KINETIC PROPERTIES OF A LOW-DENSITY PLASMA WITH LARGE RADIATION PRESSURE AND ELECTRON-PHOTON DRAG

L. É. GUREVICH and V. I. VLADIMIROV

A. F. Ioffe Physico-technical Institute, Academy of Sciences, U.S.S.R.

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We have investigated the kinetic coefficients in a low-density plasma with large radiation pressure in which the mean free paths of the electrons and photons are limited by Compton scattering. It is shown that the electron-photon drag causes a reduction in electrical conductivity and a marked increase in thermal emf as compared with a plasma characterized by Coulomb scattering alone. It is also found that scattering on photons eliminates electron loss. A resonance similar to the cyclotron resonance can occur for a wave propagating along a magnetic field, but this resonance occurs at $\omega = \Omega/2$.

At the present time a theory is available to describe the kinetic properties of a plasma consisting of electrons and ions (cf. [1]). Theories have also been developed to describe radiative thermal conductivity [2] and radiative viscosity.[3] It turns out that a photon flux can have an important effect on the conductivity and thermal emf of a plasma and that under certain conditions the electron flux can produce a transverse radiative thermal conductivity in a plasma in a magnetic field. Under these conditions an effect arises which can be called an electron-photon drag.

In an electric field the electrons acquire a momentum which is transferred to the photons, so that the photons are "dragged" by the electrons. On the other hand, a temperature gradient produces a photon "wind" which drags the electrons. In a plasma that is sufficiently rarefied or hot the electron-photon interaction is stronger than the electron-ion interaction and the mechanism indicated above can have an appreciable effect. In the present work we treat the kinetic properties of a nonrelativistic plasma in which the Compton scattering of electrons and photons is stronger than the electron bremsstrahlung due to ion collisions and the associated inverse process. The results of greatest interest are the following.

1. The thermal emf is increased by approximately the ratio $(T/\hbar c)^3/n$ where n is the electron density and T is the temperature in degrees. This ratio is extremely large for the conditions indicated here.
2. The longitudinal electric conductivity is reduced by a factor of 3 while the transverse electrical conductivity (in a magnetic field) is reduced by approximately a factor of 8.
3. The interaction with the photons eliminates electron loss.
4. Under certain conditions the mean free path of low-energy electrons can be increased appreciably.
5. There is a marked change in the dielectric constant of the plasma at external-field frequencies appreciably greater than the frequency of Coulomb collisions but much smaller than the Compton scattering frequency. Under these conditions the dielectric constant increases markedly when $\omega = \Omega/2$ (Ω is the Larmor frequency), that is to say, a resonance arises. This increased dielectric constant obtains up to very small values of the "detuning" $\delta = 2|\omega - (\Omega/2)|/\Omega$; when $\delta = 0$, however, the dielectric constant is approximately unity.

1. KINETIC EQUATIONS FOR THE ELECTRONS AND PHOTONS

We consider a hot low-density nonrelativistic plasma ($T \ll mc^2$, where m is the electron mass) in which scattering of electrons on photons is comparable with scattering of electrons on ions. We limit ourselves here to the case of a plasma in which the density is low enough so that the mean free paths of the photons and electrons are determined by Compton scattering.

The condition that must be satisfied if both scattering mechanisms are to be comparable is $\gamma \sim 1$, where γ is a parameter we introduce below

(1.10). In this case the minimum dimensions of the system (determined by the mean free path of the photons $l_f \approx 1/n\Sigma_f$, where n is the number of electrons per cubic centimeter while $\Sigma_f \approx 7 \times 10^{-25}$ cm^2 is the cross section for Compton scattering) can vary over wide limits. If $T = 10^{-10}$ (approximately 10^6 deg) it follows that $n \lesssim 2 \times 10^{10}$ while $l_f \gtrsim 6 \times 10^{14}$, that is to say we are dealing with astronomic magnitudes; when $T = 10^{-7}$ (approximately 10^9 deg), however, $n \lesssim 6 \times 10^{25}$ cm^{-3} and $l_f \gtrsim 2 \times 10^{-2}$ cm.

Assume that the plasma is in a nonequilibrium state under the effect of an electric field **E**, which is constant in time and space, a gradient of the chemical potential $\nabla\zeta$, and a temperature gradient ∇T. We use the notation $n_p = n_p^0 + f_p$ and $N_{qe} = N_q^0 + g_{qe}$ to denote the distribution functions for the photons and electrons over the respective momenta **p** and **q** and photon polarization **e**, where n_p^0 and N_q^0 are the equilibrium functions while f_p and g_q denote deviations from equilibrium. The system of time and space independent kinetic equations for electrons and photons in a magnetic field **H** is then*

$$\mathbf{v}\nabla n_p + e\mathbf{E}\nabla_p n_p + \frac{e}{c}\mathbf{v}[\mathbf{H}, \nabla_p f_p] = \frac{1}{2}\sum_{\mathbf{e},\mathbf{e}'}\int d^3q d^3q' W_{\mathbf{qq}'}$$
$$\times\{n_{p'}[N_{\mathbf{qe}}(1+N_{\mathbf{q}'\mathbf{e}'})\,\delta(\varepsilon_{p'}-\varepsilon_p+cq-cq')$$
$$+N_{\mathbf{q}'\mathbf{e}'}(1+N_{\mathbf{qe}})\,\delta(\varepsilon_{p'}-\varepsilon_p+cq'-cq)]$$
$$-n_p[N_{\mathbf{qe}}(1+N_{\mathbf{q}'\mathbf{e}'})\delta(\varepsilon_{p'}-\varepsilon_q+cq'-cq)$$
$$+N_{\mathbf{q}'\mathbf{e}'}(1+N_{\mathbf{qe}})\,\delta(\varepsilon_{p'}-\varepsilon_p+cq_p+cq')]\},$$

