ON THE THEORY OF SINGULARITIES IN NONLINEAR ELECTRODYNAMICS

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The structure of the singularities in nonlinear electrodynamics is investigated, and the equations of motion of these singularities are obtained without the use of any important assumptions concerning the concrete form of the Lagrangian for the electromagnetic field.

INTRODUCTION

THE purpose of the present paper is a general investigation of the character of the singularities in nonlinear electrodynamics with a Lagrangian having an arbitrary dependence on the two invariants $J_1 = -\frac{1}{4} F_{\mu\nu}^2$ and $J_2 = -\frac{1}{4}\epsilon_{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}$ ($F_{\mu\nu}$ is the electromagnetic field tensor). We shall study the structure of the field near individual singularities and obtain the equations of motion for interacting singularities.[1)] In the case of small accelerations of the singularities these equations are of the form of Lorentz equations with a radiative damping force. The dynamical principle from which we proceed in deriving the equations of motion of the singularities asserts the vanishing of the flux of energy and momentum through an infinitely small surface enclosing the singularity.

1. THE FIELD EQUATIONS

We obtain the equations of the electromagnetic field described by the tensor $F_{\mu\nu}$ from the variational principle

$$\delta \int L(J_1, J_2)\, d^4x = 0, \tag{1}$$

where $L(J_1, J_2)$ is the Lagrangian, which is an arbitrary function of two invariants, $J_1 = -\frac{1}{4}F_{\mu\nu}^2$ and $J_2 = -\frac{1}{4}\epsilon_{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}$ ($\epsilon_{\mu\nu\lambda\rho}$ is the completely antisymmetric tensor of the fourth rank), which satisfies only the following conditions: it is an even function of J_2, and when J_1 and J_2 are small in comparison with a characteristic quantity J_0 the Lagrangian L goes over into the usual Lagrangian of the linear theory, i.e.,

$$L(J_1, J_2) = J_1, \qquad J_1, J_2 \ll J_0. \tag{2}$$

The electromagnetic field equations derived from the variational principle (1) are of the form

$$\partial(\chi_1 F_{\lambda\nu})/\partial x_\lambda + F_{\rho\mu}\varepsilon_{\rho\mu\lambda\nu}\partial\chi_2/\partial x_\lambda = 0, \tag{3}$$

$$\partial F_{\mu\nu}/\partial x_\lambda + (\mu\nu\lambda) = 0; \qquad \chi_1 = \partial L/\partial J_1, \quad \chi_2 = \partial L/\partial J_2 \tag{3'}$$

where $(\mu\nu\lambda)$ is an abbreviated notation for the sum of the terms obtained from $\partial F_{\mu\nu}/\partial x_\lambda$ by cyclic permutation of the indices.

It is also easy to obtain the symmetric energy-momentum tensor of the electromagnetic field:

$$T_{\mu\nu} = (L - J_2\chi_2)\,\delta_{\mu\nu} + \chi_1 F_{\mu\lambda}F_{\nu\lambda}, \tag{4}$$

which, in virtue of (3) and (3′), satisfies the usual conservation law

$$\partial T_{\mu\nu}/\partial x_\mu = 0.$$

In what follows we shall be interested in the behavior of the field near certain world lines on which the field in the linear theory has singularities. Therefore it is convenient to write the field equations (3) and (3′) not in the variables x_μ, but in natural variables associated with a given world line.

Let the parametric equation of a line of singularities be of the form

$$x_\mu = \xi_\mu(\tau),$$

with the parameter τ chosen so that $\dot{\xi}^2 = -1$. Instead of the variables x_μ characterizing a point of space we introduce variables η_μ connected with x_μ by the relation

$$\eta_\mu = x_\mu - \xi_\mu(\tau), \tag{5}$$

where the parameter τ is determined from the equation

$$\eta\dot{\xi}(\tau) = 0. \tag{5'}$$

Thus η is the four-vector normal dropped from the point x onto the line $\xi_\mu(\tau)$.

From the equations (5) and (5′) it is easy to de-

[1)]A study of the structure of the field for a single stationary singularity and with a special choice of the Lagrangian has been made by Born and Infeld.[1]

termine the derivatives of τ and η_ν with respect to x_μ:

$$\frac{\partial \tau}{\partial x_\mu} = -\frac{\dot{\xi}_\mu}{1+\eta\ddot{\xi}}, \qquad \frac{\partial \eta_\nu}{\partial x_\mu} = \delta_{\mu\nu} + \frac{\dot{\xi}_\nu \dot{\xi}_\mu}{1+\eta\ddot{\xi}}.$$

Therefore for an arbitrary function $\varphi(x)$ we have

$$\frac{\partial \varphi}{\partial x_\mu} = (\delta_{\mu\nu} + \dot{\xi}_\mu \dot{\xi}_\nu)\frac{\partial \varphi}{\partial \eta_\nu} - \frac{1}{1+\eta\ddot{\xi}}\left\{\frac{\partial \varphi}{\partial \tau} + (\eta\ddot{\xi})\,\dot{\xi}_\nu \frac{\partial \varphi}{\partial \eta_\nu}\right\}\dot{\xi}_\mu.$$

In these variables the field equations (3) and (3′) take the forms

$$\begin{aligned}(\delta_{\nu\lambda} + \dot{\xi}_\nu \dot{\xi}_\lambda)&\frac{\partial}{\partial \eta_\lambda}(\chi_1 F_{\mu\nu}) \\ &- \frac{\dot{\xi}_\nu}{1+\eta\ddot{\xi}}\left\{\frac{\partial}{\partial \tau}(\chi_1 F_{\mu\nu}) + (\eta\ddot{\xi})\,\dot{\xi}_\lambda \frac{\partial}{\partial \eta_\lambda}(\chi_1 F_{\mu\nu})\right\} \\ &- F_{\alpha\beta}\varepsilon_{\alpha\beta\lambda\mu}\left\{(\delta_{\lambda\nu} + \dot{\xi}_\lambda \dot{\xi}_\nu)\frac{\partial \chi_2}{\partial \eta_\nu}\right. \\ &\left.- \frac{\dot{\xi}_\lambda}{1+\eta\ddot{\xi}}\left[\frac{\partial \chi_2}{\partial \tau} + (\eta\ddot{\xi})\,\dot{\xi}_\nu \frac{\partial \chi_2}{\partial \eta_\nu}\right]\right\} = 0, \end{aligned} \tag{6}$$

$$\begin{aligned}(\delta_{\mu\rho} + \dot{\xi}_\mu \dot{\xi}_\rho)\frac{\partial F_{\lambda\nu}}{\partial \eta_\rho} &- \frac{\dot{\xi}_\mu}{1+\eta\ddot{\xi}}\left\{\frac{\partial F_{\lambda\nu}}{\partial \tau} + (\eta\ddot{\xi})\,\dot{\xi}_\rho \frac{\partial F_{\lambda\nu}}{\partial \eta_\rho}\right\} \\ &+ (\mu\nu\lambda) = 0. \end{aligned} \tag{6′}$$

