# *PROPAGATION OF ELECTROMAGNETIC WAVES IN A MEDIUM WITH STRONG DIELECTRIC-CONSTANT FLUCTUATIONS*

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Expressions for the averaged electromagnetic field Green's function in an infinite medium with strong dielectric-constant fluctuations are deduced for the cases $ka \ll 1$ and $ka \gg 1$, where $a$ is the fluctuation correlation radius of the medium. For $ka \ll 1$ the result reduces to a variation of the propagation constant and to the appearance of longitudinal waves. For $ka \gg 1$ additional weakening of the mean field does not reduce to a change in $k$. The mean square field fluctuation for $ka \ll 1$ is also computed; if multiple scattering is taken into account it is found to be smaller than the value obtained in a perturbation-theory first approximation.

## INTRODUCTION

THE problem of electromagnetic wave propagation in a medium with fluctuations in its macroscopic parameters occurs in many different areas (plasma studies, geophysics, etc.). In most cases these problems are investigated with the help of some variant of first-order perturbation theory. However this method cannot be used to treat the propagation of waves in a medium with strong fluctuations (for example, near the plasma frequency or near a critical point) nor can it be used in a medium with slightly fluctuating parameters in those cases where the field fluctuations turn out not to be small. Several papers have appeared recently[1-3] in which attempts have been made to go beyond the limits of perturbation theory in particular cases. These papers have made widespread use of the diagram techniques of field theory. The present paper builds on the results of a preceding paper[2], which has treated the scalar wave equation for the case $ka \ll 1$ ($a$ is the correlation radius for the fluctuations in the medium); the present paper extends these results to Maxwell's equations and also treats the case $ka \gg 1$ and gives a more consistent treatment of the field fluctuations. As in [2] we limit ourselves to the case of a monochromatic source proportional to $\exp(-i\omega t)$ and consider a medium for which $\mu = 1$, $\sigma = 0$, and $\epsilon(r)$ is a random function of the coordinates.

## 1. THE PERTURBATION SERIES AND THEIR SUMMATION

If the average value of the dielectric constant $\langle\epsilon\rangle = \text{const}$, and if $\epsilon(r) = [1 + \epsilon_1(r)]\langle\epsilon\rangle$ ($\langle\epsilon_1\rangle = 0$), then, after elimination of the magnetic field, Maxwell's equations can be written in the form

$$L_{ij}(\mathbf{r}) E_j(\mathbf{r}) = \{\delta_{ij}(\Delta + k^2) - \partial^2/\partial x_i \partial x_j\} E_j = -k^2\varepsilon_1 E_i + n_i\delta(\mathbf{r} - \mathbf{r}_0), \qquad (1)$$

where $k^2 = \langle\epsilon\rangle\omega^2 c^{-2}$, and $n_i$ is the unit polarization vector of the point source. The equation $L_{ij}E_j(r) = f_i(r)$, with the radiation conditions for free space, has the solution $E_j(r) = M_{ji}(r, r')f_i(r')$ or, in inverted form,

$$M_{ji}(\mathbf{r}, \mathbf{r}') f_i(\mathbf{r}') = \int G^0_{ji}(\mathbf{r}, \mathbf{r}') f_i(\mathbf{r}')\, d^3\mathbf{r}', \qquad (2)$$

where the kernel $G^0_{ij}$ of the operator $M_{ij}$ has the form

$$G^0_{ij}(\mathbf{r}, \mathbf{r}') = G^0_{ij}(\mathbf{R}) = \left(\delta_{ij} + \frac{1}{k^2}\frac{\partial^2}{\partial R_i \partial R_j}\right) G_0(R)$$
$$= \left[\delta_{ij}\left(1 + \frac{i}{kR} - \frac{1}{k^2R^2}\right) - \frac{R_iR_j}{R^2}\left(1 + \frac{3i}{kR} - \frac{3}{k^2R^2}\right)\right] G_0(R);$$
$$G_0(R) = -\exp(ikR)(4\pi R)^{-1}, \qquad \mathbf{R} = \mathbf{r} - \mathbf{r}'. \qquad (3)$$

In the Fourier representation

$$g^0_{ij}(\varkappa) = (2\pi)^{-3} \int G^0_{ij}(\mathbf{R})\, e^{-i\varkappa \mathbf{R}}\, d^3R = [8\pi^3(k^2 - \varkappa^2)]^{-1}\left(\delta_{ij} - \frac{\varkappa_i\varkappa_j}{k^2}\right). \tag{3a}$$

Applying the operator $M_{ki}$ to (1) and making use of the equation $M_{ki}L_{ij} = \delta_{kj}$, we obtain

$$E_k(\mathbf{r}) = -k^2 M_{ki}(\mathbf{r}, \mathbf{r}')\, \varepsilon_1(\mathbf{r}')\, E_i(\mathbf{r}') + n_i G^0_{ki}(\mathbf{r}, \mathbf{r}_0). \tag{4}$$

Solving the integral equation (4) by successive iterations, we obtain the perturbation series $E_k(\mathbf{r}) = G_{kj}(\mathbf{r}, \mathbf{r}_0)\, n_j$ where

$$G_{kj}(\mathbf{r}, \mathbf{r}_0) = G^0_{kj}(\mathbf{r}, \mathbf{r}_0) - k^2 M_{ki}(\mathbf{r}, \mathbf{r}')\, \varepsilon_1(\mathbf{r}')\, G^0_{ij}(\mathbf{r}', \mathbf{r}_0) + (-k^2)^2 M_{ki}(\mathbf{r}, \mathbf{r}')\, \varepsilon_1(\mathbf{r}')\, M_{il}(\mathbf{r}', \mathbf{r}'')\, \varepsilon_1(\mathbf{r}'') \times G^0_{lj}(\mathbf{r}'', \mathbf{r}_0) + \ldots \tag{5}$$

