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FEYNMAN AMPLITUDES FOR NONRELATIVISTIC PROCESSES

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An expression $F_{nl}^{(0)}$ is found for the limiting value of the relativistic amplitude F_{nl}^{r} of an arbitrary Feynman diagram with all internal lines representing scalar particles (n is the number of internal lines and l the number of independent closed loops) for the case in which the kinetic energy transferred at external vertices and the energy developed at each vertex of the diagram are small in comparison with the masses of the virtual particles. For $n > 5l/2$ the amplitude $F_{nl}^{(0)}$ is identical with the nonrelativistic amplitude considered in [2,3]. For $n < 5l/2$ the amplitude $F_{nl}^{(0)}$ does not depend on the nonrelativistic kinematical invariants and has no singularities in the nonrelativistic region. For $n = 5l/2$ the amplitude $F_{nl}^{(0)}$ has a logarithmic dependence on the nonrelativistic invariants. The order of magnitude of the relativistic corrections to $F_{nl}^{(0)}$ is found. An explicit expression is obtained for the relativistic corrections to one-loop diagrams. The numerical value of the relativistic correction is found as a function of the energy of the incident particle x in the case of the triangular diagrams of direct nuclear reactions of the type $A + x \rightarrow B + y$, as considered in [2]. The results obtained are extended to particles of arbitrary spin.

1. INTRODUCTION

A theory of direct nuclear reactions is now being developed which is based on the idea that the direct processes are caused by singularities in the amplitude, as a function of the momentum transfer, near the region of physical values.[1] It is assumed that the contributions from the singularities to the direct-process amplitude can be represented by Feynman diagrams with vertex functions which can either be determined directly from experiment or calculated by means of nuclear models.

When direct processes in the region of low and intermediate energies are considered in the framework of this theory, an important question arises: in constructing the amplitude of a reaction can one confine oneself to using only nonrelativistic Feynman diagrams [1]; in other words, is it possible to have a closed nonrelativistic description of direct processes in the low-energy region? It is particularly important to settle this question because, as has been shown previously,[2,3] the analytic properties of nonrelativistic diagrams and their explicit calculation are radically simplified as compared with the relativistic case. To settle this question it is necessary to know what sort of diagrams make the main contribution to the amplitude of the direct process and under what conditions one can replace relativistic propagators by nonrelativistic propagators in Feynman diagrams—that is, go over from relativistic to nonrelativistic diagrams.

The Feynman diagrams for direct nuclear reactions [2] satisfy at low and moderate energies the following conditions:

A. 1) The kinetic energies transferred at the external vertices of the diagram are small in comparison with the masses of the virtual particles ($\hbar = c = 1$): $|\Delta E| \ll m_i$ (by the energy transferred we mean the difference between the kinetic energies of the external particles coming to the vertex and emerging from it);

A. 2) for a definite choice of the directions of the internal lines the energy Q developed at each vertex of the diagram is small in comparison with the masses of the virtual particles: $|Q| \ll m_i$ (Q is the difference between the masses of the particles coming to the vertex and those leaving it). Introducing a parameter β which character-

[1]By nonrelativistic (or relativistic) diagrams we mean Feynman diagrams with nonrelativistic (or relativistic) propagators.

[2]We have in mind only diagrams in which all internal lines correspond to nuclearly stable nucleonic associations.

izes the smallness of ΔE and $Q - \beta \sim (|\Delta E|/m_i)^{1/2}$ —we shall hereafter for brevity use the name β-diagrams which satisfy the conditions A.1) and A.2).

To settle the question as to which of the β-diagrams are important in the description of direct nuclear reactions we must study the asymptotic properties of the amplitudes of β-diagrams as functions of the small parameter β. These properties are determined first by the structure of the diagram (the number n of internal lines, the number l of independent closed loops, and so on), and secondly by the behavior of the vertex functions for small β. Unfortunately our information about the vertices is at present extremely limited. (The simplest three-ray vertices for direct nuclear reactions have been studied within the framework of the optical model by Shapiro[4]; it was found that when the diffuseness of the edge of the nucleus is taken into account the vertices have their own nonrelativistic singularities.) Therefore in the present paper the amplitudes F^r_{nl} of the β-diagrams have been studied only with unit vertices.

The results are as follows. For $\beta \ll 1$ the amplitudes F^r_{nl} can be represented in the form $F^r_{nl} = F^{(0)}_{nl} + F^{(1)}_{nl}$, where $F^{(0)}_{nl}$ is the main term in the expansion of F^r_{nl} in powers of the small parameter β and $F^{(1)}_{nl}$ is the relativistic correction to the main term. The explicit form of $F^{(0)}_{nl}$ has been found as an integral over Feynman parameters α_i, of a function which depends on the α_i, the external kinematic invariants, and the energies Q developed at the vertices of the diagram, and the order of magnitude in β of the relativistic correction $F^{(1)}_{nl}$ has been indicated. Furthermore it has been shown that for β-diagrams which do not contain unidirectional closed loops the $F^{(0)}_{nl}$ are identical with the nonrelativistic amplitudes F_{nl} of the β-diagrams in cases with $n > 5l/2$.

For $n \geq 5l/2$ the amplitude $F^{(0)}_{nl}$ depends on the nonrelativistic invariants. For $n < 5l/2$ it depends only on the masses of the virtual particles and does not depend on the nonrelativistic invariants, so that all of the dependence on them is contained in the small relativistic correction $F^{(1)}_{nl}$. This corresponds to the fact that for $n < 5l/2$ we can to first approximation set the momenta and kinetic energies of the external particles and the energies developed at the vertices of the diagram equal to zero in the relativistic propagators in the Feynman integral for F^r_{nl}.