$$\frac{c\mathbf{q}}{q}\nabla N_{\mathbf{qe}} = \sum_{\mathbf{e}'}\int d^3p d^3q' W_{\mathbf{qq}'} n_p\,[N_{\mathbf{q}'\mathbf{e}'}(1+N_{\mathbf{qe}})$$
$$\times\,\delta(\varepsilon_{p'}-\varepsilon_p+cq-cq') - N_{\mathbf{qe}}(1+N_{\mathbf{q}'\mathbf{e}'})$$
$$\times\,\delta(\varepsilon_{p'}-\varepsilon_p+cq'-cq)].$$

Here, $\int d^3q \equiv (2\pi\hbar)^{-3}\int dq_x dq_y dq_z$; $W_{\mathbf{qq}'} = W_{\mathbf{q}'\mathbf{q}}$ is the photon scattering probability, characterized by a momentum change from **q** to **q'**. For convenience the transformed collision integral in the first equation is symmetrized with respect to **q** and **q'**. We also neglect the collision integral corresponding to electron-ion collisions; this term will be added below.

We linearize the equations in f_p and g_q:

$$\mathbf{v}\nabla n_p + e\mathbf{E}\nabla_p n_p + \frac{e}{c}v\,[\mathbf{H}, \nabla_p f_p] = \frac{1}{2}\sum_{\mathbf{ee}'}\int d^3q d^3q' W_{\mathbf{qq}'}$$
$$\times\,\{f_{p'}\,[N_q(1+N_{q'})\,\delta(\varepsilon_{p'}-\varepsilon_p+cq-cq')$$
$$+N_{q'}(1+N_q)\,\delta(\varepsilon_{p'}-\varepsilon_p+cq'-cq)] -$$
$$-f_p\,[N_q(1+N_{q'})\,\delta(\varepsilon_{p'}-\varepsilon_p+cq'-cq)$$
$$+N_{q'}(1+N_q)\,\delta(\varepsilon_{p'}-\varepsilon_p+cq-cq')]\}$$
$$+\sum_{\mathbf{ee}'}\int d^3q d^3q' W_{\mathbf{qq}'}\,\{n_{p'}\,[g_{\mathbf{q}'\mathbf{e}'}(1+N_q)+N_{q'}g_{\mathbf{qe}}]$$
$$\times\,\delta(\varepsilon_{p'}-\varepsilon_p+cq'-cq) - n_p\,[g_{\mathbf{qe}}(1+N_{q'})$$
$$+g_{\mathbf{q}'\mathbf{e}'}N_q]\,\delta(\varepsilon_{p'}-\varepsilon_p+cq-cq')\} = A+B, \quad (1.1)$$

$$\frac{c\mathbf{q}}{q}\nabla N_q = \sum_{\mathbf{e}'}\int d^3p d^3q' W_{\mathbf{qq}'}\,\{n_p\,[g_{\mathbf{q}'\mathbf{e}'}(1+N_q)+g_{\mathbf{qe}}N_{q'})$$
$$\times\,\delta(\varepsilon_{p'}-\varepsilon_p+cq-cq') - (g_{\mathbf{qe}}(1+N_{q'})+g_{\mathbf{q}'\mathbf{e}'}N_q)$$
$$\times\,\delta(\varepsilon_{p'}-\varepsilon_p+cq'-cq)] + f_p\,[N_{q'}(1+N_q)$$
$$\times\,\delta(\varepsilon_{p'}-\varepsilon_p+cq-cq') - N_q(1+N_{q'})$$
$$\times\,\delta(\varepsilon_{p'}-\varepsilon_p+cq'-cq)]\} = C+D. \quad (1.2)$$

Here and below the equilibrium functions n_p^0 and N_q^0 will be written without the zero superscript.

Actual magnetic fields will satisfy the inequality $\hbar\Omega \ll T$, where $\Omega = eH/mc$ is the Larmor frequency, so that $H \ll mcT/e\hbar = T/\mu \approx 10^{10}$ Oe when $T = 10^{-10}$. Under these conditions the transition probability $W_{\mathbf{qq}'}$ is independent of magnetic field **H**, depending only on the cosine of the angle between the polarization vectors. We shall see below that in cases of interest the kinetic equations have solutions in which these functions are independent of polarization **e**. Hence the transition probability $W_{\mathbf{ee}'}$ is averaged over the polarizations **e** and **e'**. Thus

$$W_{\mathbf{qq}'} = \frac{r_0^2}{4}\frac{(2\pi\hbar)^3 c^2}{qq'}(1+\cos^2\mathbf{qq}'), \quad (1.3)$$

where $r_0 = e^2/mc^2$ is the electromagnetic radius of the electron. In calculating W we assume that $\epsilon = 1 - 4\pi ne^2/m\omega^2$, the dielectric constant of the plasma, is unity; this follows because for thermal photons ($\omega \sim T/\hbar$) the relation

$$4\pi ne^2/m\omega^2 = 4\pi ne^2\hbar^2/mT^2 \ll 1$$

holds for all conditions in which the effects being considered here are of importance; that is to say, the parameter γ introduced below is not smaller or greater than unity by more than three or four orders of magnitude.

In Eqs. (1.1) and (1.2) the term A describes the scattering of nonequilibrium electrons on equilibrium photons; the term B is the drag integral for equilibrium electrons dragged by nonequilibrium photons; the term C describes the scattering of nonequilibrium photons on equilibrium electrons and finally, the term D is the drag integral for equilibrium photons dragged by nonequilibrium electrons.