2. THE BOUNDARY CONDITIONS

The equations (3), (3′) or (6), (6′) describe all sorts of states of the field. In order to single out solutions corresponding to particular sets of singularities, it is necessary to impose definite boundary conditions on the solutions of these equations. Physically it is clear that far from singularities, i.e., at distances $\eta \gg r_0$ [the order of magnitude of r_0 is $r_0 \sim (e/J_0^{1/2})^{1/2}$], the field approaches zero, and consequently the usual equations of the Maxwell-Lorentz theory are valid in this region. Therefore we shall assume that for $\eta \gg r_0$ the solutions of the nonlinear equations must go over into solutions of the linear equations which correspond to the field produced by the motion of a definite type of singularity.

For example, it is known that in the linear theory the field of a charge $e = 4\pi C$ moving along the trajectory $\xi_\mu(\tau)$ is of the form, at distances $\eta \ll \rho$,[2]

$$F_{\mu\nu} = F^0_{\mu\nu} + F^1_{\mu\nu} + F^2_{\mu\nu} + \ldots,$$

$$F^0_{\mu\nu} = C\eta^{-3}(\dot{\xi}_\mu \eta_\nu - \dot{\xi}_\nu \eta_\mu),$$

$$F^1_{\mu\nu} = \tfrac{1}{2}C\eta^{-1}(\ddot{\xi}_\mu \dot{\xi}_\nu - \ddot{\xi}_\nu \dot{\xi}_\mu) - \tfrac{1}{2}C\eta^{-3}(\eta\ddot{\xi})(\dot{\xi}_\mu \eta_\nu - \dot{\xi}_\nu \eta_\mu),$$

$$\begin{aligned}F^2_{\mu\nu} = {} & \tfrac{1}{8}C\eta^{-1}\ddot{\xi}^2(\dot{\xi}_\mu \eta_\nu - \dot{\xi}_\nu \eta_\mu) + \tfrac{3}{8}C\eta^{-3}(\eta\ddot{\xi})^2(\dot{\xi}_\mu \eta_\nu - \dot{\xi}_\nu \eta_\mu) \\ & - \tfrac{3}{4}C\eta^{-1}(\eta\ddot{\xi})(\dot{\xi}_\nu \ddot{\xi}_\mu - \dot{\xi}_\mu \ddot{\xi}_\nu) + \tfrac{1}{2}C\eta^{-1}(\dddot{\xi}_\nu \eta_\mu - \dddot{\xi}_\mu \eta_\nu) \\ & \pm \tfrac{2}{3}C(\dddot{\xi}_\nu \dot{\xi}_\mu - \dddot{\xi}_\mu \dot{\xi}_\nu), \end{aligned} \tag{7}$$

where the sign (+) corresponds to the retarded solutions and the sign (−) to the advanced solutions (ρ is the radius of curvature of the four-dimensional trajectory). Therefore, if we are interested in solutions of the charge type for the nonlinear equations, for which $r_0 \ll \rho$, then for $r_0 \ll \eta \ll \rho$ these solutions must go over into the formulas (7).

Analogous boundary conditions could be formulated for other singularities, for example of the dipole type. We shall not write out the formulas here, however.

Finally we note that, as can be seen from the boundary conditions (7), the solutions of the nonlinear equations (6) and (6′) for the case of a single singularity are to be sought in the form

$$F_{\mu\nu} = F_{\mu\nu}(\eta, \dot{\xi}, \ddot{\xi}, \dddot{\xi}, \ldots), \tag{8}$$

where the functions $F_{\mu\nu}$ depend on τ only through $\dot{\xi}$, $\ddot{\xi}$, $\dddot{\xi}$, and so on.

The formulation of the boundary conditions for the case of several singularities, treated in the approximation we shall be using, will be given in Sec. 4.

3. DETERMINATION OF THE FIELD OF A POINT SINGULARITY IN ARBITRARY MOTION

1. For the solution of the field equations (6) and (6′) in the approximation $r_0 \ll \rho$ we use an expansion of the field components $F_{\mu\nu}$ of Eq. (8) in powers of $\ddot{\xi}$, $\dddot{\xi}$, ..., with the orders $\ddot{\xi} \sim 1/\rho$, $\dddot{\xi} \sim 1/\rho^2$, ... :

$$F_{\mu\nu} = F^0_{\mu\nu}(\eta; \dot{\xi}) + F^1_{\mu\nu}(\eta; \dot{\xi}, \ddot{\xi}) + F^2_{\mu\nu}(\eta; \dot{\xi}, \ddot{\xi}, \dddot{\xi}) + \ldots$$

The only second-rank antisymmetric tensor that can be constructed from η_μ, $\dot{\xi}_\mu$ is proportional to $\eta_\mu \dot{\xi}_\nu - \eta_\nu \dot{\xi}_\mu$. Therefore the zeroth-approximation solution must be of the form

$$F^0_{\mu\nu}(\eta; \dot{\xi}) = g_0(\eta)(\eta_\nu \dot{\xi}_\mu - \eta_\mu \dot{\xi}_\nu), \tag{9}$$

where $g_0(\eta)$ is as yet an unknown function of the variable η. It is easy to verify that in this approximation $F^0_{\mu\nu}$ satisfies the equation (6′) exactly.

