ON THE THEORY OF QUANTUM GENERATORS

T. I. KUZNETSOVA and S. G. RAUTIAN

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor June 13, 1962

J. Exptl. Theoret. Phys. (U.S.S.R.) **43**, 1897-1903 (November, 1962)

A plane monochromatic solution for an electromagnetic field inside a layer with a negative absorption coefficient is considered.

1. The problem of the radiation of a quantum generator can be stated as an ordinary boundary problem in electrodynamics. It is well known, however, that in its general form the solution of this kind of problem is extremely complicated and requires a reasonable simplification of the conditions. We shall consider the characteristics of a quantum generator from this point of view.

The saturation effect, which determines the steady state of generation, brings about a nonlinearity in the material equations. A change in the dielectric permeability ϵ is associated with the change in level populations in an external field. Since the relaxation time of the population is much longer than the period of vibration of the field, ϵ is determined, not by the instantaneous value of the field, but by the value of its energy averaged over a period. In this way, saturation leads only to the result that ϵ depends on the amplitude of the field as a parameter and not on time.

At the present time an expression for ϵ is known only for the case in which the strong field causing the saturation is monochromatic. [1,2] Hence, we shall consider the case of steady-state generation in a monochromatic field.

Because of diffraction at the edges of the interferometer mirrors, the field depends on all three coordinates. As calculations have shown, [3] the field amplitude can for this reason vary significantly over the mirrors of the generator. In order to simplify the problem, we shall assume below that the steady-state field varies only in a direction perpendicular to the mirrors.

The dielectric constant can be represented in the form

$$\varepsilon = \varepsilon_0 + \Delta\varepsilon, \tag{1}$$

where $\Delta\epsilon$ is determined by that pair of levels involved in the generation, and ϵ_0 is given by the remaining levels and the "solvent" (the host lattice, etc.). In the majority of cases $\Delta\epsilon$ is extremely small. Actually, the order of magnitude of the steady-state value of $\Delta\epsilon$ can be found from the familiar relation $2\pi l\lambda^{-1}\,\mathrm{Im}\,\Delta\epsilon = 1 - r$, where r is the reflection coefficient of the mirror coatings, l is the distance between the mirrors, and λ is the wavelength. In many cases r is nearly unity, so that

$$2\pi l\lambda^{-1}\,\mathrm{Im}\,\Delta\varepsilon \ll 1. \tag{2}$$

This circumstance will be exploited below.

Let a layer of a substance with a negative absorption coefficient be placed between the planes $x = 0$ and $x = l$. External to this layer there are homogeneous media with dielectric constants ϵ_1 $(x < 0)$ and ϵ_2 $(x > l)$ (Fig. 1). In actual arrangements highly reflective metallic or dielectric coatings are applied to the active layer. It is easy to see, however, that physically the essential thing is the reflection and not the means of obtaining it. Hence, in order to simplify the calculation, we shall consider the configuration of Fig. 1, in which the necessary reflection is provided by the discontinuity in the dielectric constant.

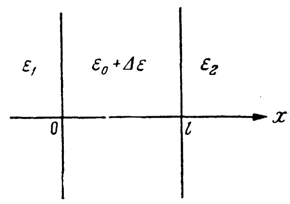

FIG. 1

The wave equation for a plane monochromatic field has the form (time dependence chosen in the form $e^{-i\omega t}$)

$$\begin{aligned} &d^2E/dx^2 + \omega^2c^{-2}\varepsilon_1 E = 0, && x < 0,\\ &d^2E/dx^2 + \omega^2c^{-2}[\varepsilon_0 + \Delta\varepsilon(|E|)]E = 0, && 0 \leqslant x \leqslant l,\\ &d^2E/dx^2 + \omega^2c^{-2}\varepsilon_2 E = 0, && x > l. \end{aligned} \tag{3}$$

In order that only receding waves propagate outside of the layer, we shall seek solutions external to the generator in the form

$$E = A_1 \exp\left(-i\frac{\omega}{c}\sqrt{\varepsilon_1}\,x\right), \quad x < 0;$$

$$E = A_2 \exp\left(i\frac{\omega}{c}\sqrt{\varepsilon_2}\,x\right), \quad x > l. \tag{4}$$

It can be seen from Eqs. (4) that the relations

$$\frac{dE}{dx} = -i\frac{\omega}{c}\sqrt{\varepsilon_1}E, \quad x < 0;$$

$$\frac{dE}{dx} = i\frac{\omega}{c}\sqrt{\varepsilon_2}E, \quad x > l. \tag{5}$$

are fulfilled outside of the layer.