*$[\mathbf{H}, \nabla] = \mathbf{H}\times\nabla$, $\mathbf{E}\nabla = \mathbf{E}\cdot\nabla$.

The momenta **q** of the thermal photons and the change $\Delta = q' - q$ due to scattering are much smaller than the momenta of the thermal electrons **p**; on the other hand, the scattering of photons on electrons is essentially elastic so that $|q' - q|/q \ll 1$. By exploiting this situation we can reduce the equation for the thermal electrons (1.1) to a differential equation in the following way. In term A (1.1) we write $N_q(1+N_{q'}) = N_{q'}(1+N_{q'})e^{c(q'-q)T^{-1}}$; furthermore $p' = p \pm \Delta$; $c(q'-q) = p\cdot\Delta/m \pm \Delta^2/2m$ (plus for the first term and minus for the second term of A) and we replace the integration over **q** by integration over **Δ** so that

$$A = \int d^3\Delta\Big[W^{(+)}(\Delta, p\Delta)\Big\{f_{p+\Delta}\exp\Big[\frac{1}{mT}\Big(p\Delta+\frac{\Delta^2}{2}\Big)\Big] - f_p\Big\} + W^{(-)}(\Delta, p\Delta)\Big\{f_{p-\Delta} - f_p\exp\Big[\frac{1}{mT}\Big(p\Delta - \frac{\Delta^2}{2}\Big)\Big]\Big\}\Big];$$

$$W^{(\pm)}(\Delta, p\Delta) = \frac{1}{2}\int d^3q W_{q,|q+\Delta|}N_{|q+\Delta|}(1+N_q) + \delta(cq - c|q+\Delta| + p\Delta/m \pm \Delta^2/2m).$$

Here, $W^{(+)}(\Delta, p\Delta) \doteq W^{(-)}(\Delta, (p+\Delta)\Delta)$.

The quantity $W^{(\pm)}(\Delta, p\Delta)$ is a sensitive function of the first argument, because it is small, but is relatively insensitive to the second, $p\cdot\Delta$. Hence, in determining the integral A with the accuracy to Δ^2 required here it is adequate to write $W_p^{(+)} = W_p^{(-)} + \nabla_p W_p^{()}\Delta$ (if W is formally expanded as a function of **p**). Expanding all other quantities appearing in A we have

$$A = \operatorname{div}_p j_{Ap}, \quad j_{Ap} = \frac{1}{3}\Big(\nabla_p f_p + \frac{p}{mT}f_p\Big)\int \Delta^2 W(\Delta, p\Delta)\,d^3\Delta,$$

where the difference between $W^{(+)}$ and $W^{(-)}$, (terms of order Δ^2 in the arguments of the δ-functions) can now be neglected. To compute the remaining integral we convert to the variables **q** and **q′** and neglect the difference $q' - q$. Using (1.3) we have

$$A = \frac{1}{\tau_f}\operatorname{div}_p(pf_p + mT\nabla_p f_p),$$

$$\tau_f = \frac{mc^2}{\Sigma_f cNT}, \quad \Sigma_f = \frac{8\pi}{3}r_0^2, \quad N = \frac{4\pi^2}{45}\Big(\frac{T}{\hbar c}\Big)^3.$$

In B we first replace **q′** and **q** in the argument of the functions $g_{q'}$ introducing the variable $\Delta = q' - q$; as before, we neglect terms of order Δ^2 in the arguments of the δ-functions and expand the integrands to second order in powers of Δ.

Then,

$$B = -\operatorname{div}_p j_{Bp}; \qquad j_{Bp} = n_p\int \Delta W_1 d\Delta,$$

$$W_1 = \int d^3q W_{q,|q+\Delta|}g_q\delta\Big(cq - c|q+\Delta| + \frac{p\Delta}{m}\Big),$$

or, for the accuracy required here,

$$j_{Bp} = 2\Sigma_f cn_p\int q g_q d^3q.$$

In computing C we make use of the fact that g_q is proportional to $q\cdot A$ where A is the perturbation vector; hence, the term containing $g_{q'}$ vanishes in the integration over **q′**. We neglect the difference in the arguments of the δ-functions in the remaining term. Then, $C = -\Sigma_f c n g_q$, where n is the number of electrons per unit volume.

Finally, in D we again neglect the difference in the arguments of the δ-functions and expand $N_{q'} = N_q + (q' - q)\,\partial N_q/\partial q$, so that

$$D = -N_q(1+N_q)\frac{\Sigma_f c}{mT}\int f_{p'}p'q\,d^3p'.$$

Collecting all terms, computing the left sides, and adding the electron-ion scattering in (2.4) (for simplicity the relaxation-time approximation is used) we have

$$-\frac{n_p}{mT}p\Big(eE - T\nabla\frac{\zeta}{T} - \frac{\varepsilon}{T}\nabla T\Big) = \operatorname{div}_p\Big[\frac{1}{\tau_f}(pf_p + mT\nabla_p f_p) - 2\Sigma_f cn_p\int q g_q d^3q\Big]\frac{e}{c}v[H, \nabla_p f_p] - \frac{f_p}{\tau_i(\varepsilon_p)}, \quad (1.4)$$

$$N_q(1+N_q)\frac{c^2 q}{T^2}\nabla T = -\Sigma_f cng_q + N_q(1+N_q)\frac{\Sigma_f c}{mT}\int f_{p'}p'q\,d^3p',$$
$$\tau_i(\varepsilon_p) = \tau_i(\varepsilon_p/T)^{3/2}, \quad \tau_i = T^{3/2}m^{1/2}/\sqrt{2}\pi e^4 n\Lambda Z^2 \quad (1.5)$$

(Λ is the Coulomb logarithm).

