$$

[the upper sign pertains to the electrons ($a = e$) and the lower to the ions ($a = i$)]. Here $(G^a)^{-1}$ is the operator inverse to G^a, $\mathbf{E}$ is the fluctuation of the electric field, and κ_{ij}^a are the tensors of the electric susceptibility of the electrons and ions of the plasma in a magnetic field. Expressing $\mathbf{E}$ in terms of $j = j^e + j^i$ with the aid of Maxwell's equations we obtain

$$j_i^a = \pm ieY_i^a - 4\pi\varkappa_{ij}^aQ_{jl}j_l,$$

$$Q_{ij} = k_ik_jk^{-2} - (\eta^2-1)^{-1}(\delta_{ij}-k^{-2}k_ik_j). \quad (24)$$

*The fluctuations of the distribution function of a gas (without account of the self-consistent fields) was investigated by Kadomtsev.[25]

†$[\mathbf{vH}_0] = \mathbf{v}\times\mathbf{H}_0$.

(The field $\mathbf{E}$ is not assumed potential, for in general the longitudinal and transverse oscillations cannot be separated in the presence of a magnetic field.)

Introducing for simplicity of notation a (2×3)-dimensional current-density vector j_α, which combines the two currents j^e ($\alpha = 1, 2, 3$) and j^i ($\alpha = -1, -2, -3$) we rewrite (24) in the form

$$\mathfrak{M}_{\alpha\beta} j_\beta = ieY_\alpha, \quad j_\alpha = \begin{pmatrix} \mathbf{j}^e \\ \mathbf{j}^i \end{pmatrix}, \quad Y_\alpha = \begin{pmatrix} \mathbf{Y}^e \\ -\mathbf{Y}^i \end{pmatrix},$$

$$\mathfrak{M}_{\alpha\beta} = \begin{pmatrix} I + 4\pi\varkappa^e Q & 4\pi\varkappa^e Q \\ 4\pi\varkappa^i Q & I + 4\pi\varkappa^i Q \end{pmatrix}. \qquad (24')$$

Recognizing that det $\mathfrak{M}$ differs from the determinant of the matrix Λ [see (2′)] only by a factor $(\eta^2 - 1)^{-2}$, we write the solution of (24′) in the form

$$j_\alpha = ie\Lambda^{-1}\mu_{\alpha\beta} Y_\beta,$$

where $\mu_{\alpha\beta}\mathfrak{M}_{\beta\gamma} = \delta_{\alpha\gamma}\Lambda$.

We must now average the products of the currents over the random forces with the aid of (17). (The fact that (23) contains additional terms due to the field $\mathbf{H}_0$, not contained in (15), does not change the expression for S, and consequently does not change the expression for $\langle y^a y^b \rangle_{\mathbf{k}\omega}$.) Noting that

$$e^2 \langle Y_i^a Y_j^b \rangle_{\mathbf{k}\omega} = -i\delta_{ab}\omega T^a (\varkappa_{ij}^a - \varkappa_{ji}^{a*}),$$

we obtain finally an expression for the correlators of the electron and ion currents

$$\langle j_i^a j_j^b \rangle_{\mathbf{k}\omega} = -i\omega |\Lambda|^{-2} \{\mu_{il}^{ae}\mu_{jm}^{be*} T^e (\varkappa_{lm}^e - \varkappa_{ml}^{e*})$$
$$+ \mu_{il}^{ai}\mu_{jm}^{bi*} T^i (\varkappa_{lm}^i - \varkappa_{ml}^{i*})\}, \qquad (25)$$

where μ_{ij}^{ab} are the elements of the matrix μ:

$$\mu = \begin{pmatrix} \mu^{ee} & \mu^{ei} \\ \mu^{ie} & \mu^{ii} \end{pmatrix}.$$

