SOVIET PHYSICS JETP VOLUME 20, NUMBER 1 JANUARY, 1965

RELATION BETWEEN THE SINGLE INTEGRALS OF THE MANDELSTAM REPRESENTATION AND DOUBLE SPECTRAL FUNCTIONS

E. I. MALKOV

A. F. Ioffe Physico-technical Institute, Academy of Sciences U.S.S.R.

Submitted to JETP editor December 27, 1963

J. Exptl. Theoret. Phys. (U.S.S.R.) 47, 188-199 (July, 1964)

It is shown that it is impossible, without violating the unitarity condition ($|a_l| \leq 1$) in three channels, to add to the two-particle amplitude satisfying the unitarity condition any functions that possess in the finite part of complex space singularities with respect to only one variable (s, t, or u) and grow more slowly than an exponential function at infinity, the only exceptions being terms corresponding to S and P waves in the three channels. This signifies that those amplitude parts which possess in the finite part of space only single-variable singularities are determined by that part which contains singularities with respect to two variables. As an example, an expression is derived for the single integrals and subtraction polynomial of the Mandelstam representation in terms of double spectral functions.

1. The unitarity condition interrelates the singularities of the amplitude of a two-particle reaction, in the sense that if the amplitude has certain singularities, then the unitarity condition stipulates that it must also have others. The opinion was advanced[1] that perhaps by specifying some of the simplest singularities (poles) it is possible to determine the remaining singularities with the aid of the unitarity condition.

We shall discuss in this paper the connection between the amplitude singularities in two variables and the single-variable singularities, and the connection between parts of the amplitude which have only single-variable (s, t, or u) singularities in the finite part of the complex space and the part having singularities in two variables. It will be shown that if the former increase at infinity (over a large circle in the complex plane) at a rate slower than exponential, then the unitarity condition ($|a_l| \leq 1$) requires that they be expressible in a part containing two-variable singularities, accurate to terms corresponding to S and P waves in three channels. In particular, the only entire function satisfying the unitarity condition and increasing at infinity more slowly than an exponential function is a constant. This connection is obtained from an examination of the behavior of the amplitude in the vicinity of an infinitely remote point, and does not depend on the properties of the amplitude in the finite part of the complex space. The requirement that the amplitude increase in the vicinity of an infinitely remote point more slowly than exponential is necessary in order for the limitations that the unitarity condition imposes on the growth of the amplitude in the physical region (to the right and to the left on the real axis) be obligatory also on approaching infinity from any direction in the complex plane.

The result obtained is a certain generalization of that of Froissart[2], who showed that the difference between two amplitudes with power-law bounds at infinity and satisfying the Mandelstam representation is equal to a constant and to six single integrals corresponding to the S and P waves in three channels, i.e., it is impossible to add to the amplitude a finite sum of single integrals and a polynomial, other than those indicated above.

2. Let A and A′ be two two-particle reaction amplitudes having the following properties:

a) They satisfy the condition of unitarity in three channels in the form $|a_l| \leq 1$.

b) They increase more slowly than any exponential at infinity (over a large circle in the complex plane):

$$|A| < C(\alpha) \exp(\alpha|s| + \alpha|t|)$$

for all $\alpha > 0$.

c) They have identical two-variable singularities in the finite part of the complex space, i.e., their difference is a sum of entire functions of one variable:

$$\Delta A = A - A' = \sum_{p=0}^{\infty} t^p f_{ps}(s) + \sum_{p=0}^{\infty} u^p f_{pt}(t) + \sum_{p=0}^{\infty} s^p f_{pu}(u) + \sum_{p,\,q=0}^{\infty} t^p u^q \sigma_{pq}; \quad (1)$$

d) With regard to the analyticity of the amplitudes in the energy, we shall assume that in a sufficiently small vicinity of an infinitely remote point all the singularities lie on the real axis, and the amplitude is analytic in z within a Mandelstam ellipse with semiaxis $z_0 = 1 + t/2q_s^2$.