These equations can not be solved without the Coulomb term; this result follows because the electrons obtain momentum from the electric field and transfer it to the photons and the momentum of the total system would increase without limit. We introduce the variable $x = \epsilon/T$ and denote the coefficients of $vn(x)$ by **A** and **B**x where

$$A = \frac{eE}{T} - \nabla\frac{\zeta}{T} - \frac{N}{n}\frac{\nabla T}{T}, \qquad B = -\frac{\nabla T}{T}.$$

Then, writing

$$f(v) = n(x)\,v\tau_i(\varphi(x) + \chi(x)), \quad (1.6)$$

where $\varphi(x)$ and $\chi(x)$ are proportional to the inhomogeneity **A** and **B**, we obtain the equations

$$(W_0 + \gamma W_1)\varphi + \gamma\tau_i[\Omega, \varphi] = J(\varphi) + \gamma A, \quad (1.7)$$

$$(W_0 + \gamma W_1)\chi + \gamma\tau_i[\Omega, \chi] = J(\chi) + \gamma Bx, \quad (1.8)$$

$$W_0 = -2x\frac{d^2}{dx^2} + (2x - 5)\frac{d}{dx} + 1, \qquad W_1 = \frac{1}{x^{3/2}}, \quad (1.9)$$

$$\gamma = \frac{\tau_f}{\tau_i} = \frac{135\Lambda}{16\pi^2}m^{5/2}c^8\hbar^3\frac{nZ^2}{T^{11/2}} \approx 2.5\cdot 10^{-65}\frac{nZ^2}{T^{11/2}}, \quad (1.10)$$

where γ is a parameter that describes the effect of the photons (τ_i is the time required for the electron energy $T\,(x = 1)$ to relax via electron-ion relaxation and T_f is the characteristic electron-photon relaxation time.)

In this work we limit ourselves to the case $\gamma \ll 1$, where electron-photon scattering is much stronger than electron-ion scattering. The quantities $J(\varphi)$ and $J(\chi)$ are integrals of the motion

$$J(\varphi) = \int_0^\infty \varphi(x) e^{-x} x^{3/2} dx \Big/ \int_0^\infty e^{-x} x^{3/2} dx.$$

As we have indicated, however, Eqs. (1.7) and (1.8) hold only for thermal electrons. They apply when $|\mathbf{p}| > |\mathbf{q}| \sim T/c$ (or $x > T/mc^2$).

Equation (1.1) is satisfied by $\varphi = \text{const}$ for electrons with the dimensionless energy $x \ll T/mc^2$. The integral of the equation is actually not changed because the photons are scattered primarily by thermal electrons. With the assumption that φ is constant, substitution of Eq. (1.8) leads, after some simple calculations, to the result ($x \ll T/mc^2$):

$$\varphi = \{\tfrac{2}{3}\sqrt{mc^2/T}\,(J(\varphi) + \gamma\mathbf{A}) + \gamma\tau_i\,[(J(\varphi) + \gamma\mathbf{A}),$$
$$\boldsymbol{\Omega}] + (3\gamma^2\tau_i/2)\sqrt{T/mc^2}\,(J(\varphi) + \gamma\mathbf{A})\,\boldsymbol{\Omega}\}$$
$$\times (\tfrac{4}{9}mc^2/T + \gamma^2\tau_i^2\Omega^2)^{-1}, \qquad (1.11)$$

if the Coulomb scattering of electrons on ions is not important at these energies.

2. SOLUTION OF THE EQUATIONS WITHOUT MAGNETIC FIELD

In the absence of a magnetic field it is convenient to replace (1.6) by

$$f_{\mathbf{p}} = \frac{\mathbf{p}\,(e\mathbf{E} - T\nabla(\zeta/T))}{mT} n(x)\,\tau_i\varphi_E(x) - \frac{\mathbf{p}\nabla Tn(x)\,\tau_i\varphi_T(x)}{mT},$$

which leads to the equations:

$$(W_0 + \gamma W_1)\,\varphi_E(x) = J(\varphi_E) + \gamma, \qquad (2.1)$$
$$(W_0 + \gamma W_1)\,\varphi_T(x) = J(\varphi_T) + \gamma\,(N/n + x). \qquad (2.2)$$

Here, the functions φ_E and φ_T have the meaning of relaxation times expressed in terms of τ_i.

To solve Eq. (2.1) we write $\varphi_E(x) = (J + \gamma)\,\varphi_1$. If $x > \gamma^{2/3}$ (in which case the Coulomb term is small) it is convenient to make the substitution $\varphi_1(x) = 1 - \gamma\psi(x)$. Neglecting terms of order γ^2 we obtain the equation $W_0\psi(x) = x^{-3/2}$; the solution that remains finite at infinity is

$$\psi(x) = -\tfrac{1}{2}x^{-3/2} + C\,(x^{-3/2} + 2x^{-1/2}). \qquad (2.3)$$

To determine C we can join (2.3) to the solution of the equation for $x < \gamma^{2/3}$ (or for $x < T/mc^2$ if $T/mc^2 > \gamma^{2/3}$). However, it is more convenient to use the equation that expresses the conservation of electron momentum in the stationary state. This equation is obtained if we substitute $g_{\mathbf{q}}$ from (1.5) in (1.4), multiply the result by $\mathbf{p}d^3\mathbf{p}$, and integrate over momentum.[1] As applied to Eq. (2.1) this procedure is equivalent to multiplying by $e^{-x}x^{3/2}dx$ and integrating over x. When $\gamma = 0$ the only nondiverging solution of Eq. (2.1) is $\varphi_1 = 1$; hence, when $\gamma \ll 1$ we can write as a first approximation

$$W_0\varphi_1 = \gamma/x^{3/2}.$$

After the indicated multiplication and integration by parts we have

$$\tfrac{3}{4}\sqrt{\pi}\,J(\varphi_1) = \tfrac{3}{4}\sqrt{\pi} - \gamma - 2x^{5/2}d\varphi/dx\,|_{x=0}.$$