From (25) we readily obtain the spectral distribution of the fluctuations of the total current density

$$\langle j_i j_j \rangle_{\mathbf{k}\omega} = -i\omega |\Lambda|^{-2} \mu_{il}^t \mu_{jm}^{t*} \{T^e (\varkappa_{lm}^e - \varkappa_{ml}^{e*}) + T^i (\varkappa_{lm}^i - \varkappa_{ml}^{i*})\}, \qquad (26)$$

where the tensor μ^t is related with the tensor

$$\mathfrak{M}_{ij}^t = (\eta^2 - 1)^{-1} (\eta^2 k^{-2} \varepsilon_{il} k_l k_j - \Lambda_{ij})$$

by the equation $\mu_{il}^t \mathfrak{M}_{lj}^t = \delta_{ij}\Lambda$ [we note that $\det \mathfrak{M}^t = (\eta^2 - 1)^{-2}\Lambda$].

Using Maxwell's equations we obtain with the aid of (25) and (26) the correlators of all the quantities of interest to us.

In particular, the correlators of the electron-density fluctuations and of the magnetic-field fluctuations have the form

$$e^2 \langle |\delta n^e|^2 \rangle_{\mathbf{k}\omega} = \frac{k_i k_j}{\omega^2} \langle j_i^e j_j^e \rangle_{\mathbf{k}\omega},$$

$$\langle \delta H_i \delta H_j \rangle_{\mathbf{k}\omega} = \left(\frac{4\pi}{\omega}\right)^2 \frac{\eta^2}{(\eta^2 - 1)^2} e_{ilm} e_{jl'm'} k^{-2} k_l k_{l'} \langle j_m j_{m'} \rangle_{\mathbf{k}\omega},$$

$$e \langle \delta n^e \delta H_j \rangle_{\mathbf{k}\omega} = -\frac{4\pi i}{c\omega} \frac{\eta^2}{\eta^2 - 1} e_{jm'l} \frac{k_m k_{m'}}{k^2} \{\langle j_m^e j_l^e \rangle_{\mathbf{k}\omega} + \langle j_m^e j_l^i \rangle_{\mathbf{k}\omega}\}, \qquad (27)$$

where $e_{\ldots}$ is a completely antisymmetrical third-rank tensor.

As already noted, the correlation functions have sharp maxima near values of ω and $\mathbf{k}$ satisfying the dispersion equation $\Lambda(\mathbf{k}, \omega_n(\mathbf{k})) = 0$ (the index n labels the type of oscillations). It is easy to establish the form of the correlation functions near such maxima. For example, for the quantity $\langle j_i j_j \rangle_{\mathbf{k}\omega}$ we have

$$\langle j_i j_j \rangle_{\mathbf{k}\omega} = B_{ij}(\mathbf{k}, \omega)\, \delta(\omega \pm \omega_n(\mathbf{k})),$$

$$B_{ij} = -i\pi\omega \left|\frac{\partial \Lambda}{\partial \omega}\right|^{-1} \mu_{il}^t \mu_{jm}^{t*} \frac{T^e (\varkappa_{lm}^e - \varkappa_{ml}^{e*}) + T^i (\varkappa_{lm}^i - \varkappa_{ml}^{i*})}{\operatorname{Im} \Lambda}.$$

If $\omega/k \gg s_e, s_i$, then this formula goes into Eq. (13) with $T = T^e$.

We note also that if $\eta \gg 1$ the correlators of the particle densities and the charge density in the presence of a magnetic field are determined by the same formulas (20) and (21) as in the case of a free plasma. Here we must take κ and ϵ to mean the longitudinal components of the corresponding tensors, for example $k^{-2} k_i k_j \epsilon_{ij}$ (this conclusion does not hold true only near the poles of the correlation functions, corresponding to the propagation of fast and slow magnetic-sound waves).*

To find the correlators of the distribution functions it is necessary to express, with the aid of Maxwell's equations, the fluctuation of the electric field in terms of the fluctuation of the distribution functions and, after substituting the resultant expression (23), to average in accordance with (17). We do not give the corresponding expressions here.