Under all the four foregoing conditions, the only nonvanishing terms in the sums (1) are those corresponding to S and P waves, i.e., terms with $p \leq 1$, while in the last sum—the term with $p = q = 0$. Let us regard ΔA as a function of t (or u) with $s = \text{const} < 0$. For the asymptotic behavior of the amplitude on the real axis in the t channel we have the limitation[3,2]

$$|A| < Ct\ln^2|A|_{max}, \tag{2}$$

where $|A|_{max}$—maximum value of the amplitude on the Mandelstam ellipse. Using the limitation b), we obtain

$$|A| < Ct^3. \tag{3}$$

A similar estimate takes place on the left cut, corresponding to the physical region of the u channel. Consequently, on a large circle surrounding infinity ΔA increases with increasing radius more slowly than exponentially, while along two radii (to the right and to the left along the real axis) it increases more slowly than t cubed. Such a function should increase more slowly than Ct^3 in any direction[4]. It follows from condition (3) that the jump on the cut increases more slowly than t^3, so that the sum

$$\sum_{p=0}^{\infty} u^p f_{pt}(t) + \sum_{p=0}^{\infty} s^p f_{pu}(u)$$

can be broken up into two terms, one bounded in the vicinity of an infinitely remote point by the quantity $t^{3+\epsilon}$, and the other an entire function, which we include in the last term of formula (1).

We rewrite (1) in the form

$$\Delta A - \sum_{p=0}^{\infty} u^p f'_{pt}(t) - \sum_{p=0}^{\infty} s^p f'_{pu}(u) = \sum_{p=0}^{\infty} t^p \left[f_{ps} + \sum_{q=0}^{\infty} u^q \sigma'_{pq}\right]. \tag{4}$$

The function is bounded from the left in the vicinity of the infinitely remote point by the quantity $t^{3+\epsilon}$, and consequently the entire function on the right is bounded by the same value, meaning that it is a polynomial of third degree in t:

$$\Delta A = \sum_{p=0}^{\infty} u^p f'_{pt}(t) + \sum_{p=0}^{\infty} s^p f'_{pu}(u) + \sum_{p=0}^{3} t^p f_{ps}(s) + \sum_{p=0}^{3} t^p \sum_{q=0}^{\infty} u^q \sigma'_{pq}. \tag{5}$$

If we consider the quantity ΔA with $t = \text{const} < 0$ and $u = \text{const} < 0$ in similar fashion, we obtain

$$\Delta A = \sum_{p=0}^{3} t^p f'_{ps}(s) + P_{stu} + \sum_{p,\,q=0}^{p+q=3} t^p u^q \sigma''_{pq}. \tag{6}$$

It then follows from condition (3) that $|f'_{pt}(t)| < t^{3+\epsilon}$, meaning that ΔA is bounded for all values of the variables by some power of t. We now repeat the initial arguments presented and impose a power-law bound on the amplitude difference in place of an exponential bound. As a result we obtain

$$\Delta A = \sum_{p=0}^{1} t^p f'_{ps}(s) + P_{stu} + C, \tag{7}$$

which is the sought result.

At a fixed angle we can obtain for ΔA an estimate analogous to that obtained by Kinoshita et al.[5]: $|\Delta A| < C\theta^{-2}\ln^{3/2} t$. Using this estimate for the functions f'_{pt}, we obtain

$$|f'_{pt}| < Ct^{-p}\ln^{3/2} t. \tag{8}$$

f'_{ps} and f'_{pu} are analogously bounded.

We now consider a less general case, when the analytic properties of the amplitude in a finite part of the space are given by the Mandelstam rules, i.e., part of the amplitude containing the singularities with respect to two variables is a double integral with double spectral functions, with a power-law bound at infinity ($|\rho_{st}| < Cs^{N-\epsilon}t^{n-\epsilon}$, etc.), and the part of the function having a singularity in the finite part of the complex space with respect to only one variable comprises a set of series of single integrals (in general, infinite, but converging on the entire plane) and an entire function.

We shall show that under a supplementary limitation on the two absorption parts (for example A_t and A_u) in the form

$$|A_t| < C(\varepsilon)\exp[\varepsilon(|s|+|t|)^{1/2}\ln(|s|+|t|)] \tag{9}$$

for all $\epsilon > 0$, the series of the single integrals are finite sums, and the entire function is a polynomial.

4. We shall show first that it follows from the unitarity condition and from the limitations b) and (9) that the functions A_t and A_u have a power-law bound for any fixed s. The absorption parts are equal to the sum of the integrals of double spectral functions over the cuts plus an entire function s. The fact that the integrals over the cuts are bounded by a power of $t(u)$ follows from the power-law boundedness of the double functions, so that it is necessary to prove only that the entire function is bounded. We denote the entire function s in A_t by $A_t^{\prime}$. We can write for it the following partial-wave expansion:

$$A_t' = \frac{t^{1/2}}{2q_t}\sum_{l=0}^{\infty}(2l+1)\,\mathrm{Im}\,a_l'(t)\,P_l(z_t). \qquad (10)$$

The reasoning is carried out in the usual manner: for $l < l_{\mathrm{eff}}$ the partial waves are bounded by the unitarity conditions, and for $l > l_{\mathrm{eff}}$ they are cut off because of the analyticity in z_t, and represent an exponentially decreasing tail.