The function $G_{kj}(\mathbf{r}, \mathbf{r}_0)$ is random, since it contains $\epsilon_1$. In what follows we will be interested in the average value $\langle G_{kj}(\mathbf{r}, \mathbf{r}_0)\rangle \equiv \widetilde{G}_{kj}(\mathbf{r}, \mathbf{r}_0)$ and also in the mean square of this function. To average equation (5) one must know the moments of the function $\epsilon_1$:

$$\langle \varepsilon_1 \rangle = 0, \qquad \langle \varepsilon_1(\mathbf{r}_1)\, \varepsilon_1(\mathbf{r}_2)\rangle \equiv B_\varepsilon(\mathbf{r}_1, \mathbf{r}_2),$$

$$\langle \varepsilon_1(\mathbf{r}_1)\, \varepsilon_1(\mathbf{r}_2)\, \varepsilon_1(\mathbf{r}_3)\rangle$$

etc. We will consider the random field $\epsilon_1$ to be Gaussian. In this case $\langle \epsilon_1(\mathbf{r}_1) \ldots \epsilon_1(\mathbf{r}_{2n+1})\rangle = 0$, and the even moments can be expressed in terms of the second moment. For example

$$\langle \varepsilon_1(\mathbf{r}_1)\, \varepsilon_1(\mathbf{r}_2)\, \varepsilon_1(\mathbf{r}_3)\, \varepsilon_1(\mathbf{r}_4)\rangle = B_\varepsilon(\mathbf{r}_1, \mathbf{r}_2)\, B_\varepsilon(\mathbf{r}_3, \mathbf{r}_4) + B_\varepsilon(\mathbf{r}_1, \mathbf{r}_3)\, B_\varepsilon(\mathbf{r}_2, \mathbf{r}_4) + B_\varepsilon(\mathbf{r}_1, \mathbf{r}_4)\, B_\varepsilon(\mathbf{r}_2, \mathbf{r}_3). \tag{6}$$

Analogous formulas hold for the higher moments as well. These relations may be obtained easily with the help of the following rule, which we explain using (6) as an example:

$$\langle \varepsilon_1(\mathbf{r}_1)\, \varepsilon_1(\mathbf{r}_2)\, \varepsilon_1(\mathbf{r}_3)\, \varepsilon_1(\mathbf{r}_4)\rangle = \dot{\varepsilon}_1(\mathbf{r}_1)\, \dot{\varepsilon}_1(\mathbf{r}_2)\, \ddot{\varepsilon}_1(\mathbf{r}_3)\, \ddot{\varepsilon}_1(\mathbf{r}_4) + \dot{\varepsilon}_1(\mathbf{r}_1)\, \ddot{\varepsilon}_1(\mathbf{r}_2)\, \dot{\varepsilon}_1(\mathbf{r}_3)\, \ddot{\varepsilon}_1(\mathbf{r}_4) + \dot{\varepsilon}_1(\mathbf{r}_1)\, \ddot{\varepsilon}_1(\mathbf{r}_2)\, \ddot{\varepsilon}_1(\mathbf{r}_3)\, \dot{\varepsilon}_1(\mathbf{r}_4). \tag{6a}$$

Here an equal number of dots is used to designate a pair of factors which occur together within the overall averaging sign. One must take the sum of the products of all possible combinations of the 2n factors taken two at a time. The number of terms in this expression is $(2n - 1)!! = 1 \cdot 3 \cdot 5 \ldots (2n - 1)$.

In averaging (5) one need keep only the even terms in the expansion. Applying (6a) we obtain

$$\begin{aligned}\widetilde{G}_{kj}(\mathbf{r}, \mathbf{r}_0) = {} & G^0_{kj}(\mathbf{r}, \mathbf{r}_0) + k^4 M_{ki}(\mathbf{r}, \mathbf{r}_1)\, \dot{\varepsilon}_1(\mathbf{r}_1)\, M_{il}(\mathbf{r}_1, \mathbf{r}_2)\, \dot{\varepsilon}_1(\mathbf{r}_2)\, G^0_{lj}(\mathbf{r}_2, \mathbf{r}_0) \\ & + k^8 M_{ki}(\mathbf{r}, \mathbf{r}_1)\, \dot{\varepsilon}_1(\mathbf{r}_1)\, M_{il}(\mathbf{r}_1, \mathbf{r}_2)\, \dot{\varepsilon}_1(\mathbf{r}_2) \times M_{lm}(\mathbf{r}_2, \mathbf{r}_3)\, \ddot{\varepsilon}_1(\mathbf{r}_3)\, M_{mn}(\mathbf{r}_3, \mathbf{r}_4)\, \ddot{\varepsilon}_1(\mathbf{r}_4)\, G^0_{nj}(\mathbf{r}_4, \mathbf{r}_0) \\ & + k^8 M_{ki}(\mathbf{r}, \mathbf{r}_1)\, \dot{\varepsilon}_1(\mathbf{r}_1)\, M_{il}(\mathbf{r}_1, \mathbf{r}_2)\, \ddot{\varepsilon}_1(\mathbf{r}_2) \times M_{lm}(\mathbf{r}_2, \mathbf{r}_3)\, \dot{\varepsilon}_1(\mathbf{r}_3)\, M_{mn}(\mathbf{r}_3, \mathbf{r}_4)\, \ddot{\varepsilon}_1(\mathbf{r}_4)\, G^0_{nj}(\mathbf{r}_4, \mathbf{r}_0) \\ & + k^8 M_{ki}(\mathbf{r}, \mathbf{r}_1)\, \dot{\varepsilon}_1(\mathbf{r}_1)\, M_{il}(\mathbf{r}_1, \mathbf{r}_2)\, \ddot{\varepsilon}_1(\mathbf{r}_2)\, M_{lm}(\mathbf{r}_2, \mathbf{r}_3) \times \ddot{\varepsilon}_1(\mathbf{r}_3)\, M_{mn}(\mathbf{r}_3, \mathbf{r}_4)\, \dot{\varepsilon}_1(\mathbf{r}_4)\, G^0_{nj}(\mathbf{r}_4, \mathbf{r}_0) + \ldots\end{aligned} \tag{7}$$

We associate Feynman diagrams with expression (7) according to the following rules.

1. The kernel of the operator $M_{ki}(\mathbf{r}_1, \mathbf{r}_2)$, i.e., $G^0_{ki}(\mathbf{r}_1, \mathbf{r}_2)$, is represented by a solid line whose ends correspond to points $\mathbf{r}_1$, $\mathbf{r}_2$ and to the indices k, i respectively.

2. The correlation function $B_\epsilon(\mathbf{r}_1, \mathbf{r}_2)$ is represented by a dotted line whose ends correspond to the points $\mathbf{r}_1$ and $\mathbf{r}_2$.

3. The factor $k^2$ is represented by a point (a vertex of the graph) at which a single dotted line and two solid lines meet.