Making use of the continuity of the tangential components of the electric and magnetic fields and the relations (5), we obtain the following boundary conditions:

$$\frac{dE}{dx} = -i\frac{\omega}{c}\sqrt{\varepsilon_1}E, \quad x = 0;$$

$$\frac{dE}{dx} = i\frac{\omega}{c}\sqrt{\varepsilon_2}E, \quad x = l. \tag{6}$$

The general solution of Eq. (3) for $0 < x < l$ can be represented as

$$E = E_0 + E_1, \qquad E_0 = A\sin\frac{2\pi x}{\lambda} + B\cos\frac{2\pi x}{\lambda},$$

$$\lambda = \frac{\omega}{c}\sqrt{\varepsilon_0}; \tag{7}$$

E_l satisfies the integral equation

$$E_1 = -\frac{2\pi}{\lambda\varepsilon_0}\int_0^x \sin\left[\frac{2\pi}{\lambda}(x-\xi)\right][E_0(\xi) + E_1(\xi)]\,\Delta\varepsilon(|E_0 + E_1|)\,d\xi. \tag{8}$$

The ratio A/B is determined from the boundary condition (6) for $x = 0$; it is $A/B = -i\sqrt{\epsilon_1/\epsilon_0}$. Consequently,

$$E_0 = A\left[\sin\frac{2\pi x}{\lambda} + i\sqrt{\frac{\varepsilon_0}{\varepsilon_1}}\cos\frac{2\pi x}{\lambda}\right]. \tag{9}$$

The boundary condition (6) for $x = l$ reduces to a system of equations in A and l:

$$A\left(1 + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right)\cos\frac{2\pi l}{\lambda} = -\frac{\lambda}{2\pi}\operatorname{Re}\frac{dE_1}{dx} - \sqrt{\frac{\varepsilon_2}{\varepsilon_0}}\operatorname{Im}E_1,$$

$$A\left(\sqrt{\frac{\varepsilon_0}{\varepsilon_1}} + \sqrt{\frac{\varepsilon_2}{\varepsilon_0}}\right)\sin\frac{2\pi l}{\lambda} = \frac{\lambda}{2\pi}\operatorname{Im}\frac{dE_1}{dx} - \sqrt{\frac{\varepsilon_2}{\varepsilon_0}}\operatorname{Re}E_1. \tag{10}$$

In accordance with Eq. (2) we shall assume that $\mu \equiv (2\pi l/\lambda)\max\Delta\epsilon \ll 1$. Then we can show from Eq. (8) that E_1 is a small correction to E_0:

$$|E_1| < \frac{\mu}{1-\mu}\left(1 + \sqrt{\frac{\varepsilon_0}{\varepsilon_1}}\right)A,$$

$$\frac{\lambda}{2\pi}\left|\frac{dE_1}{dx}\right| < \frac{\mu}{1-\mu}\left(1 + \sqrt{\frac{\varepsilon_0}{\varepsilon_1}}\right)A. \tag{11}$$

Using Eq. (11) and the boundary conditions in the form (10), we have

$$\left|\cos\left(\frac{2\pi l}{\lambda}\right)\right| < \frac{\mu}{1-\mu}\left(1 + \frac{\sqrt{\varepsilon_2/\varepsilon_0} + \sqrt{\varepsilon_0/\varepsilon_1}}{1 + \sqrt{\varepsilon_2/\varepsilon_1}}\right),$$

$$\left|\sin\left(\frac{2\pi l}{\lambda}\right)\right| < \frac{\mu}{1-\mu}\left(1 + \frac{1 + \sqrt{\varepsilon_2/\varepsilon_1}}{\sqrt{\varepsilon_2/\varepsilon_0} + \sqrt{\varepsilon_0/\varepsilon_1}}\right). \tag{12}$$

