ANALYTIC PROPERTIES OF THE AMPLITUDES OF FEYNMAN DIAGRAMS CORRESPONDING TO "MANY-POINT" FUNCTIONS WITH A SINGLE CLOSED LOOP

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The analytic properties of the amplitudes of Feynman diagrams (n-point functions) with a single internal closed loop are investigated. A concrete form of the Landau determinant conditions for the singularities of amplitudes of such diagrams is considered, as well as the related choice of invariant variables for the n-point amplitudes, which happens to coincide with the one indicated previously.[1] A connection is established between the geometric conditions, imposed on the invariant variables and due to the four-dimensionality of space (for $n \geq 6$), and the Landau conditions for the amplitude singularities; it is shown that for the case of the diagrams under consideration the geometric conditions reduce to the Landau conditions and, consequently, cannot give rise to a new type of amplitude singularities. A proof is given of the fact, that all singularities of the n-point amplitudes with one closed loop for $n \geq 6$ reduce to the singularities of reduced diagrams ("pentagon," "quadrangle," triangle," etc.); this result has also been established by Brown.[6]

THE analytic properties of the "many-point" amplitudes (Feynman diagrams with n internal lines —momenta p_i) as functions of the $3n-10$ independent variables may be studied only if the independent variables are appropriately chosen. The independent variables must be chosen in such a way as to simultaneously satisfy the following conditions: first, the geometric conditions imposed on the invariant variables (for $n \geq 6$) due to the four-dimensionality of space should be of a maximally simple symmetric form and should be easily soluble for any one of the invariants (determining thus the relation of this invariant to the remaining ones and, consequently, its range of variation) and, second, there should result simple properties and symmetric form of the Landau determinants, which determine the location of the singularities of the corresponding "many-point" amplitudes.

It follows from what has been said that the independent invariant variables, satisfying simultaneously the two requirements indicated, should be symmetric scalar combinations of the totality of 4-vectors $(p_1, p_2, \ldots, p_n)$ that characterize the n-point function. Thus, for example, the nonsymmetric choice of independent invariants composed out of the totality of 4-vectors $(p_1, p_2, \ldots, p_{n-1})$, where the vector p_n (or any other vector) has been excluded with the help of the conservation law $\sum_{i=1}^{n} p_i = 0$, is not appropriate although with this choice it is possible after a number of transformations to simplify the geometric conditions (generally speaking, equations of fifth degree) reducing them to quadratic equations in definite invariant variables.

A symmetric choice of independent invariant variables for the n-point amplitudes was given in a previous paper (see [1]), based on the requirement of maximum simplification of the geometric conditions. The latter turned out to be in the most general case systems of quadratic equations, easily soluble in a general form for any one of the invariant variables. In the present paper it is shown that the symmetric choice of independent variables is also appropriate for the study of the analytic properties of the n-point amplitudes and that, in a certain sense, it automatically follows from the form of the Landau determinants.

To simplify the study of the analytic properties of the n-point amplitudes we shall consider the example of the simplest diagrams with one closed loop (among the n internal momenta q_i) and with n external momenta p_i (in what follows we shall refer to these as loop-diagrams) for $n \geq 6$. We note, however, that as was shown by Eden et al.[2] the main and significant properties of the singularities of Feynman amplitudes are already present in the loop-diagrams ("polygons"). The analysis of the analytic properties of loop-diagrams with $n < 6$ has been given by a number of authors.[3]

1. LANDAU CONDITIONS FOR THE AMPLITUDES OF LOOP "MANY-POINT" FUNCTIONS

For the loop-diagrams there exists a simple relation between the internal 4-momenta q_i ($q_i^2 = \mu_i^2$) and the external 4-momenta p_i ($p_i^2 = m_i^2$):

$$p_i = q_i - q_{i-1}, \qquad q_0 \equiv q_n, \tag{1}$$

$$p_{i+1} + p_{i+2} + \ldots + p_{i+j} = q_{i+j} - q_i. \tag{1'}$$

The Landau condition for the n-point amplitude with one closed loop of internal 4-momenta q_i is expressed with the help of an equation of the form

$$\sum_{i=1}^{n} \alpha_i q_i = 0, \tag{2}$$

which is to be considered on the mass shell $q_i^2 = \mu_i^2$.