4. Integration is performed over the coordinates of all internal vertices of the graph, and convolution is carried out over all indices of internal vertices.

These rules set up a one-to-one correspondence between the analytic description of individual terms in the series (7) and their graphical representation and permit one to write formulas graphically. Equation (7) is represented graphically in Fig. 1, where $\widetilde{G}_{kj}$ is represented by the heavy line.

A diagram is called weakly connected if it can be divided into two parts by cutting one solid line (in Fig. 1 diagrams 3, 6, 7, 8, 9, 12 on the right hand side of the equality are weakly connected). The remaining diagrams are called strongly connected. We now consider the subsequence of strongly connected diagrams $K_{ij}(\mathbf{r}, \mathbf{r}_0)$ (which includes diagrams 2, 4, 5, 10, 11 and 13-20 in Fig. 1). Since each of the diagrams belonging to $K_{ij}$ begins and ends with the line $G^0_{ln}$, we can write for $K_{ij}$

$$K_{ij}(\mathbf{r}, \mathbf{r}_0) = \iint G^0_{il}(\mathbf{r}, \mathbf{r}_1)\, Q_{ln}(\mathbf{r}_1, \mathbf{r}_2)\, G^0_{nj}(\mathbf{r}_2, \mathbf{r}_0)\, d^3r_1 d^3r_2,$$

where the function $Q_{ln}(\mathbf{r}_1, \mathbf{r}_2)$ ("the mass operator") is represented by the graph of Fig. 2 or by means of the series

$$Q_{ij}(\mathbf{r}_1, \mathbf{r}_2) = k^4 B_\varepsilon(\mathbf{r}_1, \mathbf{r}_2)\, G^0_{ij}(\mathbf{r}_1, \mathbf{r}_2) + k^8 \iint G^0_{il}(\mathbf{r}_1, \mathbf{r}_3)\, G^0_{ln}(\mathbf{r}_3, \mathbf{r}_4)\, G^0_{nj}(\mathbf{r}_4, \mathbf{r}_2)\, B_\varepsilon(\mathbf{r}_1, \mathbf{r}_4) \times B_\varepsilon(\mathbf{r}_3, \mathbf{r}_2)\, d^3r_3 d^3r_4 + \ldots \tag{8}$$

All the weakly connected diagrams which contain only two strongly connected elements occur in the sum of graphs shown in Fig. 3a. For example,

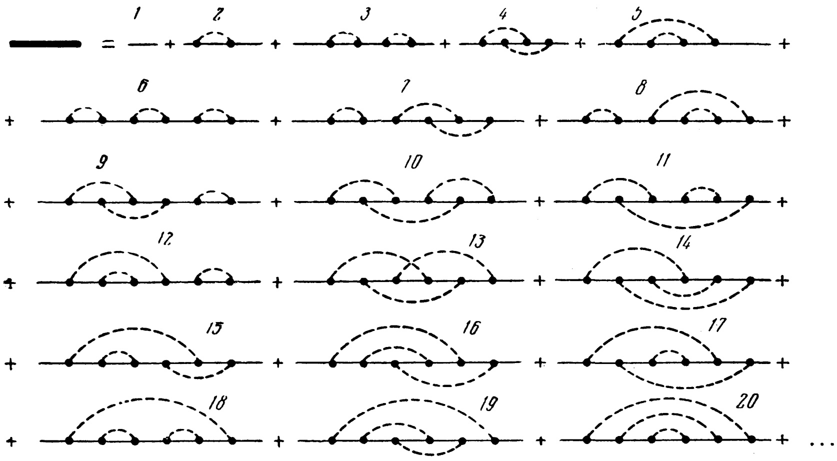


FIG. 1

FIG. 2

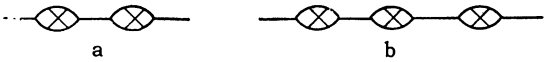


FIG. 3

graph 3 in Fig. 1 is obtained if one takes the first term in $Q_{ij}$, etc. All weakly connected graphs which contain only three strongly connected elements occur in the sum of the graphs represented in Fig. 3b, etc. It follows that $G_{ij}$ can be expanded as shown in Fig. 4.

From Fig. 4 we obtain the integral equation represented in Fig. 5 (the Dyson equation). In fact, if one solves (graphically) the equation in Fig. 5 by using successive iterations and inserts in the right hand side of this equation the sum given by the right hand part of Fig. 5 in place of the wide line, one obtains the series shown in Fig. 4. We now write the equation of Fig. 5 in analytic form:

$$\widetilde{G}_{ij}(\mathbf{r}, \mathbf{r}_0) = G^0_{ij}(\mathbf{r}, \mathbf{r}_0) + \iint G^0_{il}(\mathbf{r}, \mathbf{r}_1)\, Q_{ln}(\mathbf{r}_1, \mathbf{r}_2)\, \widetilde{G}_{nj}(\mathbf{r}_2, \mathbf{r}_0)\, d^3r_1 d^3r_2. \quad (9)$$

When the fluctuations of $\epsilon$ are statistically stationary, i.e. when $B_\epsilon(\mathbf{r}_1, \mathbf{r}_2) = B_\epsilon(\mathbf{r}_1 - \mathbf{r}_2)$, it follows from (8) that $Q_{ij}(\mathbf{r}_1, \mathbf{r}_2) = Q_{ij}(\mathbf{r}_1 - \mathbf{r}_2)$. Then we also have $\widetilde{G}_{ij}(\mathbf{r}_1, \mathbf{r}_2) = \widetilde{G}_{ij}(\mathbf{r}_1 - \mathbf{r}_2)$ and the integral in (9) is a double convolution. Hence when written in the Fourier representation, (9) has the form

$$\widetilde{g}_{ij}(\varkappa) = g^0_{ij}(\varkappa) + (2\pi)^6 g^0_{il}(\varkappa)\, q_{ln}(\varkappa)\, \widetilde{g}_{nj}(\varkappa), \quad (9a)$$

where the Fourier transforms (designated by the corresponding lower case letters) are taken as in (3a). Equation (9a) can be solved particularly easily in the case of statistically isotropic fluctuations, for which $q_{ij}$ has the form

$$q_{ij}(\varkappa) = (\delta_{ij} - \varkappa_i\varkappa_j\varkappa^{-2})\, q_1(\varkappa) + \varkappa_i\varkappa_j\varkappa^{-2} q_2(\varkappa). \quad (10)$$