These inequalities are consistent only if one of the following four conditions is fulfilled:

1) $\cos(2\pi l/\lambda) \sim \mu, \quad \sqrt{\varepsilon_1/\varepsilon_0} \sim 1/\mu, \quad \sqrt{\varepsilon_2/\varepsilon_0} \sim \mu;$

2) $\cos(2\pi l/\lambda) \sim \mu, \quad \sqrt{\varepsilon_1/\varepsilon_0} \sim \mu, \quad \sqrt{\varepsilon_2/\varepsilon_0} \sim 1/\mu;$

3) $\sin(2\pi l/\lambda) \sim \mu, \quad \sqrt{\varepsilon_1/\varepsilon_0} \sim 1/\mu, \quad \sqrt{\varepsilon_2/\varepsilon_0} \sim 1/\mu,$

4) $\sin(2\pi l/\lambda) \sim \mu, \quad \sqrt{\varepsilon_1/\varepsilon_0} \sim \mu, \quad \sqrt{\varepsilon_2/\varepsilon_0} \sim \mu.$ (13)

The physical significance of these conditions is obvious. Since μ is small all the possible conditions (13) correspond to a large discontinuity in the dielectric constant at the boundaries of the layer, which is provided by a large reflection coefficient. The first two cases correspond to an "unsymmetrical" change, i.e., either the left dielectric constant ϵ_1 is much greater than ϵ_0 and the one on the right much less, or the other way around. The latter two cases correspond to both the left and right dielectric constants being either larger or smaller than ϵ_0. The change in the form of the trigonometric function (cos or sin) is obviously associated with the so-called "half-wave loss" upon reflection from an optically denser medium. Since $\mu \ll 1$ either $\sin(2\pi l/\lambda)$ or $\cos(2\pi l/\lambda)$ should be close to zero. This means that the length of the generator should be a whole number of half-wavelengths or a whole number of half-wavelengths plus a quarter wave (to compensate the discontinuity in phase upon reflection). All the cases (13) lead to the same physical conclusions. For the sake of definiteness, we shall consider case (3) below, i.e., $\sqrt{\epsilon_0} \sim 1$, $\sqrt{\epsilon_1} \sim 1/\mu$, $\sqrt{\epsilon_2} \sim 1/\mu$.

As a consequence of the smallness of μ we can set $E_1 = 0$ in the integrand of Eq. (8). Then, to terms of the order μ^2, the field inside the generator is

$$E(x) = A\left(\sin\frac{2\pi x}{\lambda} + i\sqrt{\frac{\varepsilon_0}{\varepsilon_1}}\cos\frac{2\pi x}{\lambda}\right) - \frac{2\pi}{\lambda\varepsilon_0}\int_0^x \sin 2\pi\frac{x-\xi}{\lambda}\cdot A\sin\frac{2\pi\xi}{\lambda}\cdot\Delta\varepsilon\left(\left|A\sin\frac{2\pi\xi}{\lambda}\right|\right)d\xi. \tag{14}$$

Thus, for high reflection coefficients the field inside the layer has the form of an almost-standing wave $A\sin(2\pi x/\lambda)$. The second and third terms are small in comparison to the first; i.e., the amplitude of the field varies little.

It should be noted that because of the saturation effect the generating medium becomes inhomogeneous. In our approximation, $\Delta\epsilon$ is obviously a periodic function of x with period $\lambda/2$. This inhomogeneity is greater, the stronger the field, and the larger the saturation effect. We also note that this inhomogeneity of the medium means that the methods of geometrical optics are inapplicable.