As for the analytical properties of β-diagrams, for $n \geq 5l/2$ the main terms $F^{(0)}_{nl}$ have singularities in the nonrelativistic invariants, and the singularities are determined by the nonrelativistic Landau equations considered in [2,3]. For $n < 5l/2$ all of the nonrelativistic singularities of the amplitude F^r_{nl} appear only in the small relativistic correction $F^{(1)}_{nl}$. Therefore we can expect that the analytical properties of the amplitude of a nonrelativistic process will be determined by the singularities corresponding to diagrams with $n \geq 5l/2$.

For $\beta \to 0$ the amplitude F^r_{nl} increases without bound if $n \geq 5l/2$, and approaches a constant if $n < 5l/2$. This last result is most sensitive to the behavior of the vertex functions for $\beta \to 0$. For example, for the diagrams of Fig. 3, $n = 3l+1$, and according to Eq. (26) $F^{(0)}_{nl} = \beta^{-(l+2)}$, so that it would seem that more complicated diagrams would give larger contributions to the amplitude for scattering of a nucleon by a deuteron. If, however, we recall the well known fact that the vertex for the dissociation of a deuteron into a proton and a neutron is proportional to $(\epsilon/m)^{1/4} \sim \beta^{1/2}$, where ϵ is the binding energy of the deuteron and m is the nucleon mass, then all of the diagrams of Fig. 3 are of the same order in β ($\sim 1/\beta$). It can be seen clearly from this example that to reach a final conclusion as to which β-diagrams "survive" in the theory of nonrelativistic processes we must supplement the results of the present paper with results on the behavior of the vertex functions for $\beta \to 0$.

2. STATEMENT OF THE PROBLEM

Let us first consider spinless particles with nonzero rest mass. The diagram with n internal lines and l independent closed loops is described in the relativistic and nonrelativistic theories by the respective integrals

$$F^r_{nl} = \lim_{\delta \to +0} \int \prod_{s=1}^{l} d^4k_s \prod_{i=1}^{n} (q_i^2 + m_i^2 - i\delta)^{-1}, \tag{1}$$

$$F_{nl} = \lim_{\delta \to +0} \int \prod_{s=1}^{l} d^3k_s d\varepsilon_s \prod_{i=1}^{i} (\mathbf{q}_i^2 - 2m_i\mathscr{E}_i - i\delta)^{-1}, \tag{2}$$

where q_i, $\mathscr{E}_i$, m_i are the four-momentum, the kinetic energy, and the mass of the i-th internal line.

Let the diagram in question satisfy conditions A.1) and A.2) as formulated in the introduction—that is, let it be a β-diagram. We note that for condition A.1) to be satisfied it is not required that all of the external particles be nonrelativistic. For example, it is satisfied by photonuclear reactions with photon energies $E_\gamma \ll m_i$. It is also clear that if condition A.2) is satisfied for a given relativistic diagram this is so only for a definite choice of the directions of the internal lines; we shall regard these directions as fixed in what follows.

Let us study the limiting form of the relativistic amplitude F^r_{nl} for small β, and in particular let us find under what conditions the relativistic amplitude F^r_{nl} goes over into the nonrelativistic amplitude F_{nl}. The conditions (A) alone are insufficient for the possibility of replacing (1) and (2), as is seen already from the difference between the conditions for convergence at large momenta in the relativistic and nonrelativistic theories: the integral (1) converges for $n > 2l$, and the integral (2) for $n > 5l/2$.[3)]

It is convenient to make the study after introducing the Feynman parametrization[5] in the integrals (1) and (2) and using the integral representations for F^r_{nl} and F_{nl} that have been obtained in papers by Chisholm[6] and the present writers.[2,3]

After the introduction of the Feynman parameters α_i the integrals (1) and (2) reduce to the forms

$$F^r_{nl} = (n-1)! \lim_{\delta\to+0} \int_0^1 \prod_{i=1}^n d\alpha_i\, \delta\Big(\sum_{k=1}^n \alpha_k - 1\Big) \times \int_{-\infty}^{\infty} \prod_{s=1}^l d^4 k_s\, [Q_r(k_s, \alpha_i) - i\delta)]^{-n}, \tag{3}$$

$$F_{nl} = (n-1)! \lim_{\delta\to+0} \int_0^1 \prod_{i=1}^n d\alpha_i\, \delta \times \Big(\sum_{k=1}^n \alpha_k - 1\Big) \int_{-\infty}^{\infty} \prod_{s=1}^l d^3\mathbf{k}_s\, d\varepsilon_s\, [Q(\mathbf{k}_s, \varepsilon_s, \alpha_i) - i\delta]^{-n}; \tag{4}$$

$$Q_r(k_s, \alpha_i) = \sum_{i=1}^n \alpha_i (q_i^2 + m_i^2) = \sum_{s,t=1}^l a_{st}(\mathbf{k}_s\mathbf{k}_t - k_{s0}k_{t0}) + \sum_{s=1}^l (\mathbf{a}_s\mathbf{k}_s - a_{s0}k_{s0}) + c_r, \tag{5}$$

$$Q(\mathbf{k}_s, \varepsilon_s, \alpha_i) = \sum_{i=1}^n \alpha_i (\mathbf{q}_i^2 - 2m_i\mathscr{E}_i) = \sum_{s,t=1}^l a_{st}\mathbf{k}_s\mathbf{k}_t + \sum_{s=1}^l \mathbf{a}_s\mathbf{k}_s + c - 2\sum_{s=1}^l b_s\varepsilon_s; \tag{6}$$

$$b_s = \sum_{i=1}^n \omega_{is}\alpha_i m_i, \tag{7}$$

where $\omega_{is} = 0$ if q_i does not belong to the s-th loop, and $\omega_{is} = +1\,(-1)$ if q_i is in the s-th loop and is directed clockwise (counterclockwise).