In this case

$$\widetilde{g}_{ij}(\varkappa) = \{8\pi^3 [k^2 - \varkappa^2 - 8\pi^3 q_1(\varkappa)]\}^{-1} \left\{\delta_{ij} - \frac{\varkappa_i\varkappa_j}{k^2 - 8\pi^3 q_2(\varkappa)} \times \left[1 + 8\pi^3 \frac{q_1(\varkappa) - q_2(\varkappa)}{\varkappa^2}\right]\right\}. \quad (11)$$

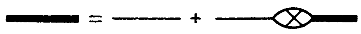

FIG. 5

Thus $\widetilde{G}_{ij}(\mathbf{R})$ has been expressed in terms of the function $Q_{ij}(\mathbf{R})$. The latter, however, is given only in the form of the series (8). One may obtain an integral equation for $Q_{ij}$ which expresses it in terms of a new unknown function (the vertex function) which leads ultimately to a nonlinear integral equation. We will not follow this course but will limit ourselves to investigation of the two limiting cases $ka \ll 1$ and $ka \gg 1$, where a is the correlation radius of the fluctuations.

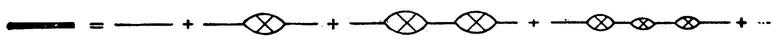

FIG. 4

## 2. THE LIMITING CASE $ka \ll 1$.

In this case we need consider only certain of the first terms in the expansion (8) of the function $Q_{ij}(\mathbf{R})$ in powers of the small parameter k. If one takes only the first term of the series for $Q_{ij}^{(1)}$ and calculates $g_{ij}^{(1)}(\kappa)$ according to (11), then this function will sum the series represented in Fig. 6; this series is obtained from Fig. 4 if in the latter one takes only the first term in the series shown in Fig. 2 in place of $Q_{ij}$.

Calculating the Fourier transform of $Q_{ij}^{(1)}(\mathbf{R}) = k^4 B_\epsilon(\mathbf{R}) G_{ij}^0(\mathbf{R})$, we can obtain for the isotropic case, after a rather complicated calculation, $q_{ij}^{(1)}(\kappa)$ in the form (10), where

$$q_1^{(1)}(\varkappa) = \frac{k^4}{2\pi^2}\int_0^\infty B_\varepsilon(r)\, G_0(r)\left[\frac{\sin\varkappa r}{\varkappa r}\left(1 + \frac{i}{kr} - \frac{1}{k^2r^2}\right) + \left(\frac{\cos\varkappa r}{\varkappa^2 r^2} - \frac{\sin\varkappa r}{\varkappa^3 r^3}\right)\left(1 + \frac{3i}{kr} - \frac{3}{k^2r^2}\right)\right] r^2 dr, \quad (12a)$$

$$q_2^{(1)}(\varkappa) = \frac{k^4}{2\pi^2}\int_0^\infty B_\varepsilon(r)\, G_0(r)\left[\frac{\sin\varkappa r}{\varkappa r}\left(1 + \frac{i}{kr} - \frac{1}{k^2r^2}\right) + \left(\frac{2\sin\varkappa r}{\varkappa^3 r^3} - \frac{2\cos\varkappa r}{\varkappa^2 r^2} - \frac{\sin\varkappa r}{\varkappa r}\right)\left(1 + \frac{3i}{kr} - \frac{3}{k^2r^2}\right)\right] r^2 dr. \quad (12b)$$

As was shown in [2], for the case of a scalar equation and a specific correlation function, the asymptote of the function $G_{ij}^{(1)}(\mathbf{R})$ for $R \gg a$ is determined by the spectrum (11), in which one must put $q_{1,2}(\kappa) \to q_{1,2}(0)$. Putting $\kappa = 0$ in (12), we obtain

$$q_1^{(1)}(0) = q_2^{(1)}(0) = \frac{k^4}{3\pi^2}\int_0^\infty B_\varepsilon(r)\, G_0(r)\, r^2 dr, \quad (13)$$

$$-\frac{\beta}{8\pi^3} \equiv \left[\frac{q_1(\varkappa) - q_2(\varkappa)}{\varkappa^2}\right]_{\varkappa=0} = \frac{k^2}{10\pi^2}\int_0^\infty B_\varepsilon(r) G_0(r)\left(1 - ikr - \frac{k^2r^2}{3}\right) r^2 dr. \quad (14)$$

Putting these expressions in (11) and introducing the notation

$$k_1^2 = k^2 - 8\pi^3 q_1^{(1)}(0) = k^2 + \tfrac{2}{3}k^4\int_0^\infty B_\varepsilon(r)\, e^{ikr}\, r dr, \quad (15)$$

we write $g_{ij}^{(1)}$ in the form

$$g_{ij}^{(1)}(\varkappa) \approx [8\pi^3(k_1^2 - \varkappa^2)]^{-1}\,[\delta_{ij} - \varkappa_i\varkappa_j k_1^{-2}(1 - \beta)]. \quad (16)$$

This expression differs from $g_{ij}^0(\kappa)$ [cf. (3a)] in that k has been replaced by $k_1$ and a factor $(1 - \beta)$ has appeared. Calculating the Fourier transform of (16) we obtain

$$G_{ij}^{(1)}(\mathbf{R}) = \left(\delta_{ij} + \frac{1-\beta}{k_1^2}\frac{\partial^2}{\partial R_i \partial R_j}\right) G_1(R),$$
$$G_1(R) = -\exp(ik_1 R)\,[4\pi R]^{-1}. \quad (17)$$

Before analyzing these formulas we will point out their limits of applicability. To do this one must calculate $k_1^2$, taking into account successive terms in the expansion of the function $Q_{ij}$. The calculations give the following expansion:

$$\tilde{k}^2 = k_1^2 + \mathrm{const}\cdot\sigma^4 k^7 a^5 + \mathrm{const}\cdot\sigma^4 k^8 a^6 \ln(1/ka) + \ldots \quad (18)$$

For $ka \ll 1$ we will have $ka\ln(1/ka) \gg 1$, so that one need keep only the second term in the above formula (the term connected with the third diagram in Fig. 2). Expanding $e^{ikr}$ in (15) we obtain

$$k_1^2 = k^2\left[1 + \frac{\sigma^2 k^2 a^2}{6\pi}(\mu + ika)\right];$$
$$\sigma^2 = \langle \varepsilon_1^2 \rangle = B_\varepsilon(0), \quad \sigma^2 a^3 \equiv 4\pi\int_0^\infty B_\varepsilon(r)\, r^2 dr,$$
$$\mu\sigma^2 a^2 = 4\pi\int_0^\infty B_\varepsilon(r)\, r dr. \quad (15a)$$

The next to last formula determines the correlation scale factor a, and the last formula is written down from dimensional considerations and includes a numerical factor $\mu$ which depends on the form of $B_\epsilon(r)$.

