DISPERSION RELATIONS FOR THE INDICES OF REFRACTION AND ABSORPTION

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The dispersion relations in macroscopic electrodynamics relate to each other the real and imaginary parts of the dielectric tensor $\epsilon_{ij}(\omega, \mathbf{k})$. On the other hand, in the study of the propagation of electromagnetic waves through a medium one is more interested in dispersion relations for the complex index of refraction of "normal waves" $\tilde{n}_l(\omega, \mathbf{s}) = n_l + i\kappa_l$ (n_l and κ_l being the refraction and absorption indices of the normal wave of type l). In an isotropic medium without spatial dispersion one has $\epsilon_{ij} = \epsilon(\omega)\delta_{ij}$, $\tilde{n}^2 = \epsilon(\omega) = \epsilon' + i\epsilon''$, and the dispersion relations for ϵ and for $\tilde{n}^2$ are essentially identical. This is usually the only case considered. In this paper the question of analytic properties and dispersion relations for $\tilde{n}_l^2$ in an anisotropic medium (in particular in a magnetically active plasma) is considered. Some comments are also made regarding spatial dispersion.

ALTHOUGH dispersion relations in electrodynamics have been established more than thirty years ago,[1] it is only recently that they have been included in textbooks and applied more or less widely.[2] It may be that it is this circumstance that explains why we were not able to find in the literature known to us the answer to one of the basic questions that arise in connection with the use of dispersion relations.

Namely, these relations are established for the complex permeability tensor $\epsilon_{ij}(\omega, \mathbf{k})$, whereas from the point of view of analysis of experimental data, as well as in a number of other cases, it is desirable to have dispersion relations connecting the index of refraction and the index of absorption, $n_l(\omega, \mathbf{s})$ and $\kappa_l(\omega, \mathbf{s})$ (the complex index of refraction $\tilde{n}_l(\omega, \mathbf{s}) = n_l + i\kappa_l$ appears in the expression for uniform plane waves $\mathbf{E}_l = \mathbf{E}_{0,l} \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$, where $\mathbf{k} = (\omega/c)\tilde{n}_l(\omega, \mathbf{s})\mathbf{s}$, $\mathbf{s} = \mathbf{k}/k$, and l is the index characterizing the normal wave, for example the ordinary or the extraordinary wave).

The relation between $\epsilon_{ij}(\omega, \mathbf{k})$ and $\tilde{n}_l(\omega, \mathbf{s})$ is obtained by making use of the field equations and has the form (see, for example, [2,3])

$$|\tilde{n}^2(\delta_{ij} - s_i s_j) - \varepsilon_{ij}(\omega, \omega\tilde{n}\mathbf{s}/c)| = \varepsilon_{ij}s_i s_j \tilde{n}^4 - [(\varepsilon_{ij}s_i s_j)\varepsilon_{ll} - s_i s_j \varepsilon_{il}\varepsilon_{lj}]\tilde{n}^2 + |\varepsilon_{ij}| = 0, \quad (1)$$

where $|a_{ij}|$ is the determinant of the matrix a_{ij}.

In an isotropic medium or in cubic crystals without spatial dispersion $\epsilon_{ij}(\omega, \mathbf{k}) = \epsilon(\omega)\delta_{ij}$ and $\tilde{n}^2 = \epsilon(\omega) \equiv \epsilon'(\omega) + i\epsilon''(\omega)$, i.e., $n^2 - \kappa^2 = \epsilon'$ and $2n\kappa = \epsilon''$. Therefore the dispersion relations connecting ϵ' and ϵ'' can be automatically expressed in terms of n and κ (for simplicity we assume that the value $\epsilon''(0)$ is finite):

$$\varepsilon'(\omega) - 1 = n^2(\omega) - \varkappa^2(\omega) - 1 = \frac{2}{\pi}\fint_0^\infty \frac{x\varepsilon''(x)}{x^2 - \omega^2}dx = \frac{4}{\pi}\fint_0^\infty \frac{xn(x)\varkappa(x)}{x^2-\omega^2}dx,$$

$$\varepsilon''(\omega) = 2n(\omega)\varkappa(\omega) = -\frac{2\omega}{\pi}\fint_0^\infty \frac{\varepsilon'(x) - 1}{x^2 - \omega^2}dx = -\frac{2\omega}{\pi}\fint_0^\infty \frac{n^2(x) - \varkappa^2(x) - 1}{x^2 - \omega^2}dx, \quad (2)$$

where by the symbol $\fint$ we understand the principal value of the integral.[1)] However even in the absence of spatial dispersion but for an optically anisotropic medium, for example a noncubic crystal, Eq. (1) leads to two different values $\tilde{n}_1^2(\omega, \mathbf{s})$ and $\tilde{n}_2^2(\omega, \mathbf{s})$ and, most important, the dispersion relations for $\epsilon_{ij}(\omega)$ are not directly expressible in terms of $\tilde{n}_{1,2}^2$. In the presence of spatial dispersion the question of how to write the dispersion relations for $\tilde{n}^2$ arises even in the simplest case of propagation of transverse waves through an iso-

1)Dispersion relations of type (2) can, of course, be written for arbitrary functions of $\epsilon(\omega)$ that have no singularities in the upper halfplane and on the real axis of the variable ω. Such relations [for example, for $\epsilon^2(\omega)$] give nothing new in comparison with the relations (2). For a medium in thermodynamic equilibrium $\epsilon(\omega)$ has no zeros in the upper halfplane and on the real axis of ω. In that case one may write dispersion relations also for the function $\tilde{n}(\omega) = \sqrt{\epsilon(\omega)}$.

tropic medium, when $\tilde{n}_\perp^2 = \epsilon_{tr}(\omega, \omega\tilde{n}_\perp(\omega)/c)$ and several solutions for $\tilde{n}_\perp^2(\omega)$ may exist.