It follows from the unitarity condition that the $\mathrm{Im}\, a_l'$ are bounded by a power of t:

$$|\mathrm{Im}\,a_l'| < |\mathrm{Im}\,a_l| + \left|\frac{2q_t\cdot 2i}{t^{1/2}\cdot 2\pi i}\int \rho_{st} Q_l(z_t)\,dz_t + \frac{2q_t\cdot 2i}{t^{1/2}\cdot 2\pi i}\int \rho_{tu} Q_l(z_t)\,dz_t\right| \qquad (11)$$

The first term in (11) is bounded by the unitarity condition ($|a_l| \leq 1$), while the second is bounded by virtue of the boundedness of the double functions (ρ_{st} etc.) and the Legendre function

$$|Q_l(z)| < \left(\frac{\pi}{2l}\right)^{1/2}\frac{|z+(z^2-1)^{1/2}|^{-(l+1/2)}}{|1-z^2|^{1/4}}. \qquad (12)$$

As a result we have

$$|\mathrm{Im}\,a_l'| < 1 + \frac{Ct^{2N-\varepsilon'}}{l^{1/2}|z_0+(z_0^2-1)^{1/2}|^{l+1/2}|1-z_0^2|^{1/2}} < \frac{Ct^{2N+1/4-\varepsilon'}}{l^{1/2}}, \qquad (13)$$

where z_0—lower integration limit in (11).

For large l, the partial waves are cut off by the condition of analyticity in z and by the boundedness of A_t as $t \to \infty$, which follows from (9). It is obvious that A_t' satisfies the condition (9), since the integrals along the cuts have a power-law bound:

$$|\mathrm{Im}\,a_l'| = \left|\frac{2q_t}{t^{1/2}\cdot 2\pi i}\oint_{z'} A_t'(t, z_t)\,Q_l(z_t)\,dz_t\right| < \frac{C\exp\left[\varepsilon(|s|+|t|)^{1/2}\ln(|s|+|t|)\right]}{l^{1/2}|z'+(z'^2-1)^{1/2}|^{l+1/2}|1-z'^2|^{1/4}}. \qquad (14)$$

The integral in (14) is taken over an ellipse with major semiaxis z′ which does not depend on t. From this we see that

$$l_{\mathrm{eff}} = \varepsilon t^{1/2}\ln t. \qquad (15)$$

We can now estimate the sum (10). For Legendre polynomials we use the bound

$$|P_l(z)| \leqslant |z+(z^2-1)^{1/2}|^l.$$

We have

$$A_t'| < C\sum_{l=0}^{l_{\mathrm{eff}}}(2l+1)\frac{t^{2N+1/4-\varepsilon'}}{\sqrt{l}}\left|1+\frac{s}{2q_t^2}+\left[\left(1+\frac{s}{2q_t^2}\right)^2-1\right]^{1/2}\right|^l < Ct^{2N+1/4-\varepsilon'}\sqrt{l_{\mathrm{eff}}}\sum_{l=0}^{l_{\mathrm{eff}}}\left|1+\frac{s^{1/2}}{q_t}\right|^l < Ct^{2N+1-\varepsilon'}\exp\left[\varepsilon\cdot s^{1/2}\ln t\right]. \qquad (16)$$

For any finite s we can take an ϵ sufficiently small so that

$$|A_t'| < Ct^{2N+1-\varepsilon''}$$

for any s; analogously we obtain for the entire part of A_u

$$|A_u'| < Cu^{2N+1-\varepsilon''}.$$

5. From the results given in Sec. 4 it follows that for any s the amplitude can be represented in the form

$$A = \frac{t^{2N+1}}{\pi}\int\frac{A_t dt'}{t'^{2N+1}(t'-t)} + \frac{u^{2N+1}}{\pi}\int\frac{A_u du'}{u'^{2N+1}(u'-u)} + \sum_{p=0}^{\infty}t^p a_p(s). \qquad (17)$$

In the last sum the only nonvanishing $a_p(s)$ are those with $p \leq 2N$. Indeed, when $s = \mathrm{const} < 0$ A is bounded in the vicinity of an infinitely remote point by the quantity t^3, as already shown for the general case (3). The integrals in (17) increase more slowly than $t^{2N+1-\epsilon}$ as t tends to infinity from any direction in the complex plane t. Therefore with increasing radius of the large circle in the complex plane t, the entire function in (17) likewise increases more slowly than $t^{2N+1-\epsilon}$, and this means that this function is a polynomial of degree 2N.