3. It is known that the dielectric constant as a function of the frequency and field depends on the line shape in the absence of external field and on the nature of the line broadening. We distinguish between broadening caused by relaxation processes (homogeneous broadening) and broadening caused by the micro-inhomogeneity of the medium, which results in a change in transition frequency from atom to atom. In this section we shall consider only homogeneous broadening; in Sec. 4 the role of inhomogeneous broadening will be analyzed.

According to [1,2] the dielectric constant ϵ has the form

$$\varepsilon = \varepsilon_0 + \Delta\varepsilon; \quad \Delta\varepsilon = \Delta\varepsilon' + i\Delta\varepsilon'', \qquad \Delta\varepsilon' = -\Delta\varepsilon''(\omega' - \omega)/\gamma,$$
$$\Delta\varepsilon'' = -\frac{\beta}{(\omega' - \omega)^2/\gamma^2 + 1 + \sigma^2 |E|^2}. \tag{15}$$

Here ω' is the transition frequency, ω is the frequency of the field, 2γ is the line width in the absence of saturation; the coefficient σ^2 is determined by the matrix element of the dipole moment of the transition under consideration and the probabilities of relaxation processes; the quantity β is related in the usual manner to the number of excitations of the system in 1 cm^3 per second and the atomic constants. For the sake of simplicity, we here consider the case when the frequency of the field coincides with the transition frequency, $\omega' = \omega$. In principle this does not change the method of calculation and the basic qualitative conclusions.

From (14) and (15) we have

$$E(x) = A\left\{\sin\frac{2\pi}{\lambda}x + i\left[\sqrt{\frac{\varepsilon_0}{\varepsilon_1}} - \frac{\beta}{\varepsilon_0\sigma^2 A^2} \times \left(1 - \frac{1}{\sqrt{1+\sigma^2 A^2}}\right)\frac{2\pi x}{\lambda}\right]\cos\frac{2\pi}{\lambda}x\right\}. \tag{16}$$

In Eq. (16) the component that oscillates with period $\lambda/2$ and does not exceed $\mu A\lambda/l$ in amplitude is neglected. Thus the amplitude varies linearly with the coordinate x. This is a consequence of our approximation ($\sqrt{\epsilon_0/\epsilon_1}$, $\sqrt{\epsilon_0/\epsilon_2} \ll 1$). In the opposite case the variation in amplitude will be nonlinear and considerably larger in magnitude.

Introducing Eq. (16) into the boundary conditions (10) for $x = l$, we arrive at the following expressions for l and A:

$$l = m\lambda/2, \tag{17}$$

$$\sigma^2 A^2 = \frac{1}{4}\left[\left(1 + \frac{8\pi l}{\lambda}\frac{\beta/\varepsilon_0}{\sqrt{\varepsilon_0/\varepsilon_1} + \sqrt{\varepsilon_0/\varepsilon_2}}\right)^{1/2} - 3\right]\left[\left(1 + \frac{8\pi l}{\lambda}\frac{\beta/\varepsilon_0}{\sqrt{\varepsilon_0/\varepsilon_1} + \sqrt{\varepsilon_0/\varepsilon_2}}\right)^{1/2} + 1\right]. \tag{18}$$

It is to be noted that $\sqrt{\epsilon_0/\epsilon_1}$ and $\sqrt{\epsilon_0/\epsilon_2}$ are associated with the Fresnel reflection coefficients at the boundaries $x = 0$ and $x = l$, respectively. For the case under consideration, that of high reflection coefficients, $\sqrt{\epsilon_0/\epsilon_1} \approx \frac{1}{4}(1 - r_1)$, $\sqrt{\epsilon_0/\epsilon_2} \approx \frac{1}{4}(1 - r_2)$. Then Eq. (18) can be written as:

$$\sigma^2 A^2 = \frac{1}{4}\left[\left(1 + \frac{16\pi l}{\lambda}\frac{\beta/\varepsilon_0}{1 - (r_1 + r_2)/2}\right)^{1/2} - 3\right]\left[\left(1 + \frac{16\pi l}{\lambda}\frac{\beta/\varepsilon_0}{1 - (r_1 + r_2)/2}\right)^{1/2} + 1\right]. \tag{19}$$

Equation (19) no longer contains the dielectric constants of the external media and as such is applicable not only to the configuration of Fig. 1, but to any other means of creating a reflection coefficient on the boundaries of the layer (for which, it is understood, $1 - r_1$, $1 - r_2 \ll 1$).