The use of dispersion relations is by far not as important in electrodynamics or acoustics [4] as in elementary particle theory. Nevertheless the question of dispersion relations for $\tilde{n} = n + i\kappa$ is of some interest (in part this is already clear from the fact of appearance of the papers by Brodin et al. [5] and Davydov [6]). It is therefore the hope of the authors that the publication of the present paper is not superfluous.

1. We consider a medium described by the tensor [3,7]

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \varepsilon'_{ij}(\omega, \mathbf{k}) + i\varepsilon''_{ij}(\omega, \mathbf{k})$$
$$= \mathrm{Re}\, \varepsilon_{ij}(\omega, \mathbf{k}) + i\, \mathrm{Im}\, \varepsilon_{ij}(\omega, \mathbf{k})$$
$$= \int_0^\infty d\tau \int d\mathbf{R} e^{-i(\mathbf{kR}-\omega\tau)} \hat{\varepsilon}_{ij}(\tau, \mathbf{R}), \tag{3}$$

where, in view of the principle of causality, the integration over τ runs from only 0 to ∞ (here ϵ'_{ij} and ϵ''_{ij} are hermitian tensors which are useful along with $\mathrm{Re}\, \epsilon_{ij}$ and $\mathrm{Im}\, \epsilon_{ij}$ when the former are not real).

If, as follows from physical considerations, the kernel $\hat{\epsilon}_{ij}(\tau, \mathbf{R})$ has no nonintegrable singularities[2)] and sufficiently rapidly, or at least without increasing, tends to zero as $\tau \to \infty$, then the functions $\epsilon_{ij}(\omega, \mathbf{k})$ are always finite in the upper half plane and on the real axis of the complex variable ω. In the case of conductors $\hat{\epsilon}_{ij}(\tau, \mathbf{R})$ tends to a finite value as $\tau \to \infty$ and therefore $\epsilon_{ij}(\omega, \mathbf{k})$ has a simple pole at $\omega = 0$ (see [2,4]). In the case of superconductors [8] there already appears a pole $\sim\omega^{-2}$; and if the medium is not only not in equilibrium but is also unstable then $\hat{\epsilon}(\tau, \mathbf{R})$ will increase with increasing τ — such a situation will not be here considered (see [9]).

If the functions $\epsilon_{ij}(\omega, \mathbf{k})$ are finite then one obtains in the usual way [1-4,7] the dispersion relations

$$\mathrm{Re}\, \varepsilon_{ij}(\omega, \mathbf{k}) - \delta_{ij} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\mathrm{Im}\, \varepsilon_{ij}(x, \mathbf{k})}{x - \omega} dx,$$

$$\mathrm{Im}\, \varepsilon_{ij}(\omega, \mathbf{k}) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\mathrm{Re}\, \varepsilon_{ij}(x, \mathbf{k}) - \delta_{ij}}{x - \omega} dx, \tag{4}$$

where it is assumed[3)] that $\epsilon_{ij}(\infty, \mathbf{k}) = \delta_{ij}$ and, as already mentioned, the pole at $\omega = 0$ is presumed absent.

It follows from their meaning that ω and x take on only real values in (4), with the wave vector $\mathbf{k}$ entering as a parameter. Since in introducing the tensor $\epsilon_{ij}(\omega, \mathbf{k})$ a Fourier transform is performed, the vector $\mathbf{k}$ in (3) appears directly only as a real vector. However in those cases when the kernel $\hat{\epsilon}_{ij}(\tau, \mathbf{R})$ decreases sufficiently rapidly, say exponentially, as $R \to \infty$, one may analytically continue the function $\epsilon_{ij}(\omega, \mathbf{k})$ into a certain region of complex values of $\mathbf{k}$ (if, for example, $\hat{\epsilon}_{ij}(\tau, \mathbf{R}) = f_{ij}(\tau) \exp(-R^2/a^2)$ then $\epsilon_{ij}(\omega, \mathbf{k})$ is analytic for arbitrary complex $\mathbf{k}$). In the usually encountered situation in optics, when the frequency ω is real, the vector $\mathbf{k} = \omega\tilde{n}(\omega, \mathbf{s})\mathbf{s}/c$ is always complex, although in the transparency region the value $\kappa = \mathrm{Im}\, \tilde{n}$ may be quite small.

The dispersion relations (4) were obtained without taking into account the fact that the speed of a signal may not exceed the speed of light in vacuum c. Nevertheless we shall not make use of the appropriate more general dispersion relations [10,7] for the following reasons. In the first place the relations (4) not only do not contradict the more general ones, but are in fact a special case of them. In the second place, we shall be interested below in dispersion relations for $\tilde{n}$ when in addition to the principle of causality use is made of relativistically invariant field equations. Under such circumstances the restrictions imposed on the speed of signals are, apparently, automatically taken into account (in this connection see [11]).