The integral representations for F^r_{nl} and F_{nl} are of the forms

$$F^r_{nl} = (i\pi^2)^l (n-2l-1)! \lim_{\delta\to+0} \int_0^1 \prod_{i=1}^n d\alpha_i\, \delta\Big(\sum_{k=1}^n \alpha_k - 1\Big)\Lambda^{-2} \times (X_r/\Lambda - i\delta)^{-(n-2l)}, \quad n > 2l, \tag{8}$$

$$F_{nl} = (i\pi^{5/2})^l\, \Gamma\Big(n - \frac{5l}{2}\Big) \lim_{\delta\to+0} \int_0^1 \prod_{i=1}^n d\alpha_i\, \delta\Big(\sum_{k=1}^n \alpha_k - 1\Big) \times \prod_{s=1}^l \delta(b_s)\, \Lambda^{-3/2}(X/\Lambda - i\delta)^{-(n-5l/2)}, \quad n > 5l/2; \tag{9}$$

$$\Lambda = \det(a_{st}), \tag{10}$$

$$\frac{X_r}{\Lambda} = -\frac{1}{4}\sum_{s,t=1}^l a_{st}^{-1}(\mathbf{a}_s\mathbf{a}_t - a_{s0}a_{t0}) + c_r, \tag{11}$$

$$\frac{X}{\Lambda} = -\frac{1}{4}\sum_{s,t=1}^l a_{st}^{-1}\mathbf{a}_s\mathbf{a}_t + c, \tag{12}$$

where (a_{st}^{-1}) is the matrix reciprocal of the matrix (a_{st}).

Let us find the connection between X_r and X. We suppose that the first l of the vectors q_i are identical with the variables of integration k_s and introduce convenient energy variables ϵ_s:

$$q_s = k_s, \quad k_{s0} = m_s + \varepsilon_s, \quad s = 1, 2, \ldots, l. \tag{13}$$

Then the q_{i0} in Eq. (5) take the form

$$q_{i0} = m_i + \mathscr{E}_i^r, \quad \mathscr{E}_i^r = \sum_{s=1}^l \omega_{is}\varepsilon_s + Q_i + E_i^r,$$

where Q_i and E_i^r are combinations of the energies developed at the vertices and the relativistic kinetic energies of the external particles corresponding to the i-th internal line. Substituting this expression for q_{i0} in Eq. (5), we get

$$Q_r = Q - \sum_{s,t=1}^l a_{st}\varepsilon_s\varepsilon_t - 2\sum_{s=1}^l d_s\varepsilon_s + d,$$

where d_s and d are given by the formulas

$$d_s = \sum_{i=1}^n \omega_{is}\alpha_i(Q_i + E_i^r),$$
$$d = -\sum_{i=1}^n \alpha_i[(Q_i + E_i^r)^2 + 2m_i(E_i^r - E_i)], \tag{14}$$

if we take into account the fact that the $\mathscr{E}_i$ in Eq. (6) are obtained from the $\mathscr{E}_i^r$ by replacing the relativistic kinetic energies E_i^r of the external particles by the nonrelativistic energies E_i.

From a comparison of this expression for Q_r with that obtained when we substitute Eq. (13) in Eq. (5) it follows that

$$c_r = c + 2\sum_{s=1}^l (b_s + d_s)\, m_s - \sum_{s,t=1}^l a_{st}m_s m_t + d,$$
$$a_{s0} = 2\Big(b_s + d_s - \sum_{t=1}^l a_{st}m_t\Big). \tag{15}$$

[3)]It is assumed that there are no divergences inside the diagram; in particular, each closed loop must contain not fewer than three lines.

From these relations and Eq. (11) and (12) we get the identical relation

$$\frac{X_r}{\Lambda} \equiv \sum_{s,t=1}^{l} a_{st}^{-1}(b_s + d_s)(b_t + d_t) + \frac{X}{\Lambda} + d. \qquad (16a)$$

We now note that for a β-diagram the quantities d_s and d are small and of the respective orders β^2 and β^4.

As can be seen from the further discussion (Sec. 3), the quantities d_s and d are of importance only in calculating the corrections of order β^2 to the main terms $F_{nl}^{(0)}$ in the expansion of F_{nl}^r in powers of β. Therefore in calculating the main terms of the amplitude of a β-diagram we can drop d_s and d and write Eq. (16a) in the form

$$X_r/\Lambda \cong \sum_{s,t=1}^{l} a_{st}^{-1} b_s b_t + X/\Lambda. \qquad (16b)$$

For β-diagrams the representation (8) takes the form

$$F_{nl}^r = (i\pi^2)^l (n - 2l - 1)! \lim_{\delta\to+0} \int_0^1 \prod_{l=1}^{n} d\alpha_l \delta\left(\sum_{k=1}^{n} \alpha_k - 1\right)$$
$$\times \Lambda^{-2}\left(\sum_{s,t=1}^{l} a_{st}^{-1} b_s b_t + \frac{X}{\Lambda} - i\delta\right)^{-(n-2l)}. \qquad (17)$$