If we now calculate from (17) the absorption part in the s channel, we see that the entire function in this part is a polynomial of degree 2N.

Indeed, the integral of A_t' over the right cut is a uniformly converging integral of the entire function s, and consequently makes no contribution to A_s; analogously, the integral of A_u' over the left cut makes no contribution to A_s. We therefore have for the absorption part A_s

$$A_s = \frac{t^{2N+1}}{\pi}\int\frac{\rho_{st}\,dt'}{t'^{2N+1}(t'-t)} + \frac{u^{2N+1}}{\pi}\int\frac{\rho_{us}\,du'}{u'^{2N+1}(u'-u)} + \sum_{p=0}^{2N}t^p\Delta a_p(s) = \frac{t^N}{\pi}\int\frac{\rho_{st}\,dt'}{t'^N(t'-t)} + \frac{u^N}{\pi}\int\frac{\rho_{us}\,du'}{u'^N(u'-u)} - \sum_{p=N}^{2N}\frac{t^p}{\pi}\int\frac{\rho_{st}\,dt'}{t'^{p+1}} - \sum_{p=N}^{2N}\frac{u^p}{\pi}\int\frac{\rho_{us}\,du'}{u'^{p+1}} + \sum_{p=0}^{2N}t^p\,\Delta a_p(s) = A_s'' + \sum_{p=0}^{2N}t^p\rho_{ps}(s), \qquad (18)$$

where A_s'' denotes the first two integrals, and all the remaining terms are lumped in the sum over the powers of t. It is therefore clear that $\rho_{ps}(s) \equiv 0$ when $p > 2N$. This means in turn that all the single integrals of degree higher than 2N vanish.

6. The unitarity condition imposes limitations

on the growth of ρ_{ps} as $s \to \infty$. Let us write down Eq. (18) for $(2N+1)$ values of the angle. We obtain a system of $(2N+1)$ linear equations with $(2N+1)$ unknowns $\rho_{ps}(s)(2q_s^2)^p$ [since $t = 2q_s^2(z-1)$]. Its solution is of the form

$$\rho_{ps}(s) = \frac{1}{(2q_s^2)^p \Delta} \sum_{i=0}^{2N} [A_s(z_i) - A_s''(z_i)] \Delta_{pi}, \quad (19)$$

where Δ —determinant of the system and Δ_{pi} — corresponding cofactor. It is obvious that the partial waves A_s and A_s'' are equal when $l > 2N$, so that we can rewrite (19) in the form

$$\rho_{ps}(s) = \frac{1}{(2q_s^2)^p \Delta} \sum_i A_{s,2N}(z_i) \Delta_{pi} - \frac{1}{(2q_s^2)^p \Delta} \sum_i A''_{s,2N}(z_i) \Delta_{pi} = \Delta\rho_{ps} + \rho_{ps}', \quad (20)$$

where $A_{s,2N}(A''_{s,2N})$ —contribution of the first 2N partial waves to $A_s(A_s'')$. Since $A_s'' = O(s^{2N-\epsilon})$, we have $\rho_{ps}' = O(s^{2N-p-\epsilon})$, and the unitarity condition ($|a_l| \le 1$) yields $\Delta\rho_{ps} = O(1/s^p)$. Thus, the principal part in the asymptotic expression for ρ_{ps} is ρ_{ps}', which is expressed in terms of double spectral functions. We shall henceforth change the notation and take ρ_{0s} and ρ_{1s} to mean their principal parts in the asymptotic expression, i.e., ρ_{0s}' and ρ_{1s}'.