Since for the field $F^0_{\mu\nu}$

$$J_1 = J_1^{(0)} = \tfrac{1}{2}\eta^2 g_0^2, \qquad J_2 = 0$$

and consequently $\chi_2(J_1, 0) = 0$, the field equations (6) take the form

$$(\delta_{\nu\lambda} + \dot{\xi}_\nu\dot{\xi}_\lambda)\frac{\partial}{\partial\eta_\lambda}\{\chi_1 g_0(\dot{\xi}_\mu\eta_\nu - \dot{\xi}_\nu\eta_\mu)\} = 0,$$

from which we have

$$\eta d(\chi_1 g_0)/d\eta + 3\chi_1 g_0 = 0.$$

The solution of this equation is of the form

$$\chi_1\left(\tfrac{1}{2}\eta^2 g_0^2, 0\right) g_0 = A/\eta^3,$$

where A is a constant of integration. Noting that according to Eq. (7) for $\eta \to \infty$ we have $g_0 = C/\eta^3$ and $\chi_1(\tfrac{1}{2}C^2/\eta^4, 0) = 1$, we find $A = C$. Introducing a new function $z = \eta g_0$, we find that z satisfies the following equation:

$$z\chi_1\left(\tfrac{1}{2}z^2, 0\right) = C/\eta^2. \tag{10}$$

2. Let us now proceed to find the electromagnetic field in first approximation.

The most general expression for the second-rank antisymmetric tensor $F^1_{\mu\nu}$ that can be constructed in first approximation from the vectors η, $\dot{\xi}$, $\ddot{\xi}$ is given by the formula

$$F^1_{\mu\nu}(\eta; \dot{\xi}, \ddot{\xi}) = g_1(\eta)(\ddot{\xi}_\mu\dot{\xi}_\nu - \ddot{\xi}_\nu\dot{\xi}_\mu) + g_2(\eta)(\eta\ddot{\xi})(\dot{\xi}_\mu\eta_\nu - \dot{\xi}_\nu\eta_\mu) + g_3(\eta)(\ddot{\xi}_\mu\eta_\nu - \ddot{\xi}_\nu\eta_\mu), \tag{11}$$

where g_1, g_2, g_3 are unknown functions of η.

It is easily verified that for the field $F^0_{\mu\nu} + F^1_{\mu\nu}$ the invariants J_1 and J_2 are

$$J_1 = J_1^{(0)} + J_1^{(1)} \equiv \tfrac{1}{2}z^2 + (\eta\ddot{\xi})g_0(\eta^2 g_2 - g_1), \qquad J_2 = 0, \tag{12}$$

and therefore in first as well as zeroth approximation we have $\chi_2 = 0$.

Using the facts that

$$\dot{\xi}_\rho \partial F^0_{\lambda\nu}/\partial\eta_\rho = g_0\dot{\xi}_\rho(\dot{\xi}_\mu\delta_{\nu\rho} - \dot{\xi}_\nu\delta_{\mu\rho}) = 0,$$

$$\dot{\xi}_\rho \partial F^1_{\lambda\nu}/\partial\eta_\rho = g_3(\ddot{\xi}_\lambda\dot{\xi}_\nu - \ddot{\xi}_\nu\dot{\xi}_\lambda)$$

and consequently

$$\dot{\xi}_\mu\dot{\xi}_\rho \partial F^1_{\lambda\nu}/\partial\eta_\rho + (\mu\lambda\nu) = 0,$$

we can put Eq. (6′) in the form

$$\partial F^1_{\mu\nu}/\partial\eta_\lambda - \dot{\xi}_\lambda \partial F^0_{\mu\nu}/\partial\tau + (\mu\nu\lambda) = 0.$$

Substituting in this equation the expressions (9) and (11) for $F^0_{\mu\nu}$ and $F^1_{\mu\nu}$, we find the following relation between the functions g_1, g_2, and g_3:

$$g_1'/\eta + g_2 + g_0 = 0. \tag{13}$$

Since up to second-order terms

$$\dot{\xi}_\lambda\frac{\partial}{\partial\eta_\lambda}\chi_1 F^0_{\mu\nu} \approx \chi_1\dot{\xi}_\lambda\frac{\partial F^0_{\mu\nu}}{\partial\eta_\lambda} = 0,$$

$$\dot{\xi}_\nu\frac{\partial}{\partial\tau}\chi_1 F^0_{\mu\nu} \approx \chi_1\dot{\xi}_\nu\frac{\partial F^0_{\mu\nu}}{\partial\tau} = \chi_1 g_0\dot{\xi}_\nu(\ddot{\xi}_\mu\eta_\nu - \ddot{\xi}_\nu\eta_\mu) = 0,$$

the equation (6) takes the form

$$(\delta_{\nu\lambda} + \dot{\xi}_\nu\dot{\xi}_\lambda)\frac{\partial F_{\mu\nu}}{\partial\eta_\lambda} + (\delta_{\nu\lambda} + \dot{\xi}_\nu\dot{\xi}_\lambda)F_{\mu\nu}\frac{\partial}{\partial\eta_\lambda}\ln\chi_1 = 0. \tag{14}$$

Using the fact that, by Eq. (10),

$$\left.\frac{\partial\chi_1}{\partial J_1}\right|_{J_1=z^2/2,\, J_2=0} = -\frac{1}{z^2 z'}\left(\frac{2}{\eta}z + z'\right)\chi_1\left(\tfrac{1}{2}z^2, 0\right),$$

we find

$$\ln\chi_1 = \ln\frac{C}{\eta^2 z} - \frac{1}{zz'}\left(\frac{2}{\eta} + \frac{z'}{z}\right)g_0(\eta^2 g_2 - g_1)(\eta\ddot{\xi}).$$

Consequently, when we include only first-order terms, Eq. (14) takes the form

$$(\delta_{\nu\lambda} + \dot{\xi}_\nu\dot{\xi}_\lambda)\frac{\partial F^1_{\mu\nu}}{\partial\eta_\lambda} + (\delta_{\nu\lambda} + \dot{\xi}_\nu\dot{\xi}_\lambda)F^1_{\mu\nu}\frac{\partial}{\partial\eta_\lambda}\ln\frac{C}{\eta^2 z} - (\delta_{\nu\lambda} + \dot{\xi}_\nu\dot{\xi}_\lambda)F^0_{\mu\nu}\frac{\partial}{\partial\eta_\lambda}\left\{\frac{1}{\eta}\left(\frac{1}{z} + \frac{2}{\eta z'}\right)(\eta^2 g_2 - g_1)(\eta\ddot{\xi})\right\} = 0.$$

Substituting in this equation the expression (11) for $F^1_{\mu\nu}$, we get

$$A_1(\eta\ddot{\xi})\eta_\mu + A_2\ddot{\xi}_\mu + A_3(\eta\ddot{\xi})\dot{\xi}_\mu = 0, \tag{15}$$

where

$$A_1 = -\frac{1}{\eta}g_3' - \frac{1}{\eta}g_3\frac{\partial}{\partial\eta}\ln\frac{C}{\eta^2 z},$$

$$A_2 = 2g_3 + \eta g_3' + \eta g_3\frac{d}{d\eta}\ln\frac{C}{\eta^2 z},$$

$$A_3 = \frac{2z}{\eta z'}\left\{g_1'' + \left(\frac{1}{\eta} - \frac{z''}{z'}\right)g_1' + \frac{1}{\eta}\left(\frac{z'}{z} - \frac{z''}{z'} - \frac{1}{\eta}\right)g_1 + z' - \frac{zz''}{z'}\right\}.$$

In the derivation of these formulas we have taken the relation (13) into account.