In Eq. (17) the quantity X/Λ is of the order β^2 relative to $\sum_{s,t} a_{st}^{-1} b_s b_t$,[2,3] the function $\sum_{s,t} a_{st}^{-1} b_s b_t$ is nonnegative for $\alpha_i > 0$ and is zero only for $b_s = 0$ $(s = 1, 2, \ldots, l)$.[6]

Changing the independent variables of integration in Eq. (17) from $\alpha_1, \ldots, \alpha_l, \alpha_{i_{l+1}}, \ldots, \alpha_{i_{n-1}}$ to $b_1, \ldots, b_l, \alpha_{i_{l+1}}, \ldots, \alpha_{i_{n-1}}$, we get

$$F_{nl}^r = (i\pi^2)^l (n - 2l - 1)!\, D\left(\frac{\alpha_s, \alpha_{i_k}}{b_s, \alpha_{i_k}}\right)$$
$$\times \lim_{\delta\to+0} \int_0^1 d\alpha_{i_{l+1}} \ldots \int_0^{1-\alpha_{i_{l+1}}-\ldots-\alpha_{i_{n-2}}} d\alpha_{i_{n-1}} J_{nl}(\alpha_{i_k}), \qquad (18)$$

$$J_{nl}(\alpha_{i_k}) = \prod_{r=1}^{l} \int_{b_r'}^{b_r''} db_r \Lambda^{-2}\left(\sum_{s,t=1}^{l} a_{st}^{-1} b_s b_t + \frac{X}{\Lambda} - i\delta\right)^{-(n-2l)} \qquad (19)$$

In Eq. (18) the Jacobian D of the transformation depends only on the masses of the virtual particles.

3. SCALAR PARTICLES. GENERAL CASE

Let us first consider β-diagrams which do not contain unidirectional closed loops [i.e., loops with all of their lines directed clockwise (or counterclockwise)], and prove that for $n > 5l/2$ such diagrams are nonrelativistic. This means that the main term in the expansion in powers of β of the amplitude F_{nl}^r, Eq. (18), is identical with the nonrelativistic amplitude F_{nl} of the diagram, Eq. (9). Since the integrand in Eq. (19) is analytic in X, it suffices to treat the case in which we have $X > 0$ in the range of integration over α_i.[4)]

The function $\Lambda(\alpha)$ is positive for $\alpha_i > 0$.[6] Therefore because of the positive definiteness of $\sum_{s,t} a_{st}^{-1} b_s b_t$ for $\beta \ll 1$ the main contribution to the integral (18) for $n > 5l/2$ is that from the range of integration over b_s near $b_s = 0$. If $b_s' < 0 < b_s''$ $(s = 1, 2, \ldots, l)$, then $J_{nl} \sim \beta^{-2(n-5l/2)}$, so that $J_n \to \infty$ for $\beta \to 0$. If, on the other hand, even one of the b_s cannot be zero $(b_s' b_s'' > 0)$, then J_{nl} remains bounded for $\beta \to 0$.

Since by hypothesis the diagram contains no unidirectional loops, in the integration over α_{ik} in Eq. (18) there is always a whole region of values of the α_{ik} in which $b_s' < 0 < b_s''$. Therefore in calculating the main term in the expansion of Eq. (18) in the parameter β we can take $b_s = 0$ $(s = 1, 2, \ldots, l)$ in the smooth functions X, Λ, and a_{st}^{-1}:

$$\widetilde{X} = X|_{b_s=0}, \qquad \widetilde{\Lambda} = \Lambda|_{b_s=0}, \qquad \widetilde{a}_{st}^{-1} = a_{st}^{-1}|_{b_s=0},$$

and extend the limits of the integrations over b_s from $-\infty$ to $+\infty$. Then Eq. (19) takes the form

$$J_{nl}^{(0)}(\alpha_{i_k}) = \int_{-\infty}^{\infty} \prod_{r=1}^{l} db_r \widetilde{\Lambda}^{-2}\left(\sum_{s,t=1}^{l} \widetilde{a}_{st}^{-1} b_s b_t + \frac{\widetilde{X}}{\widetilde{\Lambda}} - i\delta\right)^{-(n-2l)}$$
$$= \pi^{l/2} \frac{\Gamma(n - 5l/2)}{(n - 2l - 1)!} \widetilde{\Lambda}^{-3/2}\left(\frac{\widetilde{X}}{\widetilde{\Lambda}} - i\delta\right)^{-(n-5l/2)} \qquad (20)$$

Replacing $J_{nl}(\alpha_{ik})$ in Eq. (18) by the expression (20), we verify that the main term $F_{nl}^{(0)}$ in the expansion of the amplitude F_{nl}^r in powers of β is identical with the nonrelativistic amplitude (9).

We note that one can arrive at the same result in a more formal way by using the formula

$$\lim_{x\to 0} \frac{(x - i\delta)^{\nu - l/2}}{\left(\sum_{i,j=1}^{l} \alpha_{ij} t_i t_j + x - i\delta\right)^{\nu}}$$
$$= \pi^{l/2} \frac{\Gamma(\nu - l/2)}{\Gamma(\nu)} [\det(\alpha_{ij})]^{-1/2} \prod_{i=1}^{l!} \delta(t_i), \qquad (21)$$

which is valid if the matrix (α_{ij}) is positive definite and $\nu > l/2$.