The condition (2) is equivalent to the vanishing of the Landau determinant

$$|2(q_i q_k)| = 0, \qquad i, k = 1, 2, \ldots, n, \tag{3}$$

composed of the scalar products of the internal 4-momenta q_i, $(q_i q_k) = (q_k q_i)$. For $\alpha_i \neq 0$ this is the condition for the "leading" singularities of the loop n-point amplitude; the conditions for singularities of reduced diagrams, obtained from the main diagram by removing one or several internal lines and combining into one two or more vertices with external 4-momenta p_i, are similar to condition (2) with corresponding $\alpha_i = 0$ and consequently involve the principal minors of the determinant in Eq. (3). All the singularities of the n-point loop amplitude lie on hypersurfaces determined by these equations [of the type of Eq. (3)].

Let us express the elements of the determinant (3) in terms of the particle masses and the invariant scalar combinations of the external 4-momenta p_i: $s_{ik} = (p_i + p_k)^2$, $s_{ikl} = (p_i + p_k + p_l)^2$ etc., used as invariant variables for the "many-point" amplitudes. It is obvious that the elements with adjacent indices in the determinant (3) are expressible in terms of the masses alone:

$$2(q_i q_{i-1}) = \mu_i^2 + \mu_{i-1}^2 - m_i^2; \tag{4a}$$

the elements with indices differing by two units contain double invariants with adjacent indices:

$$2(q_i q_{i-2}) = \mu_i^2 + \mu_{i-2}^2 - s_{i,\,i-1}; \tag{4b}$$

the elements with indices differing by three units contain triple invariants with adjacent indices:

$$2(q_i q_{i-3}) = \mu_i^2 + \mu_{i-3}^2 - s_{i,\,i-1,\,i-2} \tag{4c}$$

etc.; finally, the elements with indices differing by n/2 units (for even n) or (n − 1)/2 units (for odd n) contain invariants with adjacent indices of multiplicity n/2 (for n even) or (n − 1)/2 (for n odd); further elements with indices differing by more units contain invariants of higher multiplicity which reduce to the above listed invariants of lower multiplicity with the help of the conservation law $\Sigma_i p_i = 0$.

As an example we consider the Landau condition for the loop 6-point function (n = 6):

$$\begin{vmatrix} 2\mu_1^2 & \mu_1^2+\mu_2^2-m_2^2 & \mu_1^2+\mu_3^2-s_{23} & \mu_1^2+\mu_4^2-s_{234} & \mu_1^2+\mu_5^2-s_{61} & \mu_1^2+\mu_6^2-m_1^2 \\ \mu_1^2+\mu_2^2-m_2^2 & 2\mu_2^2 & \mu_2^2+\mu_3^2-m_3^2 & \mu_2^2+\mu_4^2-s_{34} & \mu_2^2+\mu_5^2-s_{345} & \mu_2^2+\mu_6^2-s_{12} \\ \mu_1^2+\mu_3^2-s_{23} & \mu_2^2+\mu_3^2-m_3^2 & 2\mu_3^2 & \mu_3^2+\mu_4^2-m_4^2 & \mu_3^2+\mu_5^2-s_{45} & \mu_3^2+\mu_6^2-s_{456} \\ \mu_1^2+\mu_4^2-s_{234} & \mu_2^2+\mu_4^2-s_{34} & \mu_3^2+\mu_4^2-m_4^2 & 2\mu_4^2 & \mu_4^2+\mu_5^2-m_5^2 & \mu_4^2+\mu_6^2-s_{56} \\ \mu_1^2+\mu_5^2-s_{61} & \mu_2^2+\mu_5^2-s_{345} & \mu_3^2+\mu_5^2-s_{45} & \mu_4^2+\mu_5^2-m_5^2 & 2\mu_5^2 & \mu_5^2+\mu_6^2-m_6^2 \\ \mu_1^2+\mu_6^2-m_1^2 & \mu_2^2+\mu_6^2-s_{12} & \mu_3^2+\mu_6^2-s_{456} & \mu_4^2+\mu_6^2-s_{56} & \mu_5^2+\mu_6^2-m_6^2 & 2\mu_6^2 \end{vmatrix} = 0.$$