Comparing the second term in (18) with (15a) and requiring that their ratio be small, we obtain the condition

$$\sigma^4 k^5 a^5 \ll 1 + \sigma^2 k^2 a^2/6\pi. \quad (19)$$

When $\sigma^2 k^2 a^2 \ll 1$ (19) is satisfied automatically. For $\sigma^2 k^2 a^2 \gg 1$ the following condition follows from (19);

$$\sigma^2 k^3 a^3 \ll 1. \quad (20)$$

When (19) is satisfied it follows from (18) that

$$\tilde{k} = k_1 + \mathrm{const}\cdot\sigma^4 k^7 a^5 k_1^{-1} + \ldots$$

Since $\tilde{k}$ occurs in the exponent with the factor R, the last term may be neglected only if $\sigma^4 k^7 a^5 k_1^{-1} R \ll 1$, or

$$kR \ll (1 + \mu\sigma^2 k^2 a^2/6\pi)^{1/2}/\sigma^4 k^5 a^5. \quad (21)$$

We put $k = p + i\gamma_0$ and assume that $\gamma_0 \ll p$. Then using (15a) one may obtain an expression for $k_1$ (accurate to terms linear in $\gamma_0$);

FIG. 6

$$k_1 = pn_{\text{eff}} + i\gamma_{\text{eff}}, \tag{22}$$
$$n_э = \sqrt{1 + \mu\sigma^2 p^2 a^2/6\pi},$$
$$\gamma_{\text{eff}} = \frac{2n_{\text{eff}}^2 - 1}{n_{\text{eff}}}\gamma_0 + \frac{\sigma^2 p^4 a^3}{12\pi n_{\text{eff}}}.$$

The effective refractive index $n_{\text{eff}}$ may differ considerably from the initial value, equal to unity, since it has not been assumed that the quantity $\sigma^2 p^2 a^2$ is small. The effective absorption $\gamma_{\text{eff}}$ consists of two terms. The first is proportional to $\gamma_0$, but it is multiplied by a factor greater than one which takes account of the increase in absorption due to multiple scattering. The second term in (22c) is $\sigma_{\text{inv}}/n_{\text{eff}}$, where $\sigma_{\text{inv}}$ is the effective cross-section for single scattering into the back hemisphere. Thus taking account of multiple scattering diminishes the attenuation due to scattering. For the case $ka \ll 1$ one may obtain the following expression for the quantity $\beta$,

$$\beta = \tfrac{3}{10}(n_{\text{eff}}^2 - 1). \tag{23}$$

The coefficient $\beta$ describes the appearance of a longitudinal wave.

With the help of the general formula (17) one may calculate the average field of a point dipole in the far field. It has the perpendicular component

$$\bar{G}_\perp(R) = -[\exp(ik_1R)/4\pi R]\sin\theta, \tag{24}$$

where $\theta$ is the angle between the dipole moment and the direction from its center to the point of observation; the longitudinal component is

$$G_\parallel(R) = -\beta[\exp(ik_1R)/4\pi R]\cos\theta. \tag{24b}$$

Thus the amplitude of the longitudinal wave is comparable with the transverse component if $\beta \gtrsim 1$, i.e., if according to (23), $n_{\text{eff}}$ differs significantly from unity.

## 3. THE LIMITING CASE $ka \gg 1$.

This limiting case is essentially that of geometrical optics. As is well known, the depolarized component of the field scattered from fluctuations is of order $(ka)^{-2}$ compared to the primary polarized component, and for $ka \gg 1$ it can be neglected. Hence when $ka \gg 1$ we can neglect the depolarization term in (1) $\partial^2 E_j/\partial x_i \partial x_j$; when this is done the equation separates into three scalar equations. The Green's function of the scalar equation is $G_0(R)$ [cf. (3)]. The averaged Green's function will be represented by the series in Fig. 1, with $G_0$ in place of $G_{ij}^0$.

Let us compare the fourth order diagrams (3, 4, 5 in Fig. 1). The first of these is

$$I_{41} = k^8 \iiiint G_0(\mathbf{r}, \boldsymbol{\rho}_1) G_0(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) G_0(\boldsymbol{\rho}_2, \boldsymbol{\rho}_3) G_0(\boldsymbol{\rho}_3, \boldsymbol{\rho}_4) G_0(\boldsymbol{\rho}_4, \mathbf{r}_0) \times B_\varepsilon(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) B_\varepsilon(\boldsymbol{\rho}_3, \boldsymbol{\rho}_4)\, d^3\rho_1 \ldots d^3\rho_4,$$

and the second is

$$I_{42} = k^8 \iiiint G_0(\mathbf{r}, \boldsymbol{\rho}_1) G_0(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) G_0(\boldsymbol{\rho}_2, \boldsymbol{\rho}_3) G_0(\boldsymbol{\rho}_3, \boldsymbol{\rho}_4) G_0(\boldsymbol{\rho}_4, \mathbf{r}_0) \times B_\varepsilon(\boldsymbol{\rho}_1, \boldsymbol{\rho}_3) B_\varepsilon(\boldsymbol{\rho}_2, \boldsymbol{\rho}_4)\, d^3\rho_1 \ldots d^3\rho_4.$$

The characteristic length over which the function $G_0(R)$ varies significantly is the wavelength $\lambda$. The characteristic length over which $B_\epsilon(R)$ varies is the correlation radius a. For $ka \gg 1$, i.e., for $a \gg \lambda$, the function $B_\epsilon(R)$ is very smooth in comparison to $G_0(R)$, and the integrals $I_{41}$ and $I_{42}$ can be found approximately by taking $B_\epsilon(0)$ outside the integral sign. In this case we obtain $I_{41} \approx I_{42} \approx I_{43}$. It is clear that the same estimates are valid for diagrams of higher order. Hence one may obtain the approximate value of the averaged Green's function if each of the terms in the series in Fig. 6 is multiplied by the number of diagrams of the given order, equal to $(2n-1)!!$, and the resulting series is summed.