Calculating the Poynting vector at the boundaries $x = 0$ and $x = l$, we obtain the power radiated by the generator per unit of area:

$$S_1 = (c/32\pi)(1 - r_1)\sqrt{\varepsilon_0}A^2,$$
$$S_2 = (c/32\pi)(1 - r_2)\sqrt{\varepsilon_0}A^2. \tag{20}$$

Equations (2), obtained for the model of Fig. 1, pertain to the case when the reflecting structure does not absorb radiation. In real reflecting films there is always some absorption, which decreases S_1 and S_2. Accounting for the absorption involves replacing $1 - r_1$ and $1 - r_2$ in (2) by $1 - r_1 - a_1$ and $1 - r_2 - a_2$, where a_1 and a_2 are the absorption coefficients of the reflecting layers at $x = 0$ and $x = l$.[1)] Equations (17), (19), and (20) decide the question of the radiation of the generator in the approximation assumed.

For further analysis it is convenient to introduce the two dimensionless parameters

$$y = \sigma^2 A^2 = 16\pi\sigma^2(S_1 + S_2)/c\sqrt{\varepsilon_0}[1 - (r_1 + r_2)/2],$$
$$\eta = 2\pi l\beta/\lambda\varepsilon_0[1 - (r_1 + r_2)/2]. \tag{21}$$

The parameter y is proportional to the emitted flux and is equal to the ratio of the "saturated width" to the line width [cf. Eq. (15)] in the anti-

[1)]Absorption in the reflecting layers leads also to a discontinuity in phase upon reflection, which somewhat alters the conditions for generation (17).

node of the standing wave. The parameter η is proportional to the probability of excitation of the system, i.e., to the energy introduced into the system to obtain the population inversion, and is equal to the product of β, the maximum of the imaginary part of $\Delta\epsilon$, and the resolving power $R = \omega/\delta\omega = 2\pi l/\lambda[1-(r_1+r_2)/2]$ of a Fabry-Perot interferometer equivalent to the generator in the absence of excitation.

With this notation, Eq. (19) has the form

$$y = \tfrac{1}{4}(\sqrt{1+8\eta}-3)(\sqrt{1+8\eta}+1)$$
$$= 2\eta - \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1+8\eta}. \qquad (22)$$

The presence of a generation threshold follows from Eq. (22):

$$\eta = 1. \qquad (23)$$

Equation (23) agrees completely with the usually cited criterion for the onset of generation. This is not surprising, since the generation threshold corresponds to $A = 0$, when there is no saturation and the medium is homogeneous. From Eq. (23) it follows, among other things, that the parameter η can be considered as the excitation probability expressed in threshold units.

Figure 2 shows the graph of y as a function of η (curve 1). Since y is the square of the ratio of the saturated width (at the maximum field value) to the width γ, the ordinates of the graph represent directly the degree of saturation attained in the antinodes. It can be seen from Fig. 2 that exceeding the threshold excitation by a factor of 2 already yields a saturation of order unity and consequently a marked inhomogeneity of the generator medium.

As has been shown, [2] a splitting of the spontaneous emission line becomes noticeable at a saturation of $y = 5$. As can be seen from Fig. 2, this effect should appear at excitation probabilities several times larger than the threshold value.