As a result of the symmetry principle of the kinetic coefficients one has

$$\varepsilon_{ij}(\omega, \mathbf{k}, \mathbf{B}_0) = \varepsilon_{ji}(\omega, -\mathbf{k}, -\mathbf{B}_0), \tag{5a}$$

where $\mathbf{B}_0$ — constant magnetic induction. The relation (5a) is proved for real ω and $\mathbf{k}$ (see [7]), but it remains valid within the region of analyticity also for complex ω and $\mathbf{k}$. Since $D_i = \epsilon_{ij}E_j$ is real we have for real $\mathbf{E}$

$$\varepsilon_{ij}(\omega, \mathbf{k}, \mathbf{B}_0) = \varepsilon^*_{ij}(-\omega^*, -\mathbf{k}^*, \mathbf{B}_0). \tag{5b}$$

If wave propagation is considered and $\mathbf{k} = \omega\tilde{n}(\omega)\mathbf{s}/c$, then it follows from the same reality requirement that $k(-\omega^*) = -k^*(\omega)$, i.e.,

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2) In vacuum $\epsilon_{ij} = \delta_{ij}$ and, consequently, $\hat{\epsilon}_{ij}$ contains the term $\delta_{ij}\,\delta(\tau)\,\delta(\mathbf{R})$. If then one writes

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \delta_{ij} + 4\pi \int_0^\infty d\tau \int d\mathbf{R} e^{-i(\mathbf{kR}-\omega\tau)} \hat{\chi}_{ij}(\tau, \mathbf{R}),$$

then the function $\chi_{ij}(\tau, \mathbf{R})$ has no singularities and tends to zero as $R \to \infty$.

3) The following is known from the theory of analytic functions. If $\epsilon_{ij}(\omega, \mathbf{k}) \to \delta_{ij}$ for $\omega = \omega' + i\omega'' = \omega' \to \pm\infty$, then for $\omega \to \infty$, $\omega'' > 0$ (i.e. in the upper half-plane) $\epsilon_{ij}(\omega, \mathbf{k})$ either also tends to δ_{ij} or grows faster than a certain exponent. This result strengthens the assertion, clear from physical considerations, that $\epsilon_{ij}(\omega, \mathbf{k}) \to \delta_{ij}$ for $\omega \to \infty$, $\omega'' > 0$, and not only for $\omega = \omega' \to \infty$.

$$\varepsilon_{ij}(\omega, \mathbf{k}(\omega), \mathbf{B}_0) = \varepsilon^*_{ij}(-\omega^*, \mathbf{k}(-\omega^*), \mathbf{B}_0)$$
$$= \varepsilon^*_{ij}(-\omega^*, -\omega^*\tilde{n}^*(\omega)/c, \mathbf{B}_0),$$

and

$$\tilde{n}(\omega) = \tilde{n}^*(-\omega^*). \tag{5c}$$

For $\mathbf{B}_0 = 0$ and in the absence of spatial dispersion

$$\varepsilon_{ij}(\omega) = \varepsilon_{ji}(\omega), \quad \varepsilon'_{ij} = \mathrm{Re}\,\varepsilon_{ij}, \quad \varepsilon''_{ij} = \mathrm{Im}\,\varepsilon_{ij},$$
$$\varepsilon_{ij}(\omega) = \varepsilon^*_{ij}(-\omega). \tag{5d}$$

In the last case the relations (4) can be written in the form

$$\varepsilon'_{ij}(\omega) - \delta_{ij} = \frac{2}{\pi}\int_0^\infty \frac{x\varepsilon''_{ij}}{x^2-\omega^2}dx,$$
$$\varepsilon''_{ij}(\omega) = -\frac{2\omega}{\pi}\int_0^\infty \frac{\varepsilon'_{ij}(x) - \delta_{ij}}{x^2-\omega^2}dx. \tag{6}$$

For an isotropic medium formulae (6) go over into (2). In addition for an equilibrium isotropic medium it is known that the function $\epsilon(\omega)$ has no zeroes in the upper half-plane of the variable $\omega = \omega' + i\omega''$, and possesses also certain other interesting properties. In deriving these properties[2] one makes use, among other things, of the fact that $\mathrm{Im}\,\epsilon(\omega) = \epsilon''(\omega) > 0$ for $\omega = \omega' > 0$ (this follows from the law of increasing entropy, if one is dealing with a medium in a state of thermodynamic equilibrium or, at least, a medium that is absorptive for all $\omega = \omega' > 0$).

Analogous assertions cannot be made for all components $\epsilon_{ij}(\omega, \mathbf{k}, \mathbf{B}_0)$ since for real ω and $\mathbf{k}$ the heat released into a unit volume depends on a combination of ϵ_{ij} — it is given by [7,12]

$$q = (i\omega/16\pi)\{\varepsilon^*_{ij}(\omega, \mathbf{k}, \mathbf{B}_0) - \varepsilon_{ji}(\omega, \mathbf{k}, \mathbf{B}_0)\}E_{0i}E^*_{0j}$$
$$= (\omega/8\pi)\,\varepsilon''_{ij}(\omega, \mathbf{k}, \mathbf{B}_0)E_{0j}E^*_{0i}. \tag{7}$$

Here $\mathbf{E}_0$ is the amplitude of the electric field, $\mathbf{E} = \mathbf{E}_0 \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$ and, we remind the reader once more, $\epsilon_{ij} = \epsilon'_{ij} + i\epsilon''_{ij}$, where ϵ'_{ij} and ϵ''_{ij} are hermitian tensors.