This operation can be carried out explicitly. We use the notation $\alpha_{2n}\sigma^{2n}$ to designate the integral represented by the diagram of order 2n occurring in the series of Fig. 6 (the factor $B_\epsilon(r) = \sigma^2 b_\epsilon(r)$ occurs n times in the diagram of order 2n; $b_\epsilon(r)$ is the normalized correlation function). The sum of the series in Fig. 6 has been calculated above; it is obtained from (11) if in the latter one replaces $Q_{ij}$ by $Q_{ij}^{(1)}$. For the case of a scalar equation (cf. [2]) one has

$$G_1(R) = \sum_{n=0}^{\infty} \alpha_{2n}(R)\sigma^{2n} = G_1(R, \sigma^2) = \frac{1}{8\pi^3}\int \frac{\exp(i\varkappa R)\, d^3\varkappa}{k^2 - \varkappa^2 - \sigma^2 k^4 \int G_0(\rho)\, b_\varepsilon(\rho)\exp(-i\varkappa\rho)\, d^3\rho} \tag{25}$$

(this formula was considered to be an approximation in describing the case $ka \ll 1$, but it is the exact sum of the series in Fig. 6).

We are interested in the sum of the series

$$G_2(R) = \sum_{n=0}^{\infty} \alpha_{2n}(R)\sigma^{2n}(2n-1)!!. \tag{26}$$

Assuming that this series converges we put

$$(2n-1)!! = \frac{2^n}{\sqrt{\pi}}\Gamma\left(n + \frac{1}{2}\right) = \frac{2^n}{\sqrt{\pi}}\int_0^\infty e^{-x} x^{n-1/2}\, dx$$

and interchange the order of summation and integration:

$$G_2(R) = \int_0^\infty (\pi x)^{-1/2}\exp(-x)\Big[\sum_{n=0}^{\infty} \alpha_{2n}(R)(2x\sigma^2)^n\Big]dx.$$

Using (25) we obtain $G_1(R, 2x\sigma^2)$ for the internal sum, i.e., the function $G_1$ in which $\sigma^2$ has been replaced by $2x\sigma^2$. Introducing a new variable of integration $t^2 = 2x\sigma^2$, we obtain

$$G_2(R) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{2\sigma^2}\right) G_1(R, t^2)\, dt. \quad (27)$$

In this way $G_2(R)$ is obtained from $G_1$ by averaging over $\sigma^2$ with a Gaussian weighting factor.

The calculations that follow will be made for a particular correlation function of the type $B_\epsilon(r) = \sigma^2 \exp(-\alpha r)$. In this case the integral (25) can be calculated exactly (cf. [2]) and is given by

$$G_1(R) = C_1 R^{-1} \exp(i\varkappa_1 R) + C_2 R^{-1} \exp(i\varkappa_2 R), \quad (28)$$

$$C_{1,2}(t) = -\frac{1}{8\pi}\left[1 \pm \left(1 + \frac{4k^4t^2}{\alpha^2(\alpha - 2ik)^2}\right)^{-1/2}\right], \quad (28a)$$

$$\varkappa_{1,2}(t) = \frac{1}{\sqrt{2}}\left[k^2 - (\alpha - ik)^2 \pm \sqrt{\alpha^2(\alpha - 2ik)^2 + 4k^4t^2}\right]^{1/2}, \quad (28b)$$

where $t^2$ has been written in place of $\sigma^2$.

We will not calculate the integral (27) exactly but will rather investigate its asymptotic value as $R \to \infty$. To do this we apply the method of stationary phase to (27), since (28) contains a rapidly oscillating factor $\exp(i\kappa_{1,2}R)$. The stationary point is $t = 0$. Expanding $\kappa_{1,2}(t)$ and $C_{1,2}(t)$ is a series for $t = 0$ and then calculating the integral (27), we obtain

$$G_2(R) \approx G_0(R)\left[1 + \tfrac{1}{2}k^2\sigma^2 R\alpha^{-1}(1 + i\alpha/2k)^{-1}\right]^{-1/2}. \quad (29)$$

It follows from (29) that even for $ka \ll 1$ the average field decreases with separation more rapidly than $G_0(R)$, although this additional decrease is not exponential.

We now compare (29) with the expression obtained by applying perturbation theory to the eikonal equation. According to geometrical optics the field of a point source at the origin is given by

$$E(R) = -(4\pi R)^{-1} \exp\left\{ikR + ik\int_0^R n_1(r)\, dl\right\},$$

where $n_1$ is the deviation of the index of refraction from $\langle n\rangle = 1$, and the integration is carried out along the ray. In the case $|n_1| \ll 1$, we can put $n_1 \approx \epsilon_1/2$ and carry out the integration along a straight line. Expanding the exponent in a series accurate to terms of second order and averaging, one easily obtains a formula valid for $R \gg a$:

$$\langle E(R)\rangle = G_0(R)\left[1 - \frac{k^2R}{4}\int_0^\infty B_\varepsilon(x)\, dx + \ldots\right]. \quad (30)$$

When $B_\epsilon(x) = \sigma^2\exp(-\alpha x)$, (30) agrees with the first term in the expansion of the function $G_2(R)$ (cf. [2]) in powers of $\sigma^2$, if one neglects in (29) the small term $i\alpha/2k$ compared to unity. Hence for the exponential correlation function, (29) can also be written in the form

$$G_2(R) = G_0(R)\left(1 + \frac{1}{2}k^2R\int_0^\infty B_\varepsilon(\rho)\, d\rho\right)^{-1/2}. \quad (29a)$$

It is an open question whether or not the asymptotic expression (29a) remains valid for correlation functions of a more general type. However in any case its series expansion coincides with (30), and for $B_\epsilon(\rho) = \sigma^2\exp(-\alpha\rho)$ this expression goes over into (29).