It is easy to see that the necessary condition for Eq. (15) to be satisfied is

$$A_1 = 0, \qquad A_2 = 0, \qquad A_3 = 0$$

(we do not consider here the trivial case $\ddot{\xi} = 0$). Since $\eta^2 A_1 + A_2 = 2g_3$, we conclude that the equations $A_1 = 0$ and $A_2 = 0$ can be satisfied only for

$$g_3 \equiv 0. \tag{16}$$

It is easy to put the equation $A_3 = 0$ in the form

$$\frac{d}{d\eta}\left\{\frac{z}{z'}g_1' + \left(\frac{z}{\eta z'} - 1\right)g_1\right\} + z\left(\frac{z}{z'}\right)' = 0.$$

A first integral of this equation is given by

$$g_1' + \left(\frac{1}{\eta} - \frac{z'}{z}\right)g_1 + \frac{z'}{z}\int_A^\eta z\left(\frac{z}{z'}\right)' d\eta = 0. \tag{17}$$

We determine the constant of integration A from the boundary condition (7), according to which

$$g_0 = C/\eta^3, \qquad g_1 = C/2\eta \qquad \text{for } \eta \to \infty.$$

From this we have $A = \infty$. Integrating by parts and using the fact that $(z^2/z')\,|_{\eta=\infty} = 0$, we can put Eq. (17) in the form

$$g_1' + \left(\frac{1}{\eta} - \frac{z'}{z}\right) g_1 + z + \frac{z'}{z}\int_{\eta}^{\infty} z\, d\eta = 0.$$

Integration of this equation leads to the following result:

$$g_1(\eta) = \int_{\eta}^{\infty} z\, d\eta - \frac{z}{\eta}\int_{C_1}^{\eta} \frac{d\eta}{z}\int_{\eta}^{\infty} z\, d\eta, \tag{18}$$

where C_1 is a constant of integration.

Substituting this expression for g_1 in the relation (13), we find

$$g_2(\eta) = \frac{1}{\eta^2}\int_{\eta}^{\infty} z\, d\eta + \frac{g_0'}{\eta}\int_{C_1}^{\eta} \frac{d\eta}{z}\int_{\eta}^{\infty} z\, d\eta. \tag{19}$$

The formulas (16), (18) and (19), together with Eq. (11), solve the problem of finding the field of a single singularity in first approximation.

3. Finally, we proceed to the calculation of the second approximation to the field $F_{\mu\nu}$. Since the calculations in this case are more cumbersome than those for the first approximation, we shall indicate only the main steps.

The most general form for the second approximation is

$$\begin{aligned} F^2_{\mu\nu}(\eta; \dot\xi, \ddot\xi, \dddot\xi) = {} & f_1(\eta)(\dddot\xi_\mu\dot\xi_\nu - \dddot\xi_\nu\dot\xi_\mu) + f_2(\eta)(\dddot\xi_\mu\eta_\nu - \dddot\xi_\nu\eta_\mu) \\ & + f_3(\eta)(\eta\dddot\xi)(\dot\xi_\mu\eta_\nu - \dot\xi_\nu\eta_\mu) + f_4(\eta)(\dot\xi\dddot\xi)(\dot\xi_\mu\eta_\nu - \dot\xi_\nu\eta_\mu) \\ & + f_5(\eta)(\eta\ddot\xi)^2(\dot\xi_\mu\eta_\nu - \dot\xi_\nu\eta_\mu) + f_6(\eta)(\eta\ddot\xi)(\ddot\xi_\mu\eta_\nu - \ddot\xi_\nu\eta_\mu) \\ & + f_7(\eta)(\eta\ddot\xi)(\ddot\xi_\mu\dot\xi_\nu - \ddot\xi_\nu\dot\xi_\mu), \end{aligned} \tag{20}$$

where $f_1, \ldots, f_7$ are unknown functions of the variable η.

In this case the field equations (6′) lead to the relations

$$f_1'/\eta + f_3 = 0, \qquad f_7'/\eta + 2f_5 + g_2 - g_0 = 0. \tag{21}$$

We note that J_2 is again equal to zero in this approximation, since the only nonvanishing invariant containing $\epsilon_{\mu\nu\lambda\rho}$, namely $\epsilon_{\mu\nu\lambda\rho}\eta_\mu\dot\xi_\nu\ddot\xi_\lambda\dddot\xi_\rho$, could appear only in the third approximation. Consequently, $\chi_2 = 0$.

In second approximation the invariant J_1 is

$$J_1 = J_1^{(0)} + J_1^{(1)} + J_1^{(2)};$$

$$\begin{aligned} J_1^{(2)} = {} & -(\eta\dddot\xi)\, g_0 (f_1 - \eta^2 f_3) + \ddot\xi^2 [\tfrac{1}{2} g_1^2 + \eta^2 g_0 (f_2 - f_4)] \\ & + (\eta\ddot\xi)^2 [\tfrac{1}{2}\eta^2 g_2^2 - g_1 g_2 + g_0(\eta^2 f_5 - f_7)]. \end{aligned} \tag{22}$$

Therefore the field equations (6) take the form

$$\begin{aligned} & (\delta_{\nu\lambda} + \dot\xi_\nu\dot\xi_\lambda)\frac{\partial F^2_{\mu\nu}}{\partial\eta_\lambda} - \dot\xi_\nu\frac{\partial F^1_{\mu\nu}}{\partial\tau} + (\delta_{\nu\lambda} + \dot\xi_\nu\dot\xi_\lambda) F^0_{\mu\nu}\frac{\partial\varphi_2}{\partial\eta_\lambda} + F^1_{\mu\nu}\frac{\partial\varphi_1}{\partial\eta_\nu} \\ & - \dot\xi_\nu F^0_{\mu\nu}\frac{\partial\varphi_1}{\partial\tau} + F^2_{\mu\nu}\frac{\partial}{\partial\eta_\nu}\ln\frac{C}{\eta^2 z} = 0, \end{aligned} \tag{23}$$

where

$$\varphi_1 = \frac{(\eta\ddot\xi)}{\eta z z'}\left(\frac{2}{\eta}z + z'\right)(g_1 - \eta^2 g_2),$$

$$\varphi_2 = -\frac{J_1^{(2)}}{zz'}\left(\frac{2}{\eta} + \frac{z'}{z}\right) + \frac{J_1^{(1)2}}{z^3 z'^2}\left(\frac{z}{\eta^2} + \frac{z'}{\eta} + \frac{z'^2}{z} + \frac{zz''}{\eta z'}\right). \tag{24}$$