For an equilibrium medium $q > 0$ and consequently

$$\varepsilon''_{ij}(\omega, \mathbf{k}, \mathbf{B}_0)s_is_j > 0, \quad \omega = \omega' > 0, \tag{8}$$

where we choose for the arbitrary vector $\mathbf{E}_0$ the unit vector $\mathbf{s}$ (we are considering uniform plane waves, when $\mathbf{k} = k\mathbf{s}$, $s = 1$; it should be kept in mind that the vector $\mathbf{E}_0$ may be considered to be arbitrary, since the presence of arbitrary "external" charges and currents is allowed). In view of the hermitian nature of ϵ'_{ij} and ϵ''_{ij} the condition (8) may also be written in the form $\mathrm{Im}\,U > 0$, $\omega = \omega' > 0$, where

$$U = \varepsilon_{ij}(\omega, \mathbf{k}, \mathbf{B}_0)s_is_j. \tag{9}$$

It is easy to see that $\mathrm{Im}\,U(-\omega, \mathbf{k}, \mathbf{B}_0) < 0$ for $\omega = \omega' < 0$. Indeed, in view of (5b) and the hermitian nature of ϵ''_{ij} we have

$$\mathrm{Im}\,U(-\omega^*, \mathbf{k}, \mathbf{B}_0) = \varepsilon''_{ij}(-\omega^*, \mathbf{k}, \mathbf{B}_0)s_is_j$$
$$= -\varepsilon''^*_{ij}(\omega, -\mathbf{k}^*, \mathbf{B}_0)s_is_j = -\varepsilon''_{ij}(\omega, -\mathbf{k}^*, \mathbf{B}_0)s_is_j.$$

As already stated, relation (8) is valid for arbitrary real ω and $\mathbf{k}$, i.e.,

$$\mathrm{Im}\,U(\omega, \mathbf{k}, \mathbf{B}_0) = \varepsilon''_{ij}(\omega, \mathbf{k}, \mathbf{B}_0)s_is_j > 0.$$

Consequently, under similar conditions

$$\mathrm{Im}\,U(-\omega, \mathbf{k}, \mathbf{B}_0) = -\varepsilon''_{ij}(\omega, -\mathbf{k}, \mathbf{B}_0)s_is_j < 0.$$

If spatial dispersion is absent, or present but $\mathbf{B}_0 = 0$ (or ϵ_{ij} does not depend on the sign of $\mathbf{B}_0$), then we obtain [making use also of (5a)]

$$\mathrm{Im}\,U(-\omega, \mathbf{k}, \mathbf{B}_0) = -\mathrm{Im}\,U(\omega, \mathbf{k}, \mathbf{B}_0).$$

Moreover we always have $U \to 1$ as $\omega \to \infty$, since in that limit $\epsilon_{ij} \to \delta_{ij}$.

The enumerated properties of the function $U(\omega)$ coincide with the properties of the function $\epsilon(\omega)$ for an isotropic equilibrium medium. One may therefore assert (a proof due to one of the authors is given in [2]) that under the indicated conditions the function $U(\omega)$ does not take on real values at any point in the upper half-plane, with the exception of points on the imaginary axis[4]. Along that axis $U(\omega)$ decreases monotonically from the value $U > 1$ (dielectric) or $U = \infty$ (metals) at $\omega = i0$ to the value $U = 1$ at $\omega = i\infty$. From this it follows, in particular, that the function U has no zeros in the upper half-plane; as we have seen before $U \neq 0$ on the real axis also (for dielectrics the function $\epsilon''_{ij}(\omega) = 0$ at $\omega = 0$ but in that case it follows from thermodynamics (see [2], Sec. 14) that $U = \epsilon'_{ij}(\omega = 0)s_is_j > 1$).

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[4] Here, of course, we make use of the condition $\mathrm{Im}\,U(-\omega) = -\mathrm{Im}\,U(\omega)$, $\omega = \omega'$. If however $\mathrm{Im}\,U(\omega)$ is not symmetric on the real axis and only changes sign for $\omega = 0$ (this is possible if spatial dispersion is significant and at the same time $B_0 \neq 0$), then the proof remains valid with this difference that as a result of violation of symmetry the real values occur not along the imaginary axis but along some other unique curve in the upper ω halfplane. This curve starts out from the point $\omega = 0$ and goes off to infinity. Along the indicated curve $U(\omega)$ decreases monotonically from the value $U(i0) = U(0) > 1$ (dielectrics) or $U(+i0) = \infty$ (metals) to the value $U(\omega \to \infty) = 1$. Since zero is not included in the range of variation of U, $U(\omega) \neq 0$ for $\omega'' \geq 0$.

From the inequality $U \neq 0$ there follows (in the same region of the ω plane) the inequality[5)]

$$|\varepsilon_{ij}(\omega, \mathbf{k}, \mathbf{B}_0)| \neq 0, \quad \omega = \omega' + i\omega'', \quad \omega'' \geqslant 0. \quad (10)$$

From (10) and (1) it follows that $\tilde{n}^2(\omega, \mathbf{s}) \neq 0$, $\omega'' \geq 0$; in accordance with the basic properties of ϵ_{ij} it also follows from (10) that the inverse tensor $\epsilon_{ij}^{-1}(\omega, \mathbf{k}, \mathbf{B}_0)$ exists, and $|\epsilon_{ij}^{-1}(\omega, \mathbf{k}, \mathbf{B}_0)| \neq 0$.

The proof given here may, apparently, fail for complex values of $\mathbf{k}$ since in that case the expression (7) does not represent the released heat.[12] But in the absence of spatial dispersion in expression (7) the vector $\mathbf{k}$ may, of course, be complex and q is the heat. It is thus clear that also in the presence of spatial dispersion the indicated properties of the functions U, $|\epsilon_{ij}|$ and $|\epsilon_{ij}^{-1}|$ may persist in some region of complex values of $\mathbf{k}$.

The question of utilization of the principle of increasing entropy (i.e., the condition $q > 0$, if q is the heat) is not sufficiently clear in the general case and deserves a special study. The same is also true in general with regard to energetic considerations when spatial dispersion is taken into account and $\mathbf{k}$ is at the same time allowed to be complex (or ω is complex with real or complex $\mathbf{k}$).