It follows from the form of the Landau determinant for the 6-point function that it is necessary to choose 9 kinematically unrelated double and triple invariants with adjacent indices:

$$s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{61}; \quad s_{123} \equiv s_{456},\ s_{234} \equiv s_{561},\ s_{345} \equiv s_{612}, \tag{5}$$

as was also stated in the previous paper, [1] and impose on them one geometrical condition (which has the form of a quadratic equation with respect to any of the invariants), which reduces the 9 fixed invariants to the necessary number of independent variables for the 6-point function—that is 8. It is relevant here that the determinant Landau condition also turns out to be a quadratic equation with respect to any one of the indicated invariants. As will be shown later, this is no accident.

The principle of using invariants with adjacent indices as fixed kinematically unrelated variables, on which are imposed symmetric geometric conditions and in terms of which are expressed all the determinant conditions for singularities (3), turns out to be valid for an arbitrary loop n-point function.

2. RELATION BETWEEN THE GEOMETRIC CONDITIONS AND THE LANDAU CONDITIONS

The geometric conditions arise as a result of imposing the conditions for linear dependence (in four-dimensional space)

$$\sum_{l}^{1-4,\, l} \beta_l p_l = 0, \quad \beta_l \neq 0 \tag{6}$$

on any five external vectors: p_1, p_2, p_3, p_4, p_l ($l = 5, 6, \ldots, n$), characterizing the n-point function, with the first four (p_1, p_2, p_3, p_4) chosen as linearly independent vectors (basis) with a nonvanishing Gram determinant of fourth order: $\Delta_4(1, 2, 3, 4) = |2p_i \cdot p_k| \neq 0$ $(i, k \leq 4)$ (one of the vectors, for example p_n, may be excluded from consideration by means of the conservation law $\sum_{i=1}^{n} p_i = 0$).

The conditions (6) are equivalent to the vanishing of Gram determinants of fifth order (see [1]):

$$\Delta_5(1, 2, 3, 4, l) \equiv |2(p_i p_k)| = 0,$$
$$\Delta_5 \ (1, 2, 3, 4;\ lm) \equiv |2(p_i p_k)| = 0, \tag{7}$$

which can be constructed out of the fourth order Gram determinant by adjoining to it the last rows and columns composed of the scalar products $2p_i \cdot p_l$ $(i = 1, 2, 3, 4)$.

In the case of the loop-diagrams under consideration the external 4-momenta p_i in Eq. (6) may be replaced with the help of Eq. (1) directly by the internal 4-momenta q_i, with the result that the condition (6) takes on the form [1)]

$$\sum_{l}^{1-4,\, l} \beta_l (q_l - q_{l-1}) \equiv \sum_{l} \gamma_l q_l = 0, \tag{8}$$

where, as is easy to show,

$$\sum_{l} \gamma_l = 0. \tag{9}$$

In this form, however, the conditions (8) are equivalent to the Landau conditions (2) for the main and reduced diagrams. It therefore follows that there is an indirect connection between the Landau determinants in Eq. (3) and the determinants in the geometric conditions (7), as a result of which the geometric conditions (7) cannot produce (taking into account the quadratic nature of these conditions with respect to any one of the invariants) any additional kinematic branch points in the "many-point" amplitudes, not already present in the Landau conditions, Eq. (3).

The result here obtained may be illustrated on the example of the simplest loop "many-point" functions for n = 6 and n = 7. In the case of the 6-point function the condition (8) has the form

$$\sum_{l=1}^{6} \gamma_l q_l = 0 \tag{8'}$$

in the presence of the additional linear relation

$$\sum_{l=1}^{6} \gamma_l = 0 \tag{9'}$$

for the coefficients γ_i, which coincides in form [without taking (9′) into account] with the Landau condition for the "leading" singularities of the 6-point amplitude, i.e., the corresponding determinant conditions (3) and (7) are equivalent.