Substituting in (23) the expressions (9), (11), and (20) for $F^0_{\mu\nu}$, $F^1_{\mu\nu}$, $F^2_{\mu\nu}$, we get

$$B_1\eta_\mu + B_2\dddot\xi_\mu + B_3(\eta\ddot\xi)\ddot\xi_\mu + B_4\dot\xi_\mu = 0;$$

$$\begin{aligned} B_1 = {} & -(\eta\dddot\xi)\left[\frac{1}{\eta}f_2' - \frac{1}{\eta}\left(\frac{2}{\eta} + \frac{z'}{z}\right)f_2 + g_2 + \frac{g_0}{\eta z'}\left(\frac{2}{\eta} + \frac{z'}{z}\right)\right. \\ & \times \left.(g_1 - \eta^2 g_2)\right] - \ddot\xi^2 f_6 - (\eta\ddot\xi)^2\left[\frac{1}{\eta}f_6' - \frac{1}{\eta}\left(\frac{2}{\eta} + \frac{z'}{z}\right)f_6\right], \end{aligned}$$

$$B_2 = \eta f_2' + g_1 - \eta f_2 z'/z. \qquad B_3 = \eta f_6' + f_6 - \eta f_6 z'/z,$$

$$\begin{aligned} B_4 = {} & (\eta\dddot\xi)\left\{\eta f_3' + \eta\left(\frac{2}{\eta} - \frac{z'}{z}\right)f_3 - \frac{1}{\eta}f_1' + \frac{1}{\eta}\left(\frac{2}{\eta} + \frac{z'}{z}\right)f_1\right\} \\ & - \xi''^2\left\{\eta f_4' + \left(1 - \eta\frac{z'}{z}\right)f_4 + f_7 - f_2 + g_1\right. \\ & \left. + \frac{g_1}{\eta z'}\left(\frac{2}{\eta} + \frac{z'}{z}\right)(\eta^2 g_2 - g_1)\right\} + (\eta\ddot\xi)^2\left\{\eta f_5' + \eta\left(\frac{3}{\eta} - \frac{z'}{z}\right)f_5\right. \\ & - \frac{1}{\eta}f_7' + \frac{1}{\eta}\left(\frac{2}{\eta} + \frac{z'}{z}\right)f_7 + \frac{\varphi_1}{\eta\ddot\xi}g_2 \\ & \left. + \frac{1}{\eta}\left(\frac{\varphi_1}{\eta\ddot\xi}\right)'(\eta^2 g_2 - g_1)\right\} + g_0\eta_\lambda\frac{\partial\varphi_2}{\partial\eta_\lambda}. \end{aligned} \tag{23′}$$

It follows from Eq. (23′) that

$$B_1 = 0, \qquad B_2 = 0, \qquad B_3 = 0, \qquad B_4 = 0.$$

The first three of these equations lead to the following relations:

$$f_6 = 0, \tag{25}$$

$$\begin{aligned} & \frac{1}{\eta}f_2' + g_2 - \frac{1}{\eta}\left(\frac{2}{\eta} + \frac{z'}{z}\right)f_2 \\ & + \frac{g_0}{\eta z'}\left(\frac{2}{\eta} + \frac{z'}{z}\right)(g_1 - \eta^2 g_2) = 0, \\ & \eta f_2' + g_1 - \eta f_2 z'/z = 0. \end{aligned} \tag{25′}$$

It is easily verified that Eq. (25′) is equivalent to the system of equations

$$f_2' + \eta g_2 = 0, \qquad f_2 = g_1 - \int_{\eta}^{\infty} z\, d\eta,$$

and since $g_2 = -g_0 - g_1'/\eta$, we can verify that the equations (25′) are not inconsistent and that

$$f_2 = g_1 - \int_{\eta}^{\infty} z\, d\eta. \tag{26}$$

Noting that according to Eqs. (24), (22), and (12)

$$\varphi_2=\frac{1}{\eta z'}\left(\frac{2}{\eta}+\frac{z'}{z}\right)(f_1-\eta^2 f_3)(\eta\ddot{\xi})$$
$$-\frac{1}{zz'}\left(\frac{2}{\eta}+\frac{z'}{z}\right)\left[\frac{1}{2}g_1^2+\eta z(f_2-f_4)\right]\ddot{\xi}^2$$
$$+\left\{\frac{(\eta^2 g_2-g_1)^2}{\eta^2 z'^2}\left[\frac{1}{\eta_2}+\frac{z'^2}{2z^2}+\frac{z''}{\eta z'}\right]\right.$$
$$\left.+\frac{1}{zz'}\left(\frac{2}{\eta}+\frac{z'}{z}\right)\left[g_1^2+\frac{z}{\eta}(f_7-\eta^2 f_6)\right]\right\}(\eta\ddot{\xi})^2,$$

we find that the equation $B_4 = 0$ separates into the following three equations:

$$\frac{d}{d\eta}\left\{\frac{z}{z'}f_1'+\left(\frac{z}{\eta z'}-1\right)f_1\right\}=0,\qquad f_4'-\frac{z''}{z'}f_4+z'R_1=0,$$
$$f_7''+\left(\frac{3}{\eta}-\frac{z''}{z'}\right)f_7'+\frac{1}{\eta}\left(3\frac{z'}{z}-2\frac{z''}{z'}\right)f_7+R_2=0; \tag{27}$$

where

$$R_1=\frac{f_2-f_7-g_1}{2z}+\frac{g_1}{2\eta zz'}\left(\frac{2}{\eta}+\frac{z'}{z}\right)(g_1-\eta^2 g_2)$$
$$-\frac{1}{2}\frac{d}{d\eta}\left\{\frac{1}{zz'}\left(\frac{2}{\eta}+\frac{z'}{z}\right)\left(\frac{1}{2}g_1^2+\eta z f_2\right)\right\},$$
$$R_2=\frac{\eta z'}{z}\left\{\frac{1}{2}\eta(g_0'-g_2')+\frac{1}{2}\left(3-\eta\frac{z'}{z}\right)(g_0-g_2)\right.$$
$$+\frac{g_2}{z}f_2\left(\frac{2}{\eta}+\frac{z'}{z}\right)-\frac{z'}{z}f_2\frac{d}{d\eta}\left[\frac{f_2}{2}\left(\frac{2}{\eta}+\frac{z'}{z}\right)\right]$$
$$+\frac{z}{\eta^2}\frac{d}{d\eta}\left[\frac{\eta^2}{z^2}f_2^2\left(\frac{1}{\eta^2}+\frac{z'^2}{2z^2}+\frac{z''}{\eta z'}\right)\right.$$
$$\left.\left.+\frac{\eta^2}{zz'}\left(\frac{2}{\eta}+\frac{z'}{z}\right)\left(g_1^2-\frac{1}{2}\eta z(g_0-g_2)\right)\right]\right\}.$$