Let us note that the conditions

$$U = \varepsilon_{ij}(\omega, \mathbf{k}, \mathbf{B}_0) s_i s_j = 0,$$
$$|\varepsilon_{ij}(\omega, \mathbf{k}, \mathbf{B}_0)| = 0, \quad |\varepsilon_{ij}^{-1}(\omega, \mathbf{k}, \mathbf{B}_0)| = 0 \quad (11)$$

correspond respectively to the condition for the appearance of "fictitious" longitudinal waves, longitudinal waves, and "polarization waves" satisfying the equations of the Coulomb problem (see [3]). Therefore the above results—the impossibility of satisfying conditions (11) for $\omega'' = \mathrm{Im}\,\omega \geq 0$ —have a clear physical meaning. Namely, in a state of thermodynamic equilibrium (or, more precisely, for $q > 0$, which is a somewhat more general condition) the frequencies of the normal waves, corresponding to the Coulomb problem for real $\mathbf{k}$, should be of the form $\omega_i = \omega_i' + i\omega_i''$, $\omega_i'' < 0$, which corresponds to damping. Looking at things this way we could have written down the appropriate inequalities right away, however, as we have seen, making use of the theory of analytic functions allows us to obtain certain other results as well.

[5)]As is clear from the derivation [see in particular expression (7)] the indicated properties of the function $U(\omega)$ also persist if s is replaced by a complex vector a (in which case, of course, $s_i s_j$ must be replaced by $a_i a_j^*$). If $|\epsilon_{ij}| = 0$, then the system of equations $\epsilon_{ij} a_j = 0$ always has a solution $a_j \neq 0$ for which $U = 0$. It is for this reason that the inequality $U \neq 0$ leads to the condition $|\epsilon_{ij}| \neq 0$.

Let us consider now the quadratic form

$$\{\varepsilon_{ij}(\omega, \mathbf{k}, \mathbf{B}_0) - \tilde{n}^2(\delta_{ij} - s_i s_j)\} a_i a_j^*. \quad (12)$$

Equation (1) is the condition for the vanishing of the determinant of this quadratic form. Consequently if $\tilde{n}^2$ satisfies Eq. (1) then the form (12) for the corresponding vector a_i vanishes. But it follows from what has been said above that in an equilibrium medium and for real $\mathbf{k}$ the function $U = \epsilon_{ij}(\omega, \mathbf{k}, \mathbf{B}_0) a_i a_j^* \neq 0$ for $\omega'' \geq 0$. Moreover, $(\delta_{ij} - s_i s_j) a_i a_j^* \geq 0$, and therefore the root $\tilde{n}_l^2(\omega, \mathbf{s})$ of equation (1) cannot vanish for $\omega'' \geq 0$ in the indicated cases (equilibrium medium, $\mathbf{k}$ real).

This circumstance has already been noted above, but from the properties of the function $U(\omega)$ and the form (12) it is also clear that for $\omega'' > 0$ $\tilde{n}^2(\omega, \mathbf{s})$ takes on positive real values only on a certain line running from the point i0 to infinity (for $\mathbf{B}_0 = 0$ or in the absence of spatial dispersion this line is the imaginary half-axis).

Further,

$$\mathrm{Im}\,\tilde{n}^2(\omega, \mathbf{s}) = 2n\varkappa > 0, \quad \omega = \omega' > 0;$$
$$\mathrm{Im}\,\tilde{n}^2(\omega, \mathbf{s}) < 0, \quad \omega = \omega' < 0. \quad (13)$$

Thus $n(\omega, \mathbf{s})$ and $\kappa(\omega, \mathbf{s})$ have the same sign for $\omega = \omega' > 0$ $\kappa \neq 0$ (the wave is damped in an equilibrium medium).

Since the vector $\mathbf{k}$ turns out to be complex, the inequalities (13) themselves are only proven in the absence of spatial dispersion, when $U = \epsilon_{ij}(\omega, \mathbf{B}_0) \times s_i s_j$. In the presence of spatial dispersion the inequalities (13), mentioned earlier,[11] may be violated (this is connected with the circumstance that in that case the vector $\mathbf{k}$ and the vector of the group velocity $v_{gr} = \partial\omega/\partial\mathbf{k}$ may form an obtuse angle (see [11])).

Let us suppose that $(\delta_{ij} - s_i s_j) a_i a_j^* > 0$. It is then clear from (12) that the root $\tilde{n}^2(\omega, \mathbf{s})$ is always finite for $\omega'' \geq 0$, because the functions $\epsilon_{ij}(\omega, \mathbf{k}, \mathbf{B}_0)$ have no singularities in that region of ω. The case $(\delta_{ij} - s_i s_j) a_i a_j^* = 0$ corresponds to strictly longitudinal waves for which the electric field is $\mathbf{E} = E\mathbf{s}$. At that either the induction $D = 0$, which is possible only if $|\epsilon_{ij}| = 0$, or $\tilde{n}^2 \to 0$ and $\epsilon_{ij} s_i s_j = 0$ (see [3]). Since these requirements are not fulfilled under the conditions being discussed, it can be said that under these conditions $\tilde{n}^2$ is always finite for $\omega'' \geq 0$. This can also be seen from Eq. (1) since $\epsilon_{ij} s_i s_j$ is the coefficient of $\tilde{n}^4$.