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4)Simultaneous vanishing of X and all of the b_s in the range of integration is possible only at the singularities of the amplitude F_{nl}^r regarded as a function of the external invariants.[2,3]

Now let $2l < n < 5l/2$. In this case for $\beta \to 0$ the singularity of the integrand in Eq. (19) for $b_s = 0$ is an integrable one, and the main term of the expansion of F^r_{nl} in powers of β is obtained by neglecting X/Λ in Eq. (17):

$$F^{(0)}_{nl} = (i\pi^2)^l (n-2l-1)! \int_0^1 \prod_{i=1}^n d\alpha_i \delta \left(\sum_{k=1}^n \alpha_k - 1 \right) \Lambda^{-2} \times \left(\sum_{s,t=1}^l a_{st}^{-1} b_s b_t \right)^{-(n-2l)}. \qquad (22)$$

It can be seen from Eq. (22) that $F^{(0)}_{nl}$ is a function only of the masses of the virtual particles and does not depend on the external kinematical invariants. For $2l < n < 5l/2$ the integral (1) converges for relativistic values of the momenta and energies of the virtual particles.

β-diagrams with $n = 5l/2$ require special investigation. In this case the main term of the expansion of F^r_{nl} can be represented in the form

$$F^{(0)}_{nl} = (i\pi^{5/2})^l \lim_{\delta \to +0} \int_0^1 \prod_{i=1}^n d\alpha_i \delta \left(\sum_{k=1}^n \alpha_k - 1 \right) \prod_{s=1}^l \delta(b_s) \times \Lambda^{-3/2} \ln \frac{X/\Lambda - i\delta}{Y(\alpha_i, m_k)}, \qquad (23)$$

where Y is a function only of the variables α_i and the masses m_i of the virtual particles; explicit calculation of $Y(\alpha_i, m_k)$ for an arbitrary diagram is difficult, but $|Y| \sim m^2$, where m is an average mass of the virtual particles in the diagram.

It can be seen from Eq. (23) that for $n = 5l/2$ the main term $F^{(0)}_{nl}$ depends on the nonrelativistic invariants only through the function X, which is given, just as for nonrelativistic diagrams with $n > 5l/2$, by Eq. (12). Then the remaining terms in the expansion of F^r_{nl}, which depend on the nonrelativistic invariants, are of the order β^2 relative to $F^{(0)}_{nl}$. It follows from this fact and from the presence of the δ functions $\delta(b_s)$ in the integrand in Eq. (23) that for $n = 5l/2$ the singularities of the main terms of the expansion of F^r_{nl} in powers of β are determined by the nonrelativistic Landau equations considered previously.[2,3] It can be seen from Eq. (23) that for $\beta \to 0$ the amplitude $F^{(0)}_{nl}$ increases as $\ln \beta$.

Without going into the calculations, we present the orders of magnitude of the fractional relativistic corrections to the main terms of the expansions of the amplitudes F^r_{nl}:

$$F^r_{nl} = F^{(0)}_{nl}(1 + \delta_{nl}); \qquad (24)$$

$$\delta_{nl} \sim \begin{cases} \beta^2 & 2n - 5l = 0 \\ \beta & 2n - 5l = \pm 1 \\ \beta^2 \ln \beta & 2n - 5l = \pm 2 \\ \beta^2 & 2n - 5l = \pm 3, \pm 4, \ldots \end{cases} \qquad (25)$$

Also it has been shown above that the $F^{(0)}_{nl}$ are of the following orders of magnitude for $\beta \to 0$:

$$F^{(0)}_{nl} \sim \begin{cases} \beta^{-(2n-5l)} & n > 5l/2 \\ \ln \beta & n = 5l/2 \\ \text{const} & n < 5l/2 \end{cases}. \qquad (26)$$

Up to now we have been considering β-diagrams which do not contain unidirectional closed loops. Now suppose a β-diagram contains λ unidirectional loops composed of ν different internal lines. For $n > 5l/2$ and $n = 5l/2$ (with $l \geq 2$) such β-diagrams will be of the respective orders $\beta^{\nu-\lambda}$ and $\beta^{\nu-\lambda}(\ln \beta)^{-1}$ relative to those of β-diagrams of the same structures but not containing any unidirectional loops. Since $\nu - \lambda \geq 2$, such a β-diagram will be only a small relativistic correction to $F^{(0)}_{nl}$. For $2l < n < 5l/2$ the main term of the amplitude of a β-diagram containing unidirectional loops is given as before by Eq. (22), and no additional small factors appear.

Let us summarize some of the analytical properties of the amplitudes F^r_{nl} of β-diagrams as functions of the external kinematical invariants. Let $F^r_{nl} = F^{(0)}_{nl} + F^{(1)}_{nl}$, where $F^{(0)}_{nl}$ is the main term of the expansion of F^r_{nl} in powers of β [cf. Eqs. (9), (22), (23)]. For $n \geq 5l/2$ the singular points of $F^{(0)}_{nl}$ lie in the nonrelativistic region. For $n < 5l/2$ the amplitude $F^{(0)}_{nl}$ does not depend on the kinematical invariants, and all of the dependence on them is contained in $F^{(1)}_{nl}$, but this term is a small relativistic correction to $F^{(0)}_{nl}$. $F^{(1)}_{nl}$ can have nonrelativistic singularities as well as relativistic ones. We may suppose that diagrams with $n < 5l/2$ are unimportant in the description of relativistic processes, and in particular do not contribute to the mechanism of direct nuclear reactions.