From the first of the equations (27) and the boundary conditions (7) we get

$$f_1=\mp C\left\{1-\frac{z}{\eta}\int_{C_2}^{\eta}\frac{d\eta}{z}\right\}, \tag{28}$$

where the upper sign corresponds to retarded solutions and the lower to advanced, and C_2 is a constant of integration.

Integration of the second of the equations (27) leads to the following formula for f_4:

$$f_4=f_2+\frac{g_1^2}{4z}\left(\frac{2}{\eta}+\frac{z'}{z}\right)+\frac{1}{2}z'\int_{C_3}^{\eta}\frac{d\eta}{z}\left\{f_7-\frac{g_1}{z}\left(\frac{2}{\eta}+\frac{z'}{z}\right)f_2\right\}. \tag{29}$$

The unknown quantity f_7 appearing in this formula can be determined from the third of the equations (27).

The formulas (21) and (25)—(29), together with Eq. (20), solve the problem of finding the field of a single singularity in the second approximation.

4. STRUCTURE OF THE FIELD IN THE CASE OF SEVERAL SINGULARITIES

1. There is no superposition principle in the nonlinear theory, and therefore the inclusion of the fields of other singularities is not a trivial problem. Let us denote by $f^0_{\mu\nu}$ the electromagnetic field in the absence of the singularity under consideration. If we assume that this field is small in comparison with the proper field C^2/r_0, then in this case we must look for the total field in the form

$$F_{\mu\nu}=F^0_{\mu\nu}+f_{\mu\nu},\qquad f_{\mu\nu}(x)=\int K_{\mu\nu;\rho\sigma}\{x,x';\xi\}f^0_{\rho\sigma}(x')\,d^4x',$$

where $K_{\mu\nu;\rho\sigma}\{x, x'; \xi\}$ is a quantity to be determined, which has a functional dependence on the shape of the world line of the singularity under consideration.

For $\eta = |x - \xi| \gg r_0$ we must have

$$K_{\mu\nu;\rho\sigma}\{x,x';\xi\}=\delta(x-x')\,\delta_{\mu\rho}\delta_{\nu\sigma},$$

since in this region of space-time the linear theory holds, and along with it the principle of superposition. We shall assume, however, that the field $f^0_{\mu\nu}$ not only is small, but also changes very little in a space-time region whose radius is of the order of r_0. Therefore we shall look for the quantity $f_{\rho\sigma}$ in the form

$$f_{\rho\sigma}=h_1(\eta)f^0_{\rho\sigma}+h_2(\eta)\dot{\xi}_\alpha f^0_{\alpha\beta}\eta_\beta(\dot{\xi}_\rho\eta_\sigma-\dot{\xi}_\sigma\eta_\rho)$$
$$+h_3(\eta)(\dot{\xi}_\rho f^0_{\sigma\alpha}-\dot{\xi}_\sigma f^0_{\rho\alpha})\dot{\xi}_\alpha+h_4(\eta)(\eta_\rho f^0_{\sigma\alpha}-\eta_\sigma f^0_{\rho\alpha})\dot{\xi}_\alpha$$
$$+h_5(\eta)(\dot{\xi}_\rho f^0_{\sigma\alpha}-\dot{\xi}_\sigma f^0_{\rho\alpha})\eta_\alpha+h_6(\eta)(\eta_\rho f^0_{\sigma\alpha}-\eta_\sigma f^0_{\rho\alpha})\eta_\alpha, \tag{30}$$

where $h_1, \ldots, h_6$ are unknown functions of the variable η and $f^0_{\rho\sigma}$ can be treated as a quantity independent of η, τ. Equation (30) gives the most general form of a second-rank antisymmetric tensor constructed from the quantities $f^0_{\rho\sigma}$, η_ρ, and $\dot{\xi}_\rho$ only.

The superposition principle is valid for $\eta \gg r_0$, and therefore in this case we must take as the boundary conditions

$$h_1=1,\qquad h_2,\ldots,h_6=0,\qquad \eta\gg r_0. \tag{31}$$

2. We proceed to the determination of the functions h_i. Since the field equations (6′) are linear, they must also be satisfied by the functions (30). In the present approximation these equations are of the form

$$(\delta_{\mu\nu}+\dot{\xi}_\mu\dot{\xi}_\nu)\,\partial f_{\rho\sigma}/\partial\eta_\nu+(\mu\rho\sigma)=0.$$

When we substitute the expression (30) in this equation, we easily get the relations

$$h_5=0,\qquad h_1'/\eta+2h_6=0,\qquad h_3'/\eta+h_2-h_6=0. \tag{32}$$

The forms of the invariants J_1 and J_2 are

$$J_1\equiv J_1^{(0)}+J_1^{(f)}=\tfrac{1}{2}z^2-\tfrac{1}{2}F^0_{\mu\nu}f_{\mu\nu},$$
$$J_1^{(f)}=-g_0(\dot{\xi}_\alpha f^0_{\alpha\beta}\eta_\beta)(h_1-\eta^2 h_2+h_3-\eta^2 h_6),$$
$$J_2\equiv J_2^{(f)}=-g_0 h_1\dot{\xi}_\mu\eta_\nu f^0_{\lambda\rho}\varepsilon_{\mu\nu\lambda\rho}. \tag{32′}$$