## 4. THE CORRELATION FUNCTION OF THE ELECTROMAGNETIC FIELD

We now consider the correlation function of the field

$$B_{ij}(\mathbf{r}_1, \mathbf{r}_2) = \langle [E_i(\mathbf{r}_1) - \langle E_i(\mathbf{r}_1)\rangle]\, [E_j^*(\mathbf{r}_2) - \langle E_j^*(\mathbf{r}_2)\rangle]\rangle \quad (31)$$

and insert expression (5) for $E_i$ in the right hand side of (31). We therefore obtain the expression

$$B_{ij}(\mathbf{r}_1, \mathbf{r}_2) = W_{ij,lk}(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_0, \mathbf{r}_0)\, n_l n_k,$$

$$W_{ij,lk}(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_0, \mathbf{r}_0') = \langle [G_{il}(\mathbf{r}_1, \mathbf{r}_0) - \langle G_{il}(\mathbf{r}_1, \mathbf{r}_0)\rangle] \times [G_{jk}^*(\mathbf{r}_2, \mathbf{r}_0') - \langle G_{jk}^*(\mathbf{r}_2, \mathbf{r}_0')\rangle]\rangle. \quad (32)$$

The functions $G_{il}(\mathbf{r}, \mathbf{r}_0)$ contain the random $\epsilon_1$ which are assumed to be Gaussian, as before. In representing (32) graphically we use the same rules of correspondence given previously with a single difference: solid lines at the bottom part of a diagram represent the function $G_{ik}^{0*}(\mathbf{r}_1, \mathbf{r}_2)$, and the points on the lower lines represent the factors $k^{*2}$. The vertices of a line, and points, and dotted lines have the same meaning as before. Inserting the series (5) in (32) we can obtain a formula (cf. Fig. 7) where the function $W_{ij,lk}$ is represented on the left hand side of the equal sign. Because the average value has been separated out of $E_i$, the diagrams for W do not contain unconnected diagrams (diagrams whose various parts are not connected). Figure 7 illustrates clearly the rule for construction of diagrams of order higher than 2n. One must put a single point on the upper line and (2n − 1) points on the lower line and then connect these by dotted lines in all possible ways. One then puts two points on the upper line and (2n − 2) on the lower line, etc.

A diagram in Fig. 7 is called weakly connected if it can be divided into two parts by one cut through two solid lines (upper and lower), each of which contains at least two vertices (on either of the lines). The weakly connected diagrams in

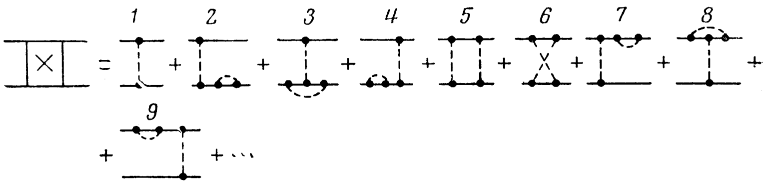

FIG. 7

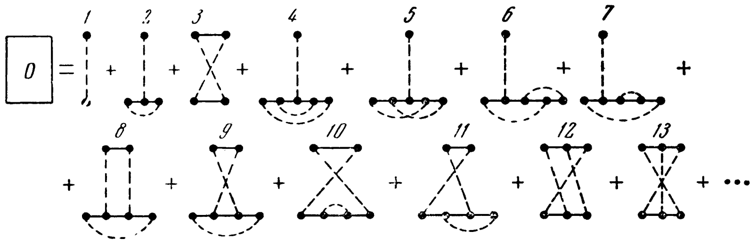

FIG. 8

Fig. 7 are 2, 4, 5, 7, 9; the diagrams 1, 3, 6, 8 are strongly connected.

We now consider the sum of the strongly connected diagrams U. Each term in U is terminated by four solid lines, i.e., U can be represented in the form

$$U_{ij,\,lk}(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_0, \mathbf{r}_0') = \iiiint G^0_{ia}(\mathbf{r}_1, \boldsymbol{\rho}_1) G^{0*}_{jb}(\mathbf{r}_2, \boldsymbol{\rho}_2)\, Q_{ab,\,pq}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \boldsymbol{\rho}_3, \boldsymbol{\rho}_4) \times G^0_{pl}(\boldsymbol{\rho}_3, \mathbf{r}_0)\, G^{0*}_{qk}(\boldsymbol{\rho}_4, \mathbf{r}_0')\, d^3\rho_1 \ldots d^3\rho_4.$$

The function $Q_{ab,pq}$ is shown in Fig. 8 (part of the diagrams of sixth order are obtained from those shown by a 180° rotation around either the horizontal or vertical axes).

The sum of the strongly connected diagrams has the form shown in Fig. 9a. The sum of all weakly connected diagrams containing only one element belonging to $Q_{ab,pq}$ has the form shown in Fig. 9b. The diagrams 2, 4, 7, 9 from Fig. 7 belong to the set in Fig. 9b.

The sum of all weakly connected diagrams containing only two elements from $Q_{ab,pq}$ and an arbitrary number of strongly connected elements from G consisting of external solid lines and solid lines uniting two elements from $Q_{ab,pq}$, has the form shown in Fig. 9c. Diagram 5 from Fig. 7 belongs to this set.

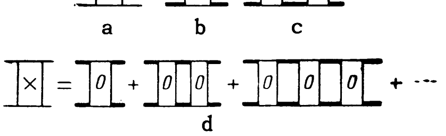

FIG. 9

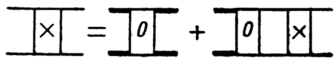

FIG. 10

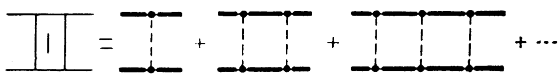

FIG. 11

Continuing this discussion we obtain the expansion for $W_{ij,lk}$ shown in Fig. 9d. In a manner analogous to the way the Dyson equation (Fig. 5) follows from the development of Fig. 4, the equation shown in Fig. 10 (analogous to the Bethe-Saltpeter equation) follows from the development of Fig. 9d. In fact, solving the equation of Fig. 10 graphically by successive iteration we obtain the development shown in Fig. 9d. We now write this equation in analytic form:

$$W_{ij,\,lk}(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_0, \mathbf{r}_0') = \iiiint \widetilde{G}_{ia}(\mathbf{r}_1, \boldsymbol{\rho}_1)\, \widetilde{G}^*_{jb}(\mathbf{r}_2, \boldsymbol{\rho}_2)\, Q_{ab,pq}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \boldsymbol{\rho}_3, \boldsymbol{\rho}_4) \times \widetilde{G}_{pl}(\boldsymbol{\rho}_3, \mathbf{r}_0)\, \widetilde{G}^*_{qk}(\boldsymbol{\rho}_4, \mathbf{r}_0')\, d^3\rho_1 \ldots d^3\rho_4 + \iiiint \widetilde{G}_{ia}(\mathbf{r}_1, \rho_1)\, \widetilde{G}^*_{jb}(\mathbf{r}_2, \boldsymbol{\rho}_2)\, Q_{ab,\,pq}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \boldsymbol{\rho}_3, \boldsymbol{\rho}_4) \times W_{pq,\,lk}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \mathbf{r}_0, \mathbf{r}_0')\, d^3\rho_1 \ldots d^3\rho_4. \qquad (33)$$