Comparing this with the expression for $J(\varphi_1)$ obtained from Eq. (2.3) we conclude that $C = 1/2$ while $J(\varphi)$, which is needed only to determine the kinetic coefficients, is $3\sqrt{\pi}/4$.

When $T/mc^2 < \gamma^2$ the solution of Eq. (2.1) is meaningful for $x > \gamma^2$. In this region the solution is of the form $\varphi_E(x) = x^{3/2}(1 + J(\varphi_E)/\gamma)$. Since $J \sim 1$, the relaxation time for the low momentum electrons is increased by the ratio γ^{-1} as a consequence of the interaction with the photons.

When $x \to \infty$ we find $\varphi_E(x) \to 3\sqrt{\pi}/4$, i.e., this quantity remains finite so that the nonstationarity associated with the loss of electrons is removed by electron scattering on photons.

To solve Eq. (2.2) we add and substract $5/2$ from the right side. The solution is then a linear combination

$$\varphi_T(x) = \left(N/n + \tfrac{5}{2}\right)(J(\varphi_T) + \gamma)\,\varphi_1(x) + \varphi_2(x), \qquad (2.4)$$

where $\varphi_1(x)$ is the known solution of (2.1) while $\varphi_2(x)$ is the solution of the equation $(W_0 + \gamma W_1)\varphi_2(x) = \gamma(x - 5/2)$.

The solution of this equation for $x \sim 1$ is

$$\varphi_2(x) = \tfrac{1}{3}\gamma\left(x - \tfrac{5}{2}\right) + C_2\,(x^{-3/2} + 2x^{-1/2}).$$

Application of the previous method gives $C_2 = 0$ to accuracy of order γ. φ_2 does not make a contribution to J and the drag integral becomes

$$J(\varphi_T) = \tfrac{3}{4}\sqrt{\pi}\left(N/n + \tfrac{5}{2}\right).$$

We now substitute the expressions $\varphi_E(x)$ and $\varphi_T(x)$ in $f_{\mathbf{p}}$ and compute $g_{\mathbf{q}}$ from Eq. (1.5); we then obtain the current $\mathbf{j}$ and energy flow $\mathbf{W}$ for $\gamma \ll 1$

$$\mathbf{j} = \sigma\left(\mathbf{E} - \frac{T}{e}\nabla\frac{\zeta}{T}\right) - \alpha\nabla T, \quad \mathbf{w} = \alpha T\left(\mathbf{E} - \frac{T}{e}\nabla\frac{\zeta}{T}\right) - \delta\nabla T;$$

$$\sigma = \tfrac{3}{4}\sqrt{\pi}\,ne^2\tau_i/m \approx 0{,}3\sigma_0, \qquad \alpha = (\sigma/e)\left(N/n + \tfrac{5}{2}\right),$$

$$\delta = \frac{N}{n}\frac{c}{\Sigma_f} + \frac{5}{2}\frac{\sigma T}{e^2}\left(\frac{N}{n} + \frac{5}{2}\right)\left(1 + \frac{4\gamma}{15\sqrt{\pi}}\right) + \frac{5}{6}\gamma\frac{\sigma T}{e^2}.$$

[1]We are indebted to O. V. Konstantinov and V. I. Perel' for pointing out this procedure to us.

Writing $j = \nabla\zeta = 0$, we obtain the thermal emf and the thermal conductivity:

$$\frac{E}{\nabla T} = \frac{1}{e}\left(\frac{N}{n} - \frac{\zeta}{T} + \frac{5}{2}\right), \quad \varkappa = \frac{N}{n}\frac{c}{\Sigma_f} + \frac{5}{6}\gamma\frac{\sigma T}{e^2}.$$

The second term gives the electron thermal conductivity, which is smaller than the photon conductivity when $\gamma \lesssim 1$. Terms proportional to γ are omitted in the expressions for σ and α.

In the other limiting case, $\gamma \gg 1$, we find $\alpha \approx N\sigma_0/ne$. Corrections of order γ^{-1} to the other kinetic coefficient are not physically meaningful in the present approximation since we have not taken account of the electron-electron interaction, which is much stronger than the electron-photon interaction in the present case. When $j = 0$ the thermal emf $E/\nabla T \cong N/en$.

The following conclusions can be drawn from these expressions.

1. In the limiting case in which scattering of electrons on photons is very strong the electrical conductivity is reduced by approximately a factor of 2.

2. Drag of electrons by photons causes a significant increase in the thermal emf, by approximately the ratio $(T/\hbar c)^3/n$ for both $\gamma \ll 1$ and $\gamma \gg 1$. This ratio can be large even when $\gamma \gg 1$. Under these conditions our results for the thermal emf hold only when the electric field is weak.

3. A feature of the Thomson scattering of photons is the result, which follows from Eq. (1.8), that the change in the photon energy flux due to the drag of photons by electrons is NTj/n; this vanishes when $j = 0$.