The total amplitude can therefore be written in the form

$$A = \left\{ \frac{s^N t^N}{\pi^2} \int \frac{\rho_{st} ds' dt'}{s'^N t'^N (s'-s)(t'-t)} + \sum_{p=0}^{2N} t^p \frac{s^{2N-p}}{\pi} \int \frac{\rho_{ps} ds'}{s'^{2N-p}(s'-s)} + P_{stu} \right\} + \sum_{p,q} t^p s^q \rho_{pq} + \left\{ \sum_{p=0}^{1} t^p \frac{s^{1-p}}{\pi} \int \frac{\Delta\rho_{ps} ds'}{s'^{1-p}(s'-s)} + P_{stu} \right\}. \quad (21)$$

Let us consider the entire function contained in (21). When $s = \text{const} < 0$, it increases along the large circle of the complex plane t more slowly than $t^{2N-\epsilon}$, and consequently this function is a polynomial in t of degree $2N-1$, and $\rho_{pq} = 0$ for $p > 2N-1$. If we consider (21) for $u = \text{const} < 0$, then analogous arguments yield $\rho_{pq} = 0$ for $p + q > 2N-1$, i.e., the entire function in the amplitude is a polynomial, q.e.d.

7. We now proceed to calculate the single integrals and the subtraction polynomial in terms of the double spectral functions. We first prove some inequalities which we need in what follows. We wish to estimate integrals of the type

$$\frac{s^{2N-p}}{\pi} \int \frac{\rho_{st} ds'}{s'^{2N-p}(s'-s)} \quad (2 \leqslant p \leqslant N-1) \text{ for } s = \text{const},\ t \to \infty.$$

We represent the absorption part of the amplitude in the t channel in the form

$$A_t = \frac{s^{2N-p}}{\pi} \int \frac{\rho_{st} ds'}{s'^{2N-p}(s'-s)} + \frac{s^{2N-p}}{\pi} \int \frac{\rho_{ut} ds'}{s'^{2N-p}(s'-s)} + \sum_{k=0}^{2N-p-1} \frac{s^k}{k!} \frac{d^k}{ds^k} A_t \Big|_{s=0}. \quad (22)$$

The second integral in (22) increases with increasing t no faster than $t^{p-\epsilon}$. Indeed,

$$\left| \frac{s^{2N-p}}{\pi} \int \frac{\rho_{tu} du'}{(1-t-u')^{2N-p}(u'-u)} \right| < \left| \frac{s^{2N-p}}{\pi} \int \frac{t^{N-\varepsilon} u'^{N-\varepsilon} du'}{t^{N-p} u'^N |u'-u|} \right| < Ct^{p-\varepsilon}. \quad (23)$$

Here we have used the limitation on the double spectral functions and the obvious inequality

$$|1-t-u'|^{2N-p} > t^{N-p} u'^N.$$

(The sum of the squares of the masses of the external particles is set equal to unity:

$$\sum_{i=1}^{4} m_i^2 = s + t + u = 1.)$$

We now estimate the derivatives of A_t, which enter in (22):

$$\frac{d^k}{ds^k} A_t \Big|_{s=0} = \frac{1}{(2q_t^2)^k} \frac{d^k}{dz^k} A_t \Big|_{z=1} = \frac{t^{1/2}}{2q_t (2q_k^2)^k} \sum (2l+1) a_l(t) \frac{d^k}{dz^k} P_l(z) \Big|_{z=1}. \quad (24)$$

For the derivatives of Legendre polynomials we have the following bound [6]:

$$\left| \frac{d^k}{dz^k} P_l(z) \right| \leqslant \frac{(l+k)!}{2^k k! (l-k)!} |z + (z^2-1)^{1/2}|^{l-k}, \quad (25)$$

hence

$$\left| \frac{d^k}{dz^k} P_l(z) \right|_{z=1} < l^{2k}. \quad (26)$$

The effective l in the sum (24) is equal to $C\sqrt{t} \times \ln t$, and therefore

$$\left| \frac{d^k}{ds^k} A_t \right|_{s=0} < \frac{C}{(2q_t^2)^k} \sum_{l=0}^{l_{\text{eff}}} l^{2k+1} < C \frac{l_{\text{eff}}^{2k+2}}{(2q_t^2)^k} = Ct \ln^{2k+2} t \quad (27)$$

(this estimate holds true also for $s < 0$).

We thus obtain

$$\frac{s^{2N-p}}{\pi} \int \frac{\rho_{st} ds'}{s'^{2N-p}(s'-s)} = A_t + O(t^{p-\varepsilon}). \quad (28)$$

Using the known bounds for A_t in the physical and in the unphysical regions of s [7,2]

$$|A_t| < \begin{cases} Ct \ln^2 t, & s \leqslant 0 \\ Ct^{1+(N-1)\sqrt{s/s_0}}, & s > 0 \end{cases}, \quad (29)$$

we can rewrite (28) in the form

$$\left|\frac{s^{2N-p}}{\pi}\int\frac{\rho_{st}ds'}{s'^{2N-p}(s'-s)}\right| < \begin{cases} Ct^{p-\varepsilon} & \text{for } s<s_p=s_0\left(\dfrac{p-1}{N-1}\right)^2. \\ Ct^{1+(N-1)\sqrt{s/s_0}} & \text{for } s\geqslant s_p \end{cases} \quad (30)$$

The other inequalities are obtained from (30) by permutations of P_{stu}.