2. The possibility of writing dispersion relations for the roots $n_l^2 = (n_l + i\kappa_l)^2$ of Eq. (1) is obviously related to the analytic properties of the functions $\tilde{n}^2(\omega, \mathbf{s})$ for $\omega'' \geq 0$. If in that region a

given root has no singular points (in particular, branch points) then we have, making use of Eq. (5c),

$$n_l^2(\omega) - \varkappa_l^2(\omega) - 1 = \frac{4}{\pi}\int_0^\infty \frac{x n_l(x)\varkappa_l(x)}{x^2-\omega^2}dx,$$

$$2n_l(\omega)\varkappa_l(\omega) = -\frac{2\omega}{\pi}\int_0^\infty \frac{n_l^2(x)-\varkappa_l^2(x)-1}{x^2-\omega^2}dx. \quad (14)$$

For brevity we have left out here the argument s in $\tilde{n}_l(\omega, s)$; we have also assumed that $\tilde{n}_l(\infty) = 1$ and $n_l(0)\kappa_l(0) = 0$, however the result can be generalized without difficulty to the case when $\tilde{n}_l^2(\infty) = n_\infty^2 < \infty$ and $n_l(\omega)\kappa_l(\omega) = A/\omega$, $A < \infty$ (for $\omega \to 0$).[6] Under the conditions when $\tilde{n}_l^2(\omega) \neq 0$ for $\omega'' > 0$ (see above) one may just as well write dispersion relations not for $\tilde{n}^2$ but for $\tilde{n}$.

We consider the case when spatial dispersion is absent, or, more precisely, when it can be neglected so that one may set $\epsilon_{ij} = \epsilon_{ij}(\omega, \mathbf{B}_0)$. In that case Eq. (1) is quadratic with respect to $\tilde{n}^2$. Since $\epsilon_{ij}(\infty) = \delta_{ij}$ it is clear that $\tilde{n}_l^2(\infty) = 1$. Further, as shown above, for an equilibrium medium the roots $\tilde{n}^2(\omega)$ become nowhere infinite—neither in the upper half-plane nor on the real axis of the variable ω. It thus remains to clarify whether the functions $\tilde{n}^2(\omega)$ might not have branch points corresponding to multiple roots. For that purpose it is most convenient to write Eq. (1) in terms of the inverse tensor $\epsilon_{ij}^{-1} = \eta_{ij}$. In view of the condition $s_i D_i = 0$ (here D is the induction; see [3,7]) in a coordinate system in which the z axis is chosen along the direction of s Eq. (1) has the form ($\tilde{m}^2 = 1/\tilde{n}^2$; below i, j = 1, 2 and we are dealing with a two-dimensional tensor η_{ij})

$$\begin{vmatrix} \eta_{11}-\tilde{m}^2 & \eta_{12} \\ \eta_{21} & \eta_{22}-\tilde{m}^2 \end{vmatrix} = \tilde{m}^4 - (\eta_{11}+\eta_{22})\tilde{m}^2 + \eta_{11}\eta_{22} - \eta_{12}\eta_{21} = 0. \quad (15)$$

Consequently the condition for the existence of a multiple root is

$$(\eta_{11}-\eta_{22})^2 + 4\eta_{12}\eta_{21} = 0. \quad (16)$$

If the matrix η_{ij} (for i, j = 1, 2) is put into diagonal form then a multiple root is possible only in the case of degeneracy (i.e., in the case of coincidence of the eigenvalues $\eta_1 = \eta_2$). The wave equation has in that case two independent solutions of the type $\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$ (two normal waves), that differ in polarization. Such a situation occurs, for example, in an isotropic medium. If instead the matrix η_{ij} does not reduce to diagonal form then a significant multiple root may appear (below, when speaking of multiple roots, we have in mind precisely this case when the matrix η_{ij} has just one eigenvector and, therefore, it is not sufficient to consider only solutions of the type $\mathbf{E} = \mathbf{E}_0 \times \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$; along with these solutions one also has solutions of the type $\mathbf{E} = \mathbf{E}_0(\mathbf{s}, \mathbf{r}) \times \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$; see [3,14,15]).

Let us write η_{ij} in the form $\eta_{ij} = \eta'_{ij} + i\eta''_{ij}$, where η'_{ij} and η''_{ij} are hermitian tensors. It is clear that a significant multiple root can occur only in the case when the principal axes of the tensors η'_{ij} and η''_{ij} do not coincide. For crystals for $\mathbf{B}_0 = 0$ such a possibility exists only in the case of triclinic, monoclinic and rhombic symmetries, because for higher symmetries the principal axes of the tensors η'_{ij} and η''_{ij} coincide (or else their choice does not matter because of degeneracy). In addition, in the absence of optical activity (for $\mathbf{B}_0 = 0$) $\epsilon_{ij}(\omega) = \epsilon_{ji}(\omega)$ and, consequently, the tensors ϵ'_{ij} and ϵ''_{ij} are real (the same is true of η'_{ij} and η''_{ij}). One may therefore choose the coordinate axes in such a way that the tensor η'_{ij} is diagonal ($\eta'_{11} = \eta'_1$, $\eta'_{22} = \eta'_2$). We then obtain from (16)

$$(\eta'_1 - \eta'_2)^2 = 4(\eta''_{12})^2, \qquad \eta''_{22} - \eta''_{11} = 0. \quad (17)$$

A similar possibility, as is known,[14,15] can occur (the case of singular optical axes) for a real frequency ω and also, apparently, for a frequency in the upper halfplane.