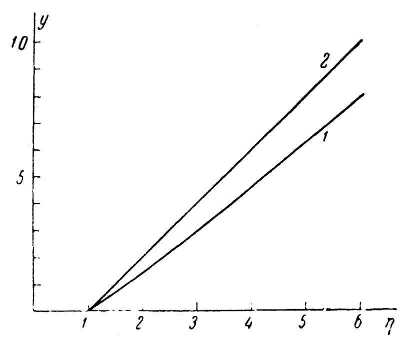

FIG. 2

It is of interest to compare Eq. (22) with the conclusions of a theory that does not take the inhomogeneity of the medium into account. Such calculations have been carried out for a microwave molecular generator [1] the dimensions of which were much smaller than the wavelength and in which there was consequently no inhomogeneity. In our notation, the analog to Eq. (22) looks like:

$$y = 2(\eta - 1) \qquad (24)$$

it is drawn in Fig. 2 for comparison (curve 2). The difference between (22) and (24) begins close to the threshold; the slopes of curves 1 and 2 differ by a factor of 1.5. The difference between 1 and 2 increases in absolute magnitude as η increases; relatively, it decreases as $1/\sqrt{\eta}$. This means that at large levels of excitation of the system the regions close to the nodes, where the radiation of the medium is small, do not play a very important role. The effect of a slight inhomogeneity of the medium brought about by saturation ($|\Delta\epsilon| \ll \epsilon_0$) is easy to understand, since the steady-state emission of the generator is determined by just this small part of ϵ, and not by ϵ_0.

4. We now consider the case when the shape of the luminescence line is determined not only by relaxation processes but also by inhomogeneous broadening. We shall assume that the transition frequencies ω' are different for different atoms of the system and that the number distribution of the atoms is given by some function $W(\omega')$. Then the dielectric constant will be expressed as

$$\Delta\varepsilon'' = -\beta \int_{-\infty}^{\infty} \frac{W(\omega')}{(\omega-\omega')^2/\gamma^2 + 1 + \sigma^2|E|^2}\, d\omega'. \qquad (25)$$

Calculating the field with the aid of Eqs. (14) and (25) and introducing the expression so obtained into the boundary conditions (10), we obtain the following equation for $y = \sigma^2 A^2$:

$$y = 2\eta\left\{1 - \int_{-\infty}^{\infty}\left[\frac{1+(\omega-\omega')^2/\gamma^2}{1+(\omega-\omega')^2/\gamma^2 + y}\right]^{1/2} W(\omega')\, d\omega'\right\}. \qquad (26)$$

By letting y go to zero, we easily obtain the generation threshold in the following form:

$$\eta \int_{-\infty}^{\infty} \frac{W(\omega')\, d\omega'}{1+(\omega-\omega')^2/\gamma^2} = 1. \qquad (27)$$

Let us consider the case of a dispersion distribution of the atoms according to frequency, such that the mean frequency coincides with the frequency of the field:

$$W(\omega') = \frac{\Gamma}{\pi}\frac{1}{(\omega-\omega')^2 + \Gamma^2}. \qquad (28)$$

The integral in Eq. (26) comes down to complete elliptical integrals of the third ($\gamma \neq \Gamma$) or second ($\gamma = \Gamma$) kind:

$$y = 2\eta\left\{1 - \frac{2}{\pi}\frac{\gamma}{\Gamma}\frac{1}{\sqrt{1+y}}\Pi\left(\frac{\pi}{2}, -\left(1-\frac{\gamma^2}{\Gamma^2}\right), \sqrt{\frac{y}{1+y}}\right)\right\}, \quad \gamma \neq \Gamma;$$

$$y = 2\eta\left\{1 - \frac{2}{\pi}\frac{1}{\sqrt{1+y}}K\left(\sqrt{\frac{y}{1+y}}\right)\right\}, \quad \gamma = \Gamma. \tag{29}$$

The threshold condition for generation (27) becomes in this case

$$\eta = 1 + \Gamma/\gamma. \tag{30}$$

As was to be expected, the threshold increases with Γ/γ.

An unexpected result associated with inhomogeneous broadening is that when the threshold is exceeded by a given amount, the degree of saturation is less, the greater Γ/γ. To illustrate this effect, Eqs. (29) were solved numerically,[2)] and graphs of $y/2\eta$ as a function of $\eta\gamma/(\gamma + \Gamma)$ were constructed (Fig. 3). The quantity $\eta\gamma/(\gamma + \Gamma)$ characterizes the excitation power of the system, expressed in threshold units. The ratio $y/2\eta$ is proportional to the probability of stimulated emission from an atom averaged over the length of the generator. It approaches unity for infinite excitation power. Thus, $y/2\eta$ can be thought of as the ratio of the mean probability for stimulated emission for a given excitation power to the limiting probability for infinite η.