3. SOLUTION OF THE EQUATIONS IN THE PRESENCE OF A MAGNETIC FIELD

If the vectors A or B are proportional to some external field E while $\nabla T \parallel H$, the solutions of Eqs. (1.7) and (1.8) are unchanged; hence we are interested only in the case in which E, $\nabla\zeta$ or ∇T (and consequently φ) are perpendicular to the magnetic field. In presence of a magnetic field the current density and energy flow density are given by the following general expressions:

$$\mathbf{j} = \sigma\mathbf{E}^* - \alpha\nabla T + \sigma' H^{-1}[\mathbf{E}^*\mathbf{H}] - \alpha' H^{-1}[\nabla T, \mathbf{H}], \quad (3.1)$$

$$\mathbf{w} = \alpha T\mathbf{E}^* - \delta\nabla T + \alpha' T H^{-1}[\mathbf{E}^*\mathbf{H}] - \delta' H^{-1}[\nabla T, \mathbf{H}], \quad (3.2)$$

$$\mathbf{E}^* = \mathbf{E} - (T/e)\nabla(\zeta/T).$$

Writing $j = 0$, with $H = H_z$ and $\nabla T = \partial T/\partial x$, we obtain the thermal emf, the Nernst field, and the energy flow

$$eE_x^* = e\frac{\alpha\sigma + \alpha'\sigma'}{\sigma^2 + \sigma'^2}\frac{\partial T}{\partial x} = \lambda\frac{\partial T}{\partial x},$$

$$eE_y^* = e\frac{\alpha\sigma' - \alpha'\sigma}{\sigma^2 + \sigma'^2}\frac{\partial T}{\partial x} = \lambda'\frac{\partial T}{\partial x},$$

$$w_x = (\alpha T\lambda + \alpha' T\lambda' - \delta)\,\partial T/\partial x = -\varkappa\partial T/\partial x,$$

$$w_y = (\alpha T\lambda' - \alpha' T\lambda + \delta')\,\partial T/\partial x = -\varkappa'\partial T/\partial x. \quad (3.3)$$

In the case of interest to us, $N \gg n$, the coefficient κ is determined to good accuracy by the photon energy flux; the electron drag does not affect this flux in Compton scattering. Hence, the electron energy flow is determined only by κ'.

Equations (1.7) and (1.8) contain two parameters, γ and $\gamma\Omega\tau_i$; hence, when $\gamma \ll 1$ we must consider two different cases, $\gamma\Omega\tau_i \gg 1$ and $\gamma\Omega\tau_i \ll 1$. When $\gamma\Omega\tau_i \gg 1$ the formal solution of Eq. (1.7) and Eq. (1.8) can be written conveniently in the form

$$\varphi(x) = [(W_0 + \gamma W_1)^2 + (\gamma\Omega\tau_i)^2]^{-1}\{(W_0 + \gamma W_1)(J(\varphi) + \gamma\mathbf{A}) + \gamma\tau_i[(J(\varphi) + \gamma\mathbf{A}), \boldsymbol{\Omega}]\}.$$

Expanding in powers of $(\gamma\Omega\tau_i)^{-1}$, keeping quadratic terms, and computing $J(\varphi)$ we have

$$\varphi(x) = \frac{A}{x^{3/2}(\gamma^{-2} + \Omega^2\tau_i^2)} + \frac{[\mathbf{A}, \boldsymbol{\Omega}]}{\Omega^2\tau_i},$$

$$\chi(x) = \frac{B}{x^{1/2}(\gamma^{-2} + \Omega^2\tau_i^2)} + \frac{5}{2}\frac{[\mathbf{B}, \boldsymbol{\Omega}]}{\Omega^2\tau_i} + \frac{4\gamma\mathbf{B}(x - 5/2)}{1 + \gamma^2\Omega^2\tau_i^2} - \frac{6(x - 5/2)}{(\gamma\Omega\tau_i)^4}(2\mathbf{B} + \tau_i[\mathbf{B}, \boldsymbol{\Omega}]).$$

The kinetic coefficients are then

$$\sigma = \frac{4}{3\sqrt{\pi}}\frac{\sigma_1}{\gamma^{-2} + (\Omega\tau_i)^2}, \quad \sigma' = \frac{\sigma_1}{\Omega\tau_i}, \quad \sigma_1 = \frac{e^2\tau_i n}{m} \approx \frac{2}{9}\sigma_0,$$

$$\alpha = \frac{\sigma}{e}\left(\frac{N}{n} + 1\right), \quad \alpha' = \frac{\sigma_1}{e\Omega\tau_i}\left(\frac{N}{n} + \frac{5}{2}\right),$$

$$\varkappa' = -\frac{25}{4}\frac{\sigma_1 T}{e^2\Omega\tau_i},$$

$$\lambda = N/n + \frac{5}{2} + 9/16\Omega^2\tau_i^2, \quad \lambda' = -9/8\Omega\tau_i.$$

The remaining coefficients can be easily obtained by means of formulas written with the help of (1.6).

In this case the fact that the photons are not in equilibrium is found to be important only in that the coefficients α, α' and λ contain the additional factor N/n as compared with the usual expressions.

When $\gamma\Omega\tau_i \ll 1$ it is convenient to consider the cases $\Omega\tau_i \gg 1$ and $\Omega\tau_i \ll 1$ separately. When $\Omega\tau_i \gg 1$ we expand the quantities φ and χ in powers of both small parameters, retaining terms of order $(\Omega\tau_i)^{-4}$ and γ^2.