From Eq. (10) and the restrictions we have placed on the Lagrangian, it follows that

$$\ln \chi_1 = \ln \frac{C}{\eta^2 z} - \frac{J_1^{(f)}}{zz'}\left(\frac{2}{\eta} + \frac{z'}{z}\right),$$
$$\chi_2 = \chi_1\left(\tfrac{1}{2} z^2, 0\right) J_2^{(f)} \psi(\eta),$$

where

$$\psi(\eta) = \frac{1}{\chi_1(z^2/2, 0)} \frac{\partial \chi_2}{\partial J_2}\Big|_{J_1 = z^2/2,\, J_2 = 0}. \quad (33)$$

In this case the field equations (6) take the form

$$(\delta_{\nu\lambda} + \dot{\xi}_\nu \dot{\xi}_\lambda) \frac{\partial f_{\mu\nu}}{\partial \eta_\lambda} + (\delta_{\nu\lambda} + \dot{\xi}_\nu \dot{\xi}_\lambda) f_{\mu\nu} \frac{\partial}{\partial \eta_\lambda} \ln \frac{C}{z\eta^2}$$
$$- (\delta_{\nu\lambda} + \dot{\xi}_\nu \dot{\xi}_\lambda) F^0_{\mu\nu} \frac{\partial}{\partial \eta_\lambda}\left\{\frac{J_1^{(f)}}{zz'}\left(\frac{2}{\eta} + \frac{z'}{z}\right)\right\}$$
$$- F^0_{\alpha\beta}\varepsilon_{\alpha\beta\lambda\mu}(\delta_{\nu\lambda} + \dot{\xi}_\nu \dot{\xi}_\lambda)\psi(\eta)\frac{\partial J_2^{(f)}}{\partial \eta_\nu} = 0.$$

Substituting in this equation the relations (30), we can put the equation in the form

$$D_1 f^0_{\rho\alpha}\eta_\alpha + D_2 f^0_{\rho\alpha}\dot{\xi}_\alpha + D_3 (\dot{\xi}_\alpha f^0_{\alpha\beta}\eta_\beta)\eta_\rho + D_4 (\dot{\xi}_\alpha f^0_{\alpha\beta}\eta_\beta)\dot{\xi}_\rho = 0; \quad (34)$$

where

$$D_1 = -\eta h_6' - 5h_6 - \frac{1}{\eta}\left(\frac{2}{\eta} + \frac{z'}{z}\right)(h_1 - \eta^2 h_6) - 4\psi g_0^2 h_1,$$

$$D_2 = -\eta h_4' + \eta \frac{z'}{z} h_4, \qquad D_3 = -\frac{1}{\eta} h_4' + \frac{1}{\eta}\left(\frac{2}{\eta} + \frac{z'}{z}\right) h_4,$$

$$D_4 = \eta h_2' + 5h_2 - \frac{1}{\eta}\left(\frac{2}{\eta} + \frac{z'}{z}\right)(\eta^2 h_2 - h_3)$$
$$- 4\psi g_0^2 h_1 + \frac{g_0}{(\dot{\xi}_\alpha f^0_{\alpha\beta}\eta_\beta)}\eta_\lambda$$
$$\times \frac{\partial}{\partial \eta_\lambda}\left\{\frac{\dot{\xi}_\alpha f^0_{\alpha\beta}\eta_\beta}{\eta z'}\left(\frac{2}{\eta} + \frac{z'}{z}\right)(h_1 - \eta^2 h_2 + h_3 - \eta^2 h_6)\right\}.$$

The equations (34) are satisfied if

$$D_1 = 0, \quad D_2 = 0, \quad D_3 = 0, \quad D_4 = 0.$$

From $D_2 = 0$ and $D_3 = 0$ it follows that

$$h_4 \equiv 0. \quad (35)$$

When we use the second of the relations (32) the equation $D_1 = 0$ takes the form

$$h_1'' + \left(\frac{2}{\eta} - \frac{z'}{z}\right)h_1' - \frac{2}{\eta}\left[\frac{2}{\eta} + \frac{z'}{z} + 4\psi\frac{z^2}{\eta}\right]h_1 = 0. \quad (36)$$

Using the relations (32), we can put the equation $D_4 = 0$ in the form

$$\frac{z'}{z}\frac{d}{d\eta}\left\{\frac{z}{z'}h_3' + \left(\frac{z}{\eta z'} - 1\right)h_3\right\} + \left(\frac{z'}{z} - \frac{z''}{z'} - \frac{1}{\eta}\right)h_1'$$
$$+ \left(3\frac{z'}{z} - \frac{z''}{z'} + \frac{3}{\eta} + 8\psi\frac{z^2}{\eta}\right)\frac{1}{\eta}h_1 = 0. \quad (37)$$

It follows from Eqs. (36) and (37) that the quantity $h = h_1 + h_3$ satisfies the equation

$$\frac{d}{d\eta}\left\{\frac{z}{z'}h' + \left(\frac{z}{\eta z'} - 1\right)h\right\} = 0.$$

Integrating this equation and using the boundary conditions (31), we get

$$h = \frac{3}{2}\left\{1 - \frac{z}{\eta}\int_{C_4}^{\eta}\frac{d\eta}{z}\right\}, \quad (38)$$

where C_4 is a constant of integration.

The formulas (32), (35), (36), and (38), together with Eq. (30), determine the effect on the structure of the field near a particular singularity caused by other singularities located sufficiently far away.

5. EQUATIONS OF MOTION OF SINGULARITIES

We can obtain the equations of motion by starting from the general dynamical principle of Lorentz, according to which, when only the electromagnetic field is present, the singularities move in such a way that the flux of energy and momentum through an infinitely small two-dimensional surface enclosing a singularity is zero.[3]

If we denote by $df^{\lambda\nu}$ an element of a surface S surrounding the singularity under consideration, the relativistically invariant formulation of this principle can be put in the form

$$\lim_{S\to 0} \dot{\xi}_\lambda \oint_S T_{\mu\nu} df^{\lambda\nu} = 0, \quad (39)$$

where the surface S lies in the space orthogonal to $\dot{\xi}$.

Taking a sphere as the surface S, we have formula $df^{\lambda\nu} = (\dot{\xi}_\lambda \eta_\nu - \dot{\xi}_\nu \eta_\lambda) dS/\eta$

and consequently

$$\dot{\xi}_\lambda df^{\lambda\nu} = -\eta_\nu dS/\eta = -\eta_\nu \eta do,$$

where $do = \eta^{-2}dS$ is the solid angle subtended by the surface element dS.