Let us consider some examples. Figure 1 shows two diagrams which describe the same reaction. Because the binding energy of the deuteron is small

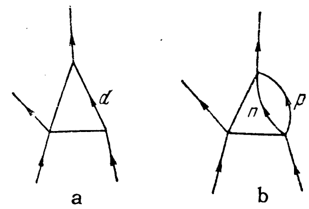

FIG. 1

these diagrams have closely similar nonrelativistic singularities. The singularities of the diagram of Fig. 1, b, however, are contained only in the relativistic correction $F_{42}^{(1)}$. Figures 2 and 3 show two types of diagrams describing the scattering of a neutron by a deuteron. The main terms $F_{nl}^{(0)}$ in the expansions for the diagrams of Fig. 2, a, b and for all of the diagrams of Fig. 3 have nonrelativistic singularities. The singularities of the diagrams of Fig. 2, c, which are more complicated than diagrams a and b, are contained in the relativistic corrections $F_{nl}^{(1)}$.

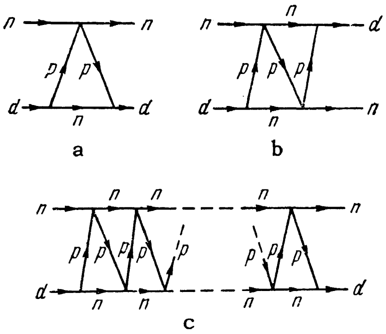

FIG. 2

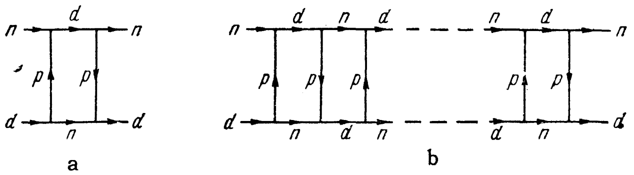

FIG. 3

4. SCALAR PARTICLES. ONE-LOOP DIAGRAMS

Let us apply the results of Sec. 3 to one-loop diagrams. Since the conditions for convergence of relativistic and nonrelativistic one-loop diagrams are the same ($n \geq 3$), one-loop nonunidirectional β-diagrams are always nonrelativistic. For a relativistic one-loop diagram the forms of Λ and X_r are as follows [7]:

$$\Lambda = \sum_{i=1}^{n} \alpha_i, \qquad X_r = \sum_{i,k=1}^{n} \alpha_i \alpha_k m_i m_k y_{ik};$$
$$y_{ik} = y_{ki} = (m_i^2 + m_k^2 + p_{ik}^2)/2m_i m_k, \quad y_{ii} = 1. \tag{27}$$

For a β-diagram the y_{ik} can be represented in the form

$$y_{ik} = \omega_i \omega_k \left(1 + \eta_{ik} + \eta_{ik}^{(4)} + O(\beta^6)\right). \tag{28}$$

Here η_{ik} are the nonrelativistic invariants of order β^2 given by Eq. (13) of [2], and the quantities $\eta_{ik}^{(4)}$ of order β^4 are given by the formula

$$\eta_{ik}^{(4)} = \omega_i \omega_k \left[Q_{ik}^2 - E_{ik}^2 - 2M_{ik} E_{ik}^{(4)} \right] / 2m_i m_k;$$

ω_i, Q_{ik}, E_{ik}, and M_{ik} are defined in Sec. 4 of [2], and the quantity $E_{ik}^{(4)}$ is made up of the fourth-order terms in the expansion in powers of (v/c) of the relativistic kinetic energies of the external particles in the same way that E_{ik} is made up of the nonrelativistic kinetic energies. We get from Eqs. (27) and (28)[5]

$$X_r = b^2 + X_1 + d^{(4)} + O(\beta^6);$$
$$b = \sum_{i=1}^{n} \alpha_i \omega_i m_i, \qquad X_1 = \sum_{i,k=1}^{n} \alpha_i \alpha_k \omega_i \omega_k m_i m_k \eta_{ik},$$
$$d^{(4)} = \sum_{i,k=1}^{n} \alpha_i \alpha_k \omega_i \omega_k m_i m_k \eta_{ik}^{(4)}. \tag{29}$$

The amplitude F_{n1}^r takes the form

$$F_{n1}^r = i\pi^2 (n-3)! \lim_{\delta \to +0} \int_0^1 \prod_{i=1}^{n} d\alpha_i \delta \left(\sum_{k=1}^{n} \alpha_k - 1 \right) \times \left[b^2 + X_1 + d^{(4)} + O(\beta^6) - i\delta \right]^{-(n-2)}. \tag{30}$$

For $n \geq 4$ the calculation of the main and first-correction terms in the expansion of F_{n1}^r in powers of β leads to the result

$$F_{n1}^r = F_{n1}^{(0)} + F_{n1}^{(1)} = i\pi^{5/2} \Gamma\left(n - \frac{5}{2}\right) \lim_{\delta \to +0} \int_0^1 \prod_{i=1}^{n} d\alpha_i \times \delta\left(\sum_{k=1}^{n} \alpha_k - 1\right) \delta(b) (X_1 - i\delta)^{-(n-5/2)} \times \left\{ 1 + \left[\left(n - \frac{5}{2}\right) \left(\frac{1}{4} \left(\frac{\partial X_1}{\partial b} \right)^2 - d^{(4)} \right) (X_1 - i\delta)^{-1} - \frac{1}{4} \frac{\partial^2 X_1}{\partial b^2} \right] \right\}. \tag{31}$$

The first term in the curly brackets in Eq. (31) corresponds to $F_{n1}^{(0)}$ and is identical with the nonrelativistic amplitude F_{n1} considered in Sec. 4 of [2]. In accordance with Eq. (25) $F_{n1}^{(1)}/F_{n1}^{(0)} \sim \beta^2$.