7. We consider now the scattering of electromagnetic waves by fluctuations in a plasma. In a free plasma this scattering is determined only by the fluctuations of the electron density; for a plasma in a magnetic field $\mathbf{H}_0$ it is necessary to take additional account, generally speaking, of the magnetic-field fluctuations $\delta\mathbf{H}$ (the fluctuations of the ion density are insignificant because of the large ion mass). The electric field of the scattered waves obviously satisfies the equation

$$\left(\operatorname{rot}\operatorname{rot} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}' = -\frac{4\pi e}{c^2}\frac{\partial}{\partial t}(n_0 \mathbf{v}' + \delta n^e \mathbf{v}),$$

---

*These waves were investigated by Stepanov[22,24] and Bernstein.[23]

where **v** is the average electron velocity due to the field of the incident wave $\mathbf{E} = \mathbf{E}^0 \exp(i\mathbf{k}_0\cdot\mathbf{r} - i\omega_0 t)$:

$$\mathbf{v} = i\frac{e}{m}\frac{\omega_0^2\mathbf{E} - \omega_H^2(\mathbf{hE})\,\mathbf{h} - i\omega_0\omega_H[\mathbf{hE}]}{\omega_0(\omega_0^2 - \omega_H^2)}, \qquad \mathbf{h} = \frac{\mathbf{H}_0}{H_0}$$

and **v′** is the electron velocity associated with the scattered wave:

$$\frac{d\mathbf{v}'}{dt} = \frac{e}{m}\mathbf{E}' + \frac{e}{mc}[\mathbf{v}'\mathbf{H}_0] + \frac{e}{mc}[\mathbf{v}\delta\mathbf{H}]$$

(We assume that $v, v' \ll c$). From these relations follows an equation for the field of the scattered wave with frequency $\omega = \omega_0 + \Delta\omega$ and a wave vector $\mathbf{k} = \mathbf{k}_0 + \mathbf{q}$:

$$\Lambda_{ij}(\mathbf{k}, \omega) E_j'(\mathbf{k}, \omega) = E_j^0 \left\{\frac{\omega_0}{\omega}(\delta_{ij} - \varepsilon_{ij}^0)\frac{\delta n^e(\mathbf{q}, \Delta\omega)}{n_0} + i\frac{e}{mc}\frac{\omega_0}{\Omega^2}(\delta_{il} - \varepsilon_{il}) e_{lkm}(\delta_{kj} - \varepsilon_{kj}^0)\,\delta H_m(\mathbf{q}, \Delta\omega)\right\},$$

where $\epsilon_{ij} = \epsilon_{ij}(\omega)$ and $\epsilon_{ij}^0 = \epsilon_{ij}(\omega_0)$ [the tensor ϵ_{ij} is determined from (12)].

To find the scattering coefficient we must divide the intensity of the scattered wave by the $\mathbf{k}_0$-component of the Poynting vector of the incident wave

$$S_0 = \frac{c}{8\pi}\eta_0 |E^0|^2 \left(1 - \frac{|\mathbf{e}_0\mathbf{k}_0|^2}{|\mathbf{e}_0|^2 k_0^2}\right);$$

$$\mathbf{e}_0 = \left(i;\ -\frac{\varepsilon_2^0}{\eta_0^2 - \varepsilon_1^0};\ \frac{i}{2}\frac{\eta_0^2 \sin 2\vartheta_0}{\eta_0^2\sin^2\vartheta_0 - \varepsilon_3^0}\right), \qquad \eta_0 = \frac{ck_0}{\omega_0}$$

(ϑ_0 is the angle between $\mathbf{H}_0$ and $\mathbf{k}_0$). We give only the final results.

In the absence of a magnetic field, the differential scattering coefficient for an unpolarized wave is

$$d\Sigma = \frac{1}{4\pi}\left(\frac{e^2}{mc^2}\right)^2\left(\frac{\omega}{\omega_0}\right)^2\sqrt{\frac{\varepsilon}{\varepsilon_0}}(1 + \cos^2\theta)\langle|\delta n^e|^2\rangle_{\mathbf{q}\Delta\omega}\,do\,d\omega, \tag{28}$$

where θ is the scattering angle, do the element of solid angle $\mathbf{k}$, $\epsilon \equiv \epsilon(\omega) = 1 - \Omega^2/\omega^2$, and $\epsilon_0 = \epsilon(\omega_0)$. We note that this formula admits of arbitrary frequency variations. When $\Delta\omega \ll \omega_0$ the factor $(\omega/\omega_0)^2\sqrt{\epsilon/\epsilon_0}$ becomes equal to unity, and (28) goes into the well-known formula for the scattering on fluctuations with small frequency variation (see, for example,[7]).