The solution is

$$\varphi(x) = \frac{J(\psi)}{(\Omega\tau_i)^2}\left[1 - \frac{J^2(\psi)}{(\Omega\tau_i)^2} - (\gamma\Omega\tau_i)^2\right]\mathbf{A}$$
$$+\left(1 - \frac{J^2(\psi)}{(\Omega\tau_i)^2} - 2(\Omega\tau_i\gamma)^2 - \gamma\psi\right)\frac{[\mathbf{A}, \mathbf{\Omega}]}{\Omega^2\tau_i},$$
$$\chi(x) = \frac{5}{2}\chi_1(x) - \frac{\gamma}{3}\left(x - \frac{5}{2}\right)\mathbf{B}$$

[$\chi_1(x)$ is obtained from $\varphi(x)$ by replacing **A** by **B**]. Here, ψ is determined from Eq. (2.3) and $J(\psi) = 4/3\sqrt{\pi}$, while the kinetic coefficients are:

$$\sigma = \frac{\sigma_1 J(\psi)}{\Omega^2\tau_i^2}\left[1 - \frac{J^2(\psi)}{(\Omega\tau_i)^2} - (\gamma\Omega\tau_i)^2\right],$$
$$\sigma' = \frac{\sigma_1}{\Omega\tau_i}\left[1 - \frac{J^2(\psi)}{(\Omega\tau_i)^2} - 2(\gamma\Omega\tau_i)^2 - \gamma J(\psi)\right],$$
$$\alpha = \frac{\sigma_1 J(\psi)}{e(\Omega\tau_i)^2}\left[\frac{N}{n} + \frac{5}{2} - \frac{N}{n}\frac{J^2(\psi)}{(\Omega\tau_i)^2} - \frac{N}{n}(\gamma\Omega\tau_i)^2\right],$$
$$\alpha' = \frac{\sigma_1}{e\Omega\tau_i}$$
$$\times\left[\frac{N}{n} + \frac{5}{2} - \frac{N}{n}\frac{J^2(\psi)}{(\Omega\tau_i)^2} - \frac{2N}{n}(\gamma\Omega\tau_i)^2 - \frac{N}{n}\gamma J(\psi)\right],$$
$$\lambda = \frac{N}{n} + \frac{5}{2}, \qquad \lambda' = -\frac{J(\psi)}{\Omega\tau_i}\frac{5}{2}[(\gamma\Omega\tau_i)^2 + \gamma J(\psi)],$$
$$\varkappa' = -2.5\sigma'\gamma T/e^2.$$

In this case, in addition to the changes in the coefficients α, α' and λ, similar to those obtained earlier, we may note certain features of the coefficients λ' and κ'. These quantities contain the factors γ and γ^2 so that they become smaller as scattering on photons becomes more important. When $\gamma\Omega^2\tau_i^2 > 1$ we have $\lambda' \sim \Omega$ in place of the usual relation $\lambda' \sim \Omega^{-1}$.

When $\Omega\tau_i \ll 1$ we need only retain terms of order unity, γ, and $\gamma\Omega\tau_i$ in the expansion of $\varphi(x)$ and $\chi(x)$. We then have

$$\varphi(x) = \frac{A}{J(\psi)} + \frac{[\mathbf{A}, \mathbf{\Omega}]\tau_i}{J^2(\psi)}$$
$$+\gamma\mathbf{A}\left(1 - \frac{\psi}{J(\psi)}\right) + \frac{\gamma\tau_i[\mathbf{A}, \mathbf{\Omega}](J(\psi) - \psi)}{J^2(\psi)},$$
$$\chi(x) = \tfrac{5}{2}\chi_1(x) - \tfrac{1}{3}\gamma\left(\tfrac{5}{2} - x\right)\mathbf{B},$$

where $\chi_1(x)$ differs from $\varphi(x)$ in that **A** is replaced by **B**. In this case

$$\sigma = \tfrac{3}{4}\sqrt{\pi}\sigma_1, \quad \sigma' = \tfrac{9}{10}\pi\sigma_1\Omega\tau_i, \quad \alpha = \frac{\sigma}{e}\left(\frac{N}{n} + \frac{5}{2}\right),$$
$$\alpha' = \frac{\sigma'}{e}\left(\frac{N}{n} + \frac{5}{2}\right), \quad \varkappa' \approx -2.5\frac{\sigma'\gamma T}{e^2}, \quad \lambda' = 0.$$

The most interesting new features of the kinetic coefficients in this case are the reduction in the longitudinal electrical conductivity by approximately a factor of 2 and in the transverse conductivity by a factor of 8 as compared with the usual case. The other feature that merits attention is the fact that the coefficient λ' vanishes to order γ and $(\Omega\tau_i)^2$. In the other limiting case ($\gamma \gg 1$) it can be stated, without solving the kinetic equations, that the coefficients α, α' and λ are proportional to N/n.