The linear integral equation (33) connects the correlation function W with the function Q. In contrast to (9) this equation cannot be solved explicitly by a Fourier transformation. If we take for Q only the first term in the series, i.e., if we put

$$Q_{ab,\,pq}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \boldsymbol{\rho}_3, \boldsymbol{\rho}_4) \approx Q^{(1)}_{ab,\,pq}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \boldsymbol{\rho}_3, \boldsymbol{\rho}_4) = |k|^4\, \delta_{ap}\delta_{bq}\,\delta(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_3)\,\delta(\boldsymbol{\rho}_2 - \rho_4)\, B_\varepsilon(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2),$$

we obtain the so-called "staircase" approximation to the Bethe-Saltpeter equation, summing the diagrams of Fig. 11.

The "staircase" approximation gives satisfactory results only when $ka \ll 1$, since the successive terms in the series for Q contain higher powers of k. However even in the "staircase" approximation, (33) may only be solved approximately using the limitation $ka \ll 1$. This approximate solution is based on the fact that for the case $B_\epsilon(\mathbf{r}) = \sigma^2 a^3 \delta(\mathbf{r})$ one may easily obtain formally an exact solution, which may be used as the initial approximation for solution with a different correlation function $B_\epsilon(\mathbf{r})$ which is not a delta function. However, analysis of the solution obtained in this way leads to the conclusion that it has meaning only when one takes only the first term of the series in Fig. 11; i.e., when one may restrict oneself to the approximation

$$W_{ij,\,lk}(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_0, \mathbf{r}_0') \simeq |k|^4 \iint \widetilde{G}_{ia}(\mathbf{r}_1, \boldsymbol{\rho}_1)\, \widetilde{G}^*_{jb}(\mathbf{r}_2, \boldsymbol{\rho}_2) \times B_\varepsilon(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)\, \widetilde{G}_{al}(\boldsymbol{\rho}_1, \mathbf{r}_0)\, \widetilde{G}^*_{bk}(\boldsymbol{\rho}_2, \mathbf{r}_0')\, d^3\rho_1 d^3\rho_2. \quad (34)$$

This expression differs from the first term of the perturbation theory series (Fig. 7) in replacing function $G^0_{ij}$ by $\widetilde{G}_{ij}$ and hence it sums a certain infinite subsequence of perturbation series. If one uses the function $\widetilde{G}^{(1)}_{ij}(\mathbf{r}_1, \mathbf{r}_2)$ found above for $\widetilde{G}_{ij}$ then diagrams 1, 2, 4, 7, 9 of Fig. 7 will occur in (34) as well as all diagrams containing elements of Fig. 6 in arbitrary combinations on the external lines. Expression (17) must be used for $G^{(1)}_{ij}$.

To simplify calculation of the mean-squared fluctuations, the field was treated as a scalar field. This calculation does not allow one to take account of depolarization effects but it correctly takes into account the effect of multiple scattering. The calculation gives the formula

$$A \equiv \frac{\langle |E(\mathbf{r}) - \langle E(\mathbf{r})\rangle|^2\rangle}{|\langle E(\mathbf{r})\rangle|^2} = \frac{|k|^4\sigma^2 a^3}{8\pi\gamma_{\text{eff}}} \quad (35)$$

for the relative magnitude of the field fluctuations. Equation (35) is valid for $A \ll 1$, in which case one need take account only of the first term of the series in Fig. 11. This condition is equivalent to requiring that the first term in (22c) for $\gamma_{\text{eff}}$ (due to true absorption) considerably exceed the second term (due to scattering); it also imposes a limitation on the absorption $\gamma_0$. Using this fact, (35) may be written in the form

$$A = \frac{|k|^4\sigma^2 a^3}{8\pi\gamma_0} \cdot \frac{n_{\text{eff}}}{2n^2_{\text{eff}} - 1}, \quad (35a)$$

where the first factor is the value of A found by first order perturbation theory. For $\sigma^2 k^2 a^2 \gg 1$ the quantity $n_{\text{eff}} \gg 1$, so that $n_{\text{eff}}(2n^2_{\text{eff}} - 1)^{-1} \approx (2n_{\text{eff}})^{-1} \backsim (\sigma a |k|)^{-1}$. In this case (35a) becomes

$$A \sim |k|^3 a^2 \sigma / \gamma_0, \quad (36)$$

i.e., in the case of strong fluctuations of the dielectric constant, the fluctuations of the field are proportional to the third power of the frequency (in contrast to the Rayleigh formula) and are proportional only to the first power of $\sigma$. Physically the slow increase in the field fluctuations with increasing $\sigma$ is due to the increase in absorption $\gamma_{\text{eff}}$ due to multiple scattering, which causes a decrease in the region from which waves are "gathered" at the point of observation of the scattered wave.

Note added in proof (Feb. 28, 1964). It has been shown by V. N. Finkel'berg [JETP 46, 725 (1954) Soviet Phys. JETP 19, 494 (1964)] that a term $3^{-1}k^2\delta_{ij}(R)$ must be added to the right hand side of (3). Taking this term into account has the result that the quantity $k^2 = \omega^2 c^{-2}\langle\epsilon\rangle$ in the final formulas must be replaced by $k_0^2 = \omega^2 c^{-2}\epsilon_0$ where $\epsilon_0$ is a root of the equation $\langle(\epsilon - \epsilon_0)(\epsilon + 2\epsilon_0)^{-1}\rangle = 0$, where in determining $\epsilon_0$ one must take account of the deviation from a Gaussian distribution for the random quantity $\epsilon(\mathbf{r})$. The author is grateful to Yu. A. Ryzhov, V. N. Tamoĭkin and V. M. Finkel'berg for pointing this out.

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