In the presence of a magnetic field, $d\Sigma$ has the form

$$d\Sigma = \frac{1}{2\pi}\left(\frac{e^2}{mc^2}\right)^2\left(\frac{\omega_0\omega}{\Omega^2}\right)^2 R\left\{|\xi|^2\langle|\delta n^e|^2\rangle_{\mathbf{q}\Delta\omega} - \frac{en_0}{mc}\frac{\omega}{\Omega^2}\operatorname{Im}(\xi A_i\langle\delta n^e\delta H_i\rangle_{\mathbf{q}\Delta\omega}) + \frac{n_0}{4\pi mc^2}\frac{\omega^2}{\Omega^2}A_i^* A_j\langle\delta H_i\delta H_j\rangle_{\mathbf{q}\Delta\omega}\right\}do\,d\omega,$$

$$R = \eta^3\left\{\eta_0\left(|\mathbf{e}_0|^2 - \frac{|\mathbf{e}_0\mathbf{k}_0|^2}{k_0^2}\right)\varepsilon_{ij}e_i^* e_j^*\right\}^{-1}, \quad \xi = (\varepsilon_{ij}^0 - \delta_{ij})e_i^* e_j^0,$$

$$A_i = (\varepsilon_{kl} - \delta_{kl})e_k^* e_{lmi}(\varepsilon_{mj}^0 - \delta_{mj})e_j^0, \tag{29}$$

where **e** is the polarization vector of the scattered wave

$$\mathbf{e} = \left(\frac{\varepsilon_2}{\eta^2 - \varepsilon_1}\sin\varphi + i\cos\varphi;\ -\frac{\varepsilon_2}{\eta^2 - \varepsilon_1}\cos\varphi + i\sin\varphi;\ \frac{i}{2}\frac{\eta^2\sin 2\vartheta}{\eta^2\sin^2\vartheta - \varepsilon_3}\right)$$

[ϑ is the angle between $\mathbf{H}_0$ and **k**, while φ is the angle between the planes $(\mathbf{k}_0, \mathbf{H}_0)$ and $(\mathbf{k}, \mathbf{H}_0)$]. The quantities $\langle|\delta n^e|^2\rangle$, $\langle\delta n^e\delta H_i\rangle$ and $\langle\delta H_i\delta H_j\rangle$ are determined from (27).

8. If the electron and ion temperatures are equal and there is no magnetic field* the spectrum of the scattered radiation consists of the Doppler-broadened principal line ($\Delta\omega \lesssim qs_i$) and of sharp maxima at $\Delta\omega = \pm\Omega$ if $aq \ll 1$. If in addition $\omega_0 \gg \Omega$, then the integral scattering coefficient (in a given angle interval) has the form

$$d\Sigma = \frac{n_0}{4}\left(\frac{e^2}{mc^2}\right)^2\frac{1 + 2(aq)^2}{1 + (aq)^2}(1 + \cos^2\theta)\,do; \quad q = \frac{2\omega_0}{c}\sin\frac{\theta}{2}. \tag{30}$$

Whether the electron and ion temperatures are equal or not, the coefficient of scattering is determined for small changes in temperature ($\Delta\omega \ll qs_i$) by the formula

$$d\Sigma = \frac{n_0}{2\sqrt{\pi}}\left(\frac{e^2}{mc^2}\right)^2\frac{a}{s_i}\frac{(aq)^3}{[1 + (aq)^2]^2}(1 + \cos^2\theta)\,do\,d\omega. \tag{31}$$