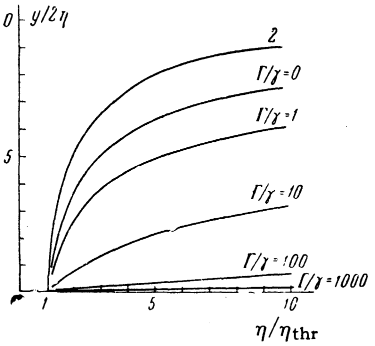

FIG. 3

For comparison, the graph of $y/2\eta$ for a homogeneous generator is also shown in Fig. 3 (curve 2, which corresponds to curve 2 in Fig. 2). From the figure it can be seen that the curves $y/2\eta$ and η/η_{thr} the more gradually approach unity, the greater Γ/γ.

In order to make a quantitative estimate of this dependence, we make use of an approximate formula, which is valid for $y \gg 1$ (practically, for $y \gtrsim 10$):

$$\frac{y}{2\eta} = 1 - \frac{2}{\pi}\frac{\arcsin\sqrt{1 - y\gamma^2/\Gamma^2}}{\sqrt{1 - y\gamma^2/\Gamma^2}}. \tag{31}$$

From Eq. (31) it can be seen that $y/2\eta$ is a function of $y\gamma^2/\Gamma^2$ or $(\gamma/\Gamma)(\gamma/\Gamma + 1)\eta/\eta_{thr}$. Consequently, under these conditions it is necessary that the degree of exceeding the excitation threshold be proportional to Γ/γ in order to attain a given ratio $y/2\eta$. For example, in order to obtain $y/2\eta \sim 1$, it is necessary that $y\gamma^2/\Gamma^2 \sim 1$, i.e., $\eta/\eta_{thr} \sim \Gamma/\gamma$. And, if η/η_{thr} is of the order of a few units and $y\gamma^2/\Gamma^2 \ll 1$, then

$$\frac{y}{2\eta} = \frac{1}{\pi}\frac{\gamma}{\Gamma}\frac{\eta}{\eta_{thr}} \tag{32}$$

Physically this means that for $\Gamma \neq 0$ it is necessary to supply considerable excitation (in threshold units) to the system in order that practically all of it can be channelled into stimulated transitions instead of radiationless or emissive relaxation processes.

These rules remain qualitatively the same for any bell-shaped function $W(\omega')$ —only the numerical coefficients change, and the dependence on the ratio of the widths is as before.

In conclusion, it is appropriate to comment on the possibility of comparing theory and experiment. It is possible to compare the absolute value of the generated flux or to measure the relative power as a function of η. In the second, experimentally more simple variant, difficulties arise because of the necessity of varying the excitation over wide limits (of the order of $10\eta_{thr}$) in order to demonstrate the nonlinearity of the function $y(\eta)$. In the first variant relatively difficult measurements of y, excitation power, and a number of other parameters (γ, Γ, r) are necessary, for which data on the damping time and the intensity and width of the spontaneous emission line can be utilized. The greatest difficulty, however, is in the determination of the lifetimes of the lower levels. Indeed, one can think of stating the question in reverse —the rules described above are so sensitive to the ratio γ/Γ that they can be used to determine it experimentally.

[2)]Complete elliptic integrals of the third kind are tabulated in [4].

[1]N. G. Basov and A. M. Prokhorov, UFN **57**, 485 (1955).

[2]S. G. Rautian and I. I. Sobel'man, JETP **41**, 456 (1961), Soviet Phys. JETP **14**, 328 (1962).

[3]A. G. Fox and T. Li, Bell System Tech. J. **40**, 453 (1961).

[4]C. Heuman, J. Math. and Phys. **20**, 127, 336 (1941).

Translated by L. M. Matarrese
320