The expansion in powers of β of the amplitude of a triangular diagram is of the form

$$F_{31}^r = F_{31}(x_{23}) \left[1 + \delta_{31} + O(\beta^2) \right], \tag{32}$$

where the fractional correction δ_{31} is given by the formula

$$\delta_{31} = \frac{1}{\pi \sqrt{2}} \left(\frac{m_2}{m_1 + m_2} |\eta_{12}|^{1/2} + \frac{m_3}{m_1 + m_3} |\eta_{13}|^{1/2} \right) \times \left[(m_1 + m_3) \ln \frac{m_1}{m_2} - (m_1 + m_2) \ln \frac{m_1}{m_3} \right] \times (m_2 - m_3)^{-1} \varphi(z); \tag{33}$$

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[5]We take this opportunity to note that in the expression for X in Eq. (16) of [2] a term was omitted which, being proportional to b, does not contribute to the amplitude (17), nor to Eq. (19) in [2].

$$\varphi(z) = \begin{cases} 2\sqrt{-z} / \ln \dfrac{1+\sqrt{-z}}{1-\sqrt{-z}}, & -1 < z < 0 \\ \sqrt{z} / \tan^{-1}\sqrt{z}, & z > 0; \end{cases}$$
$$z = (\eta_{23} - \eta_{23}^0)/(\eta_{23}^0 - \eta_{23}^-). \quad (34)$$

Equations (33), (34) correspond to the triangular diagram of Fig. 2, a in [2] and are written in the notations of Sec. 5 of that paper [Eqs. (26)—(28)]. It can be seen from Eq. (33) that $\delta_{31} \sim \beta$, which is in accordance with Eq. (25).

For triangular diagrams of direct nuclear reactions of the type $A + x \rightarrow B + y$, which have been considered in [2], we have found the numerical value of the relativistic correction δ_{31} and its dependence on the energy of the incident particle x. Diagrams were taken whose singularities in t_{xy} [see Eq. (32) in [2]] lie closest to the physical region [which is located in $t_{xy} < 0$]. The calculation was made by the formula (33), in which $z = (t_{xy}^0 - t_{xy})/(t_{xy}^- - t_{xy}^0)$, where t_{xy}^- is the position of the singularity in the t_{xy} plane and t_{xy}^0 corresponds to the point η_{23}^0 defined in [2]:

$$t_{xy}^0 = -2\,(m_2 m_3 \eta_{23}^0 + (M_x - M_y)\,Q),$$
$$Q = M_x + m_3 - M_y - m_2.$$

The point t_{xy}^0 lies to the right of or on the boundary of the physical region in the t_{xy} plane; since, however, $\delta_{31}(z)$ changes slowly with increasing z, the value of $\delta_{31}(0)$ at $z = 0$ characterizes the magnitude of the correction near the right-hand edge of the physical region in the t_{xy} plane.

The results of the calculations are given in the table, which gives the position of the singularity in the t_{xy} plane, the function $\delta_{31}(0)$, the values of t_{xy} at which the correction δ_{31} reaches 10 and 15 percent, and the values of the energy of particle x in the laboratory system for which the middle of the physical region is at the values of t_{xy} given in the fourth and fifth columns (these energies are entered in columns 6 and 7). The diagrams which describe the reaction are shown in Fig. 4.

We see from the table that for the diagrams in question the accuracy of the nonrelativistic approximation is 10 percent or better over a wide range of energies of the incident particles.

Diagram describing reaction	t_{xy}^-, MeV·amu	$-\delta_{31}(0)$, %	t_{xy}, MeV·amu		E_L, MeV	
			$\lvert\delta_{31}\rvert = 10\%$	$\lvert\delta_{31}\rvert = 15\%$	$\lvert\delta_{31}\rvert = 10\%$	$\lvert\delta_{31}\rvert = 15\%$
I	163	5.8	—590	—1850	120	380
II	258	4.4	—2060	—5710	420	1180
III	97.8	3.9	—790	—2150	80	230
IV	277	6.9	—500	—1850	50	200

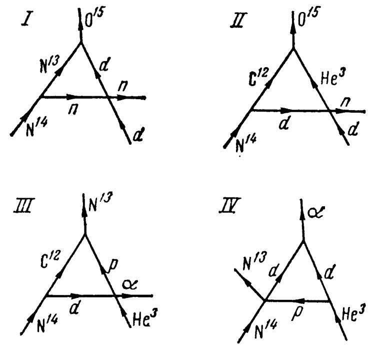

FIG. 4

5. INCLUSION OF SPIN

We shall now extend the results of Secs. 2—4 to the case of particles with spin. Out of the n internal lines of a β-diagram let ν_S lines correspond to particles of spin S which have the propagation function

$$G_S(q_i) = \frac{d_S(q_i)}{q_i^2 + m_i^2 - i\delta}, \quad (35)$$

where $d_S(q_i)$ is a polynomial in q_i of degree 2S. For

$$n > 2l + \sum_{S \geqslant 1/2} S\nu_S$$

the relativistic amplitude of the diagram converges and is of the form

$$F^r_{nl(\nu_S)} = \lim_{\delta \to +0} \int \prod_{t=1}^{l} d^4k_t P(q) \prod_{i=1}^{n} (q_i^2 + m_i^2 - i\delta)^{-1}, \quad (36)$$
$$P(q) = \prod_{(i_{1/2})} d_{1/2}(q_{i_{1/2}}) \prod_{(i_1)} d_1(q_{i_1}) \ldots \quad (37)$$