4. HIGH-FREQUENCY CONDUCTIVITY

Assume that the system is in an external electric field $\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$ where $\mathbf{k}\cdot\mathbf{E} = 0$ and $\omega\tau_i \gg 1$. We can then neglect the Coulomb term in Eq. (1.7) and find

$$W_0\varphi + \tau_f[\Omega, \varphi] - i\omega\tau_f\varphi = \mathbf{A} + J(\varphi)Y; \tag{4.1}$$
$$Y = \frac{3}{4\sqrt{\pi}}\int\frac{\cos^2 \mathbf{q}\mathbf{E}\, do}{1 - i\omega\tau_f + ick\tau_f\cos\mathbf{kq}}. \tag{4.2}$$

Here, do is the element of solid angle in **q** space; the factor Y arises as a consequence of the need for taking account of both time and spatial dispersion for the photons in Eq. (1.8). Calculation yields

$$Y = \frac{1}{ck\tau_f}\left\{\frac{i\omega\tau_f - 1}{ck\tau_f} - i\left[1 + \frac{(1 - i\omega\tau_f)^2}{(ck\tau_f)^2}\right.\right.$$
$$\left.\left.\times\ln\frac{ick\tau_f - i\omega\tau_f + 1}{\sqrt{(1 - i\omega\tau_f)^2 + (ck\tau_f)^2}}\right]\right\}. \tag{4.3}$$

Since the solutions of the homogeneous equation are damped we shall only be interested in the particular solution of the inhomogeneous equation (4.1) φ = const, which is

$$\varphi = \{(1 - i\omega\tau_f)^2 + \Omega^2\tau_f^2\}\{[(1 - i\omega\tau_f)^2$$
$$+ \Omega^2\tau_f^2 - (1 - i\omega\tau_f)Y]^2 + \Omega^2\tau_f^2Y^2\}^{-1}\{(1 - i\omega\tau_f)\mathbf{A}$$
$$+ [\mathbf{A}, \mathbf{\Omega}]\tau_f\} = K\{(1 - i\omega\tau_f)\mathbf{A} + \tau_f[\mathbf{A}, \mathbf{\Omega}]\}, \tag{4.4}$$

whence

$$\sigma = (ne^2\tau_f/m)K(1 - i\omega\tau_f - Y), \qquad \sigma' = (ne^2\tau_f/m)K\Omega\tau_f.$$

The dielectric constant is

$$\varepsilon_{ik} = \delta_{ik} + 4\pi i\sigma_{ik}/\omega. \tag{4.5}$$

The tensor ϵ_{ik} is of the same form as σ_{ik}, that is to say, it consists of a symmetric diagonal part and an antisymmetric nondiagonal part; when $\mathbf{H} = \mathbf{H}_z$ this tensor, like σ_{ik}, can be expressed in terms of three scalar quantities:

$$\varepsilon = 1 + 4\pi i\sigma/\omega, \qquad \varepsilon' = 4\pi i\sigma'/\omega, \qquad \varepsilon_{zz} = \varepsilon_0,$$

where ϵ_0 is the dielectric constant for $\mathbf{H} = 0$. These relations are an equation for the tensor ϵ_{ik}. In the limiting cases, which we now consider, they are simplified considerably.

We consider the case of a wave propagating along the magnetic field. When $\tau_i^{-1} \ll \omega \ll \tau_f^{-1}$, we have

$$\varepsilon = 1 - \frac{2\omega_n^2 (1 + \Omega^2\tau_f^2)}{(2i\omega + \Omega^2\tau_f)^2 + \Omega^2 (1 + i\omega\tau_f)},$$
$$\varepsilon' = \frac{i\omega_n^2\tau_f}{\omega} \frac{1 + \Omega^2\tau_f^2}{(2i\omega\tau_f + \Omega^2\tau_f^2)^2 + \Omega^2\tau_f^2 (1 + i\omega\tau_f)},$$
$$\omega_n^2 = 4\pi ne^2/m.$$

When $\Omega \sim \omega$, but is not too close to $\omega = \Omega/2$, we can neglect the small terms $\Omega\tau_f$, thus obtaining

$$\varepsilon = 1 + \frac{2\omega_n^2}{\Omega^2 - (2\omega)^2}, \qquad \varepsilon' = \frac{i\omega_n^2}{\Omega^2 - (2\omega)^2}\frac{\Omega}{\omega}. \qquad (4.6)$$

These expressions apply almost up to the point $\omega = \Omega/2$. Actually, writing $\omega = (1 - \delta)\Omega/2$ we see that damping can be neglected so long as

$$\delta > \omega\tau_f (\omega_n/\omega)^2. \qquad (4.7)$$

Thus, up to very small values of the deviation the dielectric constant is essentially real and increases, reaching maximum values in the immediate vicinity of the point $\omega = \Omega/2$. It might be said that there is a resonance close to this point.

The Faraday effect exhibits the following behavior. The angle of rotation per unit length is

$$\frac{\omega}{2c}\left[\left(1 + \frac{4\omega_n^2}{\Omega^2 - (2\omega)^2}\right)^{1/2} - 1\right].$$

However, the increase in refractive index as ω approaches the value $\Omega/2$ continues only up to deviations that satisfy (4.7). Calculation shows that for smaller deviations the absolute value of the complex dielectric constant is of order unity.

It is evident that photon drag due to electrons is no longer important ($Y = 0$) when $\omega\tau_f \gg 1$; also

$$\varepsilon = 1 - (\omega_n^2\tau_f/i\omega)(1 - i\omega\tau_f)[(1 - i\omega\tau_f)^2 + \Omega^2\tau_f^2]^{-1},$$
$$\varepsilon' = (i\omega_n^2\tau_f/\omega)[(1 - i\omega\tau_f)^2 + \Omega^2\tau_f^2]^{-1}\Omega\tau_f. \qquad (4.8)$$

The easiest way to check the validity of these qualitative considerations is to carry out the calculations in the absence of magnetic field, in which case ϵ_{ik} becomes a scalar. Then, $k = \omega\sqrt{\epsilon}/c$ and (4.5) becomes a quadratic equation in ϵ. One of its solutions corresponds to a damping in a time much smaller than the period while the other coincides with Eq. (5.7) for $\Omega = 0$.

Neglecting electron-electron interactions we find that the longitudinal waves are damped out in a time smaller than one period.

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