If the medium is optically active as a result of the presence of a magnetic field $\mathbf{B}_0$ (magneto-active medium), $\epsilon_{ij}(\omega, \mathbf{B}_0) = \epsilon_{ji}(\omega, -\mathbf{B}_0)$ and for a given B_0 the hermitian tensors ϵ'_{ij} and ϵ''_{ij} should not be real. The most interesting and important example of such a medium is that of a plasma in a magnetic field. In that case, in the so-called elementary theory approximation, the tensor $\epsilon_{ij}(\omega)$ has the form (the z axis is taken along the direction of the external magnetic field $\mathbf{H}_0 \equiv \mathbf{B}_0$; see [16])

$$\varepsilon_{xx} \pm i\varepsilon_{xy} = \varepsilon_{yy} \mp i\varepsilon_{yx} = 1 - \frac{\omega_0^2}{\omega^2 \mp \omega\omega_H + i\nu\omega},$$

$$\varepsilon_{zz} = 1 - \frac{\omega_0^2}{\omega^2 + i\nu\omega}, \quad \omega_H = \frac{|e|H_0}{mc}, \quad \omega_0^2 = \frac{4\pi e^2 N}{m}. \quad (18)$$

[6]The problem of dispersion relations for the velocities of propagation of normal acoustic waves $\tilde{u}$ is posed in the same way as the problem of dispersion relations for $\tilde{n}^2$. However previously[4] only the case of an isotropic medium has been considered, so that right away dispersion relations were discussed for $\tilde{u}^2$, and not for the components of the elastic modulus tensor λ_{ijlm}. But in the anisotropic case one must start precisely with dispersion relations for λ_{ijlm}. Spatial dispersion plays a smaller part in acoustics[13] than in electrodynamics, and in this respect the acoustic case is simpler. But the necessity of separately discussing the high frequency region makes the acoustic problem on the whole more complicated.

The solution of Eq. (1) with such a tensor ϵ_{ij} gives rise to two roots:

$$\tilde{n}_{1,2}^2 = 1 - \frac{2v(1+is-v)}{2(1+is)(1+is-v) - u\sin^2\alpha \pm \sqrt{u^2\sin^4\alpha + 4u(1+is-v)^2\cos^2\alpha}}; \quad (19)$$

$$v = \omega_0^2/\omega^2, \quad u = \omega_H^2/\omega^2, \quad s = \nu/\omega;$$

where ν is the effective number of collisions and α is the angle between $\mathbf{s}$ and $\mathbf{H}_0$. It is clear that we have a double root if $u\sin^4\alpha + 4(1+is-v)^2 \times \cos^2\alpha = 0$, i.e., for

$$\omega_{k,1,2} = -i(\nu - \nu_k)/2 \pm \sqrt{\omega_0^2 - (\nu-\nu_k)^2/4},$$

$$\omega_{k,3,4} = -i(\nu + \nu_k)/2 \pm \sqrt{\omega_0^2 - (\nu+\nu_k)^2/4},$$
$$\nu_k = \omega_H \sin^2\alpha / 2|\cos\alpha|. \quad (20)$$

For $\nu = \nu_k$ the multiple roots lie at the points $\omega_{k,1,2} = \pm\omega_0$ and $\omega_{k,3,4} = -i\nu \pm (\omega_0^2 - \nu_k^2)^{1/2}$. If instead $\nu > \nu_k$ then all the points ω_k lie in the lower half-plane. Therefore for $\nu \geq \nu_k$ the dispersion relations (14) should be valid for both roots $\tilde{n}_1^2$ and $\tilde{n}_2^2$ [(see 20); the appearance of a branch point on the real axis, which occurs for $\nu = \nu_k$, changes nothing].

On the other hand when $\nu < \nu_k$ the points $\omega_{k,1,2}$ lie in the upper half-plane.[7)] Under such conditions the dispersion relations in the form (14) are, generally speaking, not valid. One can, of course, write analogous relations if one integrates along a contour that excludes the points ω_k in the upper half-plane. For a known function $\epsilon_{ij}(\omega)$ or $\tilde{n}^2(\omega)$ this can be done, but in that case the dispersion relations themselves are not needed. On the other hand we see no possibility of estimating from some sort of general considerations what the contribution might be of the integral along the contour L enclosing the roots ω_k (see, for example, the figure which depicts the case when in (20) $\nu < \nu_k$, $\omega_0 > (\nu - \nu_k)/2$). And to perform a measurement of $\epsilon_{ij}(\omega, \mathbf{B}_0)$ or $\tilde{n}^2(\omega)$ for complex frequencies $\omega = \omega' + i\omega''$, $\omega'' > 0$, although possible in principle [it is necessary to utilize a field $E = E_0\exp(\omega'' t) \times \exp(-i\omega' t)$ or, more precisely, a pulse whose form in a certain interval is close to the indicated one], is hardly likely in the interesting cases.

In the presence of spatial dispersion Eq. (1) may have all kinds of roots $\tilde{n}_l$ with all sorts of singularities; this depends on the character of the dependence of the function $\epsilon_{ij}(\omega, \omega\tilde{n}(\omega, \mathbf{s})\mathbf{s}/c)$ on the argument $\tilde{n}(\omega, \mathbf{s})$. For this reason the relations (14) are in the presence of spatial dispersion generally speaking, not valid. They may however be fully valid if the circumstances happen to be such that the root $\tilde{n}_l^2(\omega, \mathbf{s})$ under consideration has no singularities in the upper half of the ω plane. This is the case, in particular, when the spatial dispersion is weak and one is dealing with the "ordinary" roots $\tilde{n}^2$, which are present also when spatial dispersion is ignored.