The relation (39) therefore takes the form

$$\lim_{\eta\to 0} \eta \overline{T_{\mu\nu}\eta_\nu} = 0, \quad (39')$$

where the bar indicates averaging over angles:

$$\bar{f} = \frac{1}{4\pi}\oint f do.$$

The accuracy with which we have carried out the determination of the field near the singularities obviously allows us in the calculation of $T_{\mu\nu}$ to keep only terms of orders 1, $\ddot{\xi}$, $\dddot{\xi}$, f^0 (obviously we must neglect terms of the form $f^0\ddot{\xi}$).

According to Eq. (4), to determine $T_{\mu\nu}$ we must calculate the quantities $F_{\mu\nu}F_{\nu\lambda}$, $J_2\chi_2$, L, χ_1. Keeping only terms of the indicated orders in the calculation of these quantities, we get

$$F_{\mu\lambda}F_{\nu\lambda}=F^0_{\mu\lambda}F^0_{\nu\lambda}+\{F^0_{\mu\lambda}F^1_{\nu\lambda}+F^1_{\mu\lambda}F^0_{\nu\lambda}\}+\{F^0_{\mu\lambda}f_{\nu\lambda}+f_{\mu\lambda}F^0_{\nu\lambda}\}$$
$$+\{F^0_{\mu\lambda}F^2_{\nu\lambda}+F^1_{\mu\lambda}F^1_{\nu\lambda}+F^2_{\mu\lambda}F^0_{\nu\lambda}\},$$
$$J_2\chi_2=0,\ L=L(J_1^{(0)},0)+(J_1^{(1)}+J_1^{(2)}+J_1^{(f)})\chi_1(J_1^{(0)},0)$$
$$+\frac{1}{2}J_1^{(1)2}\partial\chi_1(J_1^{(0)},0)/\partial J_1,$$
$$\chi_1=\chi_1(J_1^{(0)},0)+(J_1^{(1)}+J_1^{(2)}+J_1^{(f)})\partial\chi_1(J_1^{(0)},0)/\partial J_1$$
$$+\frac{1}{2}J_1^{(1)2}\partial^2\chi_1(J_1^{(0)},0)/\partial J_1^2. \tag{40}$$

Noting that

$$\overline{\eta_\mu}=0,\qquad \overline{\eta_\mu\eta_\nu}=\frac{1}{3}(\delta_{\mu\nu}+\dot{\xi}_\mu\dot{\xi}_\nu)\eta^2,\qquad \overline{\eta_\mu\eta_\nu\eta_\lambda}=0,$$

and using the relations (4), (40), (12), (22), and (32′), and also the formulas (9), (11), (20), and (30), which give the quantities $F^0_{\mu\nu}$, $F^1_{\mu\nu}$, $F^2_{\mu\nu}$, and $f_{\mu\nu}$, we get after simple but rather cumbersome calculations

$$\eta\overline{T_{\mu\nu}\eta_\nu}=\frac{2}{3}C\left\{g_1+\frac{1}{\eta}\frac{z}{z'}(\eta^2g_2-g_1)\right\}\ddot{\xi}_\mu$$
$$+\frac{2}{3}C\left\{f_1+\frac{1}{\eta}\frac{z}{z'}(\eta^2f_3-f_1)\right\}$$
$$\times(\dddot{\xi}_\mu-\ddot{\xi}^2\dot{\xi}_\mu)+\frac{2}{3}C\left\{-h_1-h_3\right.$$
$$\left.+\frac{1}{\eta}\frac{z}{z'}(h_1+h_3-\eta^2h_2-\eta^2h_6)\right\}f^0_{\mu\lambda}\dot{\xi}_\lambda. \tag{41}$$

Be means of the formulas (18), (19), (28), and (21), which give the quantities g_1, g_2, f_1, and f_3, we find

$$g_1+\frac{1}{\eta}\frac{z}{z'}(\eta^2g_2-g_1)=\int_\eta^\infty z\,d\eta,$$
$$f_1+\frac{1}{\eta}\frac{z}{z'}(\eta^2f_3-f_1)=\mp C.$$

By using the formulas (32) and (38) we can put the coefficient of $f^0_{\mu\nu}\dot{\xi}_\lambda$ in Eq. (41) in the form

$$-h_1-h_3+\frac{1}{\eta}\frac{z}{z'}(h_1+h_3-\eta^2h_2-\eta^2h_6)$$
$$=\frac{z^2}{\eta z'}\frac{d}{d\eta}\left(\frac{\eta h}{z}\right)=-\frac{3}{2}.$$

Thus to find the equations of motion it is not necessary to solve the equation (36) for the quantity h_1, since the answer involves only the quantity h, for which the explicit expression (38) has been found. The result is that Eq. (41) takes the form

$$\eta\overline{T_{\mu\nu}\eta_\nu}=\ddot{\xi}_\mu\frac{2}{3}C\int_\eta^\infty z\,d\eta-Cf^0_{\mu\lambda}\dot{\xi}_\lambda\mp\frac{2}{3}C^2(\dddot{\xi}_\mu-\ddot{\xi}^2\dot{\xi}_\mu),$$

and, according to the dynamical principle (39′), the equations of motion are of the form

$$\frac{2}{3}C\int_0^\infty z\,d\eta\ddot{\xi}_\mu=Cf^0_{\mu\lambda}\dot{\xi}_\lambda\pm\frac{2}{3}C^2(\dddot{\xi}_\mu-\ddot{\xi}^2\dot{\xi}_\mu). \tag{42}$$

We still have to connect the quantity

$$\frac{2}{3}C\int_0^\infty z\,d\eta$$

with the mass of the singularity. To do this we calculate the integral $-\int T_{44}\mathbf{dx}$ for the field of a single stationary singularity. Since in this case, according to Eqs. (4) and (9),

$$T_{44}=L+\chi_1F^2_{4k},\qquad F_{4k}=ig_0x_k$$

and $\eta = x$, $\eta_4 = 0$, when we integrate by parts and use Eq. (10) we can verify that

$$\int T_{44}dx=-\frac{8\pi C}{3}\int_0^\infty z\,d\eta.$$

Consequently the mass of the singularity is given by

$$m=-\int T_{44}dx=\frac{8\pi C}{3}\int_0^\infty z\,d\eta, \tag{43}$$

and the equations of motion (42) take the usual form

$$m\ddot{\xi}_\mu=ef^0_{\mu\lambda}\dot{\xi}_\lambda\pm\frac{2}{3}\frac{e^2}{4\pi}(\dddot{\xi}_\mu-\ddot{\xi}^2\dot{\xi}_\mu). \tag{42′}$$

Here we have used the fact that the constant C is connected with the charge of the singularity by the relation

$$C=e/4\pi.$$

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