We shall now show that in convergent β-diagrams which contain no unidirectional loops, for $n > 5l/2$ the relativistic propagators (35) for particles with spin can be replaced by nonrelativistic propagators $(q_i^2 - 2m_i\mathscr{E}_i - i\delta)^{-1}$ which do not depend on the spin. Using the Feynman parametrization, we put Eq. (36) in the form

$$F^r_{nl(\nu_S)} = (n-1)! \lim_{\delta \to +0} \int_0^1 \prod_{i=1}^{n} d\alpha_i \delta\Big(\sum_{k=1}^{n} \alpha_k - 1\Big)$$
$$\times \int_{-\infty}^{\infty} \prod_{t=1}^{l} d^4k_t P(q)\,(Q_r - i\delta)^{-n}, \quad (38)$$

where Q_r is given by Eq. (5); q_i is of the form

$$q_i = \sum_{t=1}^{l} \omega_{it} k_t + p_i,$$

where p_i is a linear combination of the four-momenta of the external particles.

By a shift of origin in k-space we eliminate from Q_r the terms linear in k_t

$$k_t \to k'_t = k_t + \frac{1}{2}\sum_{u=1}^{l} a_{tu}^{-1} a_u, \qquad (39)$$

$$q_i = \sum_{t=1}^{l} \omega_{it} k'_t + p'_i; \quad p'_i = p_i - \frac{1}{2}\sum_{t,u=1}^{l} \omega_{it} a_{tu}^{-1} a_u, \qquad (40)$$

$$Q_r = \sum_{t,u=1}^{l} a_{tu} k'_t k'_u + \frac{X_r}{\Lambda} \qquad (41)$$

and turn the path of integration over k'_{s0}, so as to go over in the usual way[8] to a Euclidean space of variables k''_s. Then, using Eq. (16c) (sic), we get

$$F^r_{nl(\nu_S)} = i^l (n-1)! \lim_{\delta\to+0} \int_0^1 d\alpha_{i_{l+1}} \ldots \int_0^{1-\alpha_{i_{l+1}}-\ldots-\alpha_{i_{n-2}}} d\alpha_{i_{n-1}}\, D$$

$$\times \prod_{\sigma=1}^{l} \int_{b'_\sigma}^{b''_\sigma} db_\sigma \int_{-\infty}^{\infty} \prod_{\tau=1}^{l} d^4 k''_\tau P\Big(\sum_{r=1}^{l} \omega_{ir} k''_r + p'_i\Big)$$

$$\times \Big\{\sum_{t,u=1}^{l} (a_{tu} k''_t k''_u + a_{tu}^{-1} b_t b_u) + \frac{X}{\Lambda} - i\delta\Big\}^{-n}. \qquad (42)$$

The quadratic form

$$\sum_{t,u=1}^{l} (a_{tu} k''_t k''_u + a_{tu}^{-1} b_t b_u)$$

in the $5l$-dimensional space of the variables ($\mathbf{k}''_t$, k''_{t0}, b_t) is now positive definite, and therefore for $n > 5l/2$ and $\beta \to 0$ the integral (42) involves a pole at $k''_\sigma = b_\sigma = 0$, which makes the main contribution to the integral when

$$n > 2l + \sum_S S\nu_S.$$

Using Eq. (21) in the $5l$-dimensional space of the variables ($\mathbf{k}''_\sigma$, $k''_{\sigma 0}$, b_σ), we find the main term in the expansion of the expression (42) in powers of β:

$$F^{(0)}_{nl(\nu_S)} = (i\pi^{5/2})^l \Gamma\Big(n - \frac{5l}{2}\Big) \lim_{\delta\to+\infty} \int_0^1 \prod_{i=1}^{n} d\alpha_i$$

$$\times\, \delta\Big(\sum_{k=1}^{n} \alpha_k - 1\Big) \prod_{t=1}^{l} \delta(b_t)\, P(p'_i)\, \Lambda^{-3/2} \Big(\frac{X}{\Lambda} - i\delta\Big)^{-(n-5l/2)}. \qquad (43)$$

It follows from Eqs. (40) and (15) that

$$p'_{i0} = p_{i0} - \sum_{t,u=1}^{l} \omega_{it} a_{tu}^{-1}(b_u + d_u) + \sum_{t=1}^{l} \omega_{it} m_t$$

$$= p_{i0} + \sum_{t=1}^{l} \omega_{it} m_t - \sum_{t,u=1}^{l} \omega_{it} a_{tu}^{-1} d_u,$$

because of the presence of the δ function $\delta(b_u)$ in Eq. (43). On the other hand, if we set all the $\epsilon_t = 0$, then for the β-diagram $q_{i0} = m_i + 0(\beta^2)$, and therefore

$$p_{i0} = m_i - \sum_{t=1}^{l} \omega_{it} m_t + O(\beta^2).$$

It follows that in Eq. (43) $p'_{i0} = m_i + 0(\beta^2)$. The spatial components of the vector p'_i are of the order of βm. Therefore $P(p'_i) = P(m_i) + 0(\beta)$, and the main term of the expansion of the relativistic amplitude in powers of β is of the form

$$F^{(0)}_{nl(\nu_S)} = P(m_i)\, F_{nl}, \qquad (44)$$

where F_{nl} is the nonrelativistic amplitude (2) for the given diagram and $P(m_i)$ is a constant which depends only on the masses of the virtual particles.

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