In the case of the 7-point function we obtain in accordance with Eq. (8) the set of conditions of the form

$$\text{a)} \sum_{l=1}^{7} \gamma'_l q_l = 0, \quad \text{b)} \sum_{l}^{1-5,7} \gamma''_l q_l = 0 \tag{8''}$$

with the additional relations for the coefficients γ'_i and γ''_i

$$\text{a)} \sum_{l=1}^{7} \gamma'_l = 0, \quad \text{b)} \sum_{l}^{1-5,7} \gamma''_l = 0. \tag{9''}$$

It is obvious that conditions (8″) [leaving out (9″)] are equivalent in form to the Landau conditions for the "leading" singularities of the 7-point amplitude for the singularities of the reduced 7-point amplitude (with the internal line q_6 removed). In an analogous manner one establishes the equivalence of conditions (7) and (3) for loop diagrams with n > 7. Let us emphasize here that it is the conditions (7) that reduce to the conditions (3).

We note that the above indicated method may also be utilized in the analysis of the geometric conditions of more complicated diagrams of "many-point" functions with several internal loops. At that, by considering a number of diagrams with two and three internal loops we again find that the geometric conditions introduce no changes into the picture of singularities of the "many-point" amplitude, obtained on the basis of the Landau conditions (3). In connection with the result here established the assertion by Chan Hong-Mo [4] that the geometric conditions give rise to a new class of kinematic branch points of the amplitudes of Feynman diagrams of "many-point" functions must be, apparently, false (in any case for the amplitudes of loop "many-point" functions).

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1)This condition, as will be seen in what follows, is equivalent to the contraction of one of the sides of the "polygon" (loop) and coalescence of the two corresponding vertices.

3. ANALYTIC PROPERTIES OF THE LOOP n-POINT AMPLITUDE FOR $n \geq 6$

The concrete choice of invariant variables for the loop "many-point" amplitude and the establishment of a connection between the geometric conditions (7) and the Landau conditions for the singularities of the amplitude, Eq. (3), allow one to arrive at the conclusion that for $n \geq 6$ all the singularities of the n-point function with a loop of internal momenta coincide with the singularities of the reduced, down to and including the "pentagon," diagrams obtained from the main diagram by crossing out internal lines and coalescing vertices with external momenta. This result follows from the well-known property of vanishing of Gram determinants [5] if any of its principal minors vanish.

Indeed, let us consider the conditions (8) into which are transformed the conditions for linear dependence of the vectors (6), together with the conditions for the coefficients γ_i, Eq. (9). Among the conditions (8) for any loop n-point function there is always one that contains no more than six internal momenta q_i [in the case of the 6-point function—condition (8′), in the case of the 7-point function—condition (8″b)]. The existence of the additional relation among the coefficients γ_i in the form of conditions (9), (9′), and (9″) allows one to conclude that along with the linear dependence of the 6 vectors q_i in the form of Eqs. (8), (8′), and (8″) the 5 vectors q_j are also linearly dependent [which five vectors may be obtained from the 6 vectors q_i by simultaneous translation of all vectors by the length of one of the q_i: $q_j = q_i - q_k$ $(j = 1, 2, \ldots, 5)$].

The condition of linear dependence of 5 vectors q_j of the form

$$\sum_{j}^{(5)} \delta_j q_j = 0, \quad \delta_j \neq 0 \tag{10}$$

for an arbitrary loop n-point function means, on the other hand, the vanishing of the fifth order Gram determinant with the elements $2 q_i \cdot q_k$, which is the principal minor of fifth order of the Landau determinant (3), characterizing the n-point function reduced to a "pentagon" and causing, in view of the above mentioned property of Gram determinants, the vanishing of the determinant (3) itself and all its principal minors of order higher than fifth.

In this manner the singularities of an arbitrary loop n-point function (for $n \geq 6$) reduce to the singularities of the "pentagon" (and also the "quadrangle," "triangle," etc.). The same conclusion was arrived at by Brown [6] who proceeded by a different method (based on an analysis of Feynman integrals for "many-point" functions).

In making practical use of the Landau conditions for the singularities of the loop n-point functions $(n \geq 6)$ it is necessary, in view of the above established results, to consider only the set of fifth order minors, and also minors of fourth, third and second order, of the main determinant (3) (taking into account the above indicated translation) which, when equated to zero, represent second order equations with respect to any one of the invariants contained in them. As a result the invariant variables connected by the indicated equations are broken up into symmetric subgroups that enter into different fifth order minors. In particular, in the case of the 6-point function one must consider fifth order minors giving rise to a set of quadratic equations connecting various sets of six out of the nine fixed invariant variables of the 6-point function.

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