Let us consider, for example, an isotropic nongyrotropic medium, for which

$$\varepsilon_{ij}(\omega, k\mathbf{s}) = \varepsilon_{tr}(\omega, k)(\delta_{ij} - s_i s_j) + \varepsilon_l(\omega, k) s_i s_j,$$
$$n_\perp^2 = \varepsilon_{tr}(\omega, \omega\tilde{n}_\perp(\omega)/c), \quad \varepsilon_l(\omega, \omega\tilde{n}_\parallel(\omega)/c) = 0, \quad (21)$$

where $\tilde{n}_\perp$ and $\tilde{n}_\parallel$ are the indices of refraction for respectively transverse and longitudinal waves. When spatial dispersion is neglected $\epsilon_{tr}(\omega, 0) = \epsilon_l(\omega, 0) = \epsilon(\omega)$. Let us suppose now that at all frequencies one may set $\epsilon_{tr}(\omega, k) = \epsilon(\omega) + a(\omega)\tilde{n}_\perp^2(\omega)$, $a < 1$, and, consequently, $\tilde{n}_\perp^2(\omega) = \epsilon(\omega)/(1 - a(\omega))$. It is obvious that in this case the dispersion relations (14) for $\tilde{n}_\perp^2$ are valid.[8)]

In the special case of the roots $\tilde{n}_\perp^2(\omega)$ for an isotropic medium (or also for an anisotropic medium provided that one is dealing with the propagation of transverse waves along certain symmetry axes) one may arrive at the relations (14) also in a somewhat different way. Namely in an isotropic medium [see (4) and (21); $\epsilon' = \operatorname{Re}\epsilon$, $\epsilon'' = \operatorname{Im}\epsilon$]

$$\varepsilon'_{tr}(\omega, k) - 1 = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\varepsilon''_{tr}(x, k)}{x - \omega}dx,$$

$$\varepsilon''_{tr}(\omega, k) = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\varepsilon'_{tr}(x, k) - 1}{x - \omega}dx. \quad (22)$$

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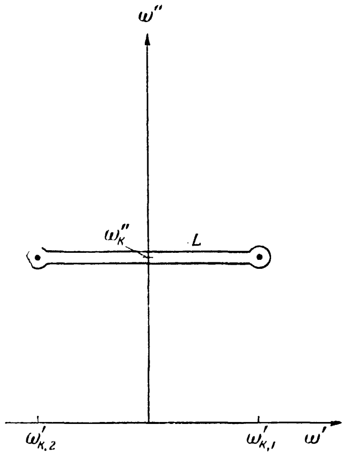
7)The location of the points $\omega_{k,1,2}$ plays a significant role in the study of the "triplication" effect of signals reflected by the ionosphere (see [16], sec. 28).

8)The conclusion on the validity of the dispersion relation (14) for the ordinary root $\tilde{n}^2$ in an isotropic medium with weak spatial dispersion taken into account has been previously reached by Davydov.[6] We note that Davydov is wrong when he remarks that absorption was not taken into account in the article of one of the authors.[17] In fact in [17], and then in more detail in [3], the substantial role of absorption in the consideration of effects of spatial dispersion near the line of exciton absorption is especially emphasized (see [17], p. 1598). Also there [17] the frequency region where absorption becomes relatively weak is estimated, and only graphs of the function $\tilde{n}^2$, of illustrative character, are given in the case when absorption is negligible.



These relations are, of course, valid in the special case when one is dealing with wave propagation, i.e., $k = \omega\tilde{n}_{\perp}(\omega)/c$.

Let us consider now

$$\int_{-\infty}^{+\infty} \frac{\varepsilon_{tr}(x, \omega\tilde{n}_{\perp}(\omega)/c) - \varepsilon_{tr}(x, x\tilde{n}_{\perp}(x)/c)}{x - \omega}\, dx.$$

If $\epsilon_{tr}(x, x\tilde{n}_{\perp}(x)/c)$ has no singularities for Im $x > 0$ (on the real axis ϵ_{tr} may have only integrable singularities) and

$$[\varepsilon_{tr}(x, \omega\tilde{n}_{\perp}(\omega)/c) - \varepsilon_{tr}(x, x\tilde{n}_{\perp}(x)/c)] \to 0 \quad \text{as} \quad x \to \infty,$$

then the indicated integral vanishes. We now take into account the fact that (for real ω) $\tilde{n}_{\perp}(\omega) = \tilde{n}_{l}^{*}(-\omega)$, which fact is clear from (5c) and was already used in deriving the relations (14). It is then easy to transform (22) to the form (14) with $\tilde{n}_{l}(\omega) = \tilde{n}_{\perp}(\omega)$. The possibility of violation of relations (14) for some ("extraordinary") root of the equation $\tilde{n}_{\perp}^{2} = \epsilon_{tr}(\omega, \omega\tilde{n}_{\perp}/c)$, as is clear from what has been said above, is due only to the fact that for the corresponding values of k the function $\epsilon_{tr}(\omega, k)$ either does not exist or has singularities as a function of k.

And so the center of gravity of the question of dispersion relations for $\tilde{n}_{l}^{2}(\omega, \mathbf{s})$ lies in the region of analysis of the concrete form of this function.

Violation of relations (14) for a given root $\tilde{n}_{l}^{2}(\omega, \mathbf{s})$ indicates the appearance for that root of singularities in the upper halfplane or on the real axis. For triclinic, monoclinic and rhombic crystals, and also for magneto-active plasmas, in a state of thermodynamic equilibrium, the singularity of $\tilde{n}^{2}(\omega, \mathbf{s})$ may be a multiple root with $\omega_{k}'' > 0$, which already appears in the absence of spatial dispersion. For crystals with higher symmetries, however, (and also for the indicated media under conditions when the multiple root in the absence of spatial dispersion plays no role or is absent) violation of relations (14) for an equilibrium medium may be due only to the effects of spatial dispersion. Since this dispersion in optics is weak,[3] it appears to be, generally speaking, possible to clarify the dependence of ϵ_{ij} on **k**, at least in that region of variation of the variables that corresponds to the ordinary roots $\tilde{n}_{l}^{2}$. Consequently the analysis of the question of dispersion relations and their utilization within certain limits is possible and of value also in the presence of spatial dispersion.

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