If $\Delta\omega \gg qs_e$ the scattering is only on the Langmuir oscillations. In this region, for arbitrary relation between T^e and T^i, the form of $d\Sigma$ is

$$d\Sigma = \frac{e^2 T^e q^2}{16\pi(mc^2)^2}\frac{\omega^2}{\omega_0^2}\sqrt{\frac{\varepsilon}{\varepsilon_0}}(1 + \cos^2\theta)\{\delta(\Delta\omega - \Omega) + \delta(\Delta\omega + \Omega)\}\,do\,d\omega. \tag{32}$$

In a strongly nonisothermal plasma ($T^e \gg T^i$) an additional sharp maximum appears in the frequency interval between the central (Doppler) maximum and the Langmuir satellites when $\Delta\omega = \omega_s(q)$. This maximum is connected with the possibility of propagation of specific sound oscillations in the plasma. In the case of greatest interest, when $\Delta\omega/qs_i \gg \ln(T^e/T^i)$, the scattering coefficient for $\Delta\omega \sim \omega_s(q)$ is equal to

$$d\Sigma = \frac{e^2 k_e^4 T^e(1 + \cos^2\theta)}{16\pi(mc^2)^2(k_e^2 + q^2)}\{\delta(\Delta\omega - \omega_s(q)) + \delta(\Delta\omega + \omega_s(q))\}. \tag{33}$$

9. In the presence of a magnetic field, as follows from the results of Sec. 6, the scattered-radiation spectrum contains, along with the Doppler-broadened principal lines, sharp maxima connected with the possibility of propagation of

*Various particular cases occurring under these conditions have been studied in detail for small changes of frequency ($\Delta\omega \ll \omega_0$) by Dougherty and Farley.[3]

various natural modes in the plasma (two types of Langmuir oscillations, Alfven waves, and also fast and slow magnetic-sound waves in the case when $T^e \gg T^i$). We give here the final expression only for the scattering coefficient on the Langmuir oscillations ($\Delta\omega \sim \omega_\pm$), which is simplest in form

$$d\Sigma = \frac{e^2 q^2 T^e}{8\pi (mc^2)^2} \frac{\omega^2 \omega_0^2}{\Omega^6} R \,|\,\xi\,|^2 \frac{(\Delta\omega)^2 [(\Delta\omega)^2 - \omega_H^2]^2}{(\Delta\omega)^4 \cos^2\tilde{\vartheta} + [(\Delta\omega)^2 - \omega_H^2]^2 \sin^2\tilde{\vartheta}}$$
$$\times \{\delta(\Delta\omega - \omega_+) + \delta(\Delta\omega + \omega_+) + \delta(\Delta\omega - \omega_-) + \delta(\Delta\omega + \omega_-)\}\, do\, d\omega, \qquad (34)$$

where $\tilde{\vartheta}$ is the angle between $\mathbf{q}$ and $\mathbf{H}_0$.

When $\omega_0 \gg \omega_+$ and $T^e = T^i$, we can determine the integral scattering coefficient (in a given angle interval). It is then sufficient to retain in (29) only the first term proportional to $\langle |\delta n^e|^2 \rangle$; the relative contributions of the second and third terms to the integral coefficient amounts to s/c and $(s/c)^2$. If $(\Delta\omega)_{\rm eff}$ denotes the effective frequency interval in which $\langle |\delta n^e|^2 \rangle$ is different from zero [we note that $(\Delta\omega)_{\rm eff} \lesssim \omega_+$], then the integral scattering coefficient, for $(\Delta\omega)_{\rm eff} \ll \omega_0 \sin\theta$, will be

$$d\Sigma = \frac{n_0}{4}\left(\frac{e^2}{mc^2}\right)^2 \left(\frac{\omega_0}{\Omega}\right)^4 (R\,|\,\xi\,|^2)_{\omega=\omega_0} \frac{1 + 2(aq)^2}{1 + (aq)^2}\, do. \qquad (35)$$

[We note that in general the second and third terms in the differential scattering coefficient (29) cannot be neglected.]

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