8. The subsequent reasoning is as follows: we take the Mandelstam representation, written in the usual form (21), and transform it algebraically in order to obtain a one-dimensional dispersion relation, for example for fixed $s \le 0$. This yields two dispersion integrals over the left and right cuts, written with two subtractions ($|A| < Ct \ln^2 t$ in the t channel and $|A| < Cu \ln^2 u$ in the u channel), and a polynomial in t. This polynomial is found to be of degree $(2N-1)$, but the limitation on the asymptotic behavior of the amplitude, which follows from the unitarity condition, requires vanishing of all the coefficients of the powers of t above the first. Equating the resultant expressions to zero, we obtain the sought-for connection of the single integrals of degree larger than first and the coefficients of the subtraction polynomial, ρ_{pq} with $p > 1$, with the double spectral functions.

Relegating the cumbersome algebraic derivations to the appendix, we present the final result. For $p \ge N$ we have

$$a_p(s)=\frac{s^{2N-p}}{\pi}\int\frac{\rho_{ps}ds'}{s'^{2N-p}(s'-s)} + \sum_{q=0}^{2N-1-p} s^q\rho_{pq} + P_{2N-1-p}(s). \quad (31)$$

Equating this expression to zero, we get

$$\rho_{ps}\equiv 0 \quad \text{for } p\geqslant N; \qquad \sum_{q=0}^{2N-1-p} s^q\rho_{pq} = -P_{2N-1-p}(s), \quad (32)$$

i.e., only the single integrals of degree smaller than N differ from zero. For $2 \le p \le N-1$ we have

$$a_p(s)=\frac{s^{2N-p}}{\pi}\int\frac{\rho_{ps}ds'}{s'^{2N-p}(s'-s)} - \frac{s^{2N-p}}{\pi^2}\int\frac{dt'}{t'^{p+1}}\int\frac{\rho_{st}ds'}{s'^{2N-p}(s'-s)} - (-1)^p\frac{s^{2N-p}}{\pi^2}\sum_{l=p}^{N-1}\binom{l}{p}\int\frac{du'}{u'^{l+1}}\int\frac{(1-s')^{l-p}\rho_{us}ds'}{s'^{2N-p}(s'-s)} + \sum_{q=0}^{2N-1-p} s^q\rho_{pq} + P_{2N-1-p}(s). \quad (33)$$

The integrals in the right side converge when $s < s_p$ in accordance with (30). Equating (33) to zero, we obtain

$$\frac{s^{2N-p}}{\pi}\int\frac{\rho_{ps}ds'}{s'^{2N-p}(s'-s)} = \frac{s^{2N-p}}{\pi^2}\int\frac{dt'}{t'^{p+1}}\int\frac{\rho_{st}ds'}{s'^{2N-p}(s'-s)} + (-1)^p\frac{s^{2N-p}}{\pi^2}\sum_{l=p}^{N-1}\binom{l}{p}\int\frac{du'}{u'^{l+1}}\int\frac{(1-s')^{l-p}\rho_{us}ds'}{s'^{2N-p}(s'-s)}, \quad (34)$$

which is the sought expression for the single integrals in terms of double spectral functions. For the coefficients of the subtraction polynomial we get

$$\sum_{q=0}^{2N-1-p} s^q\rho_{pq} = -P_{2N-1-p}(s). \quad (35)$$

Let us consider the expressions obtained for the single integrals. When $s < s_p$ these integrals converge uniformly and the integrands are analytic functions of s. Consequently, the single integrals have no singularities in this region. When $s > s_p$, the singularities of the obtained expressions are made up of singularities of the integrand function (cut along the real positive axis from s_0 to infinity) and the singularities due to the behavior of the integrand at infinity. Under the integral signs of (34), on the other hand, we have the sum of the absorption part and the function (28) which converges at infinity. Let, for example, the absorption part A_t behave at infinity like $f(s)t^{l(s)}$, where $l(s)$ is a real growing function below the threshold s_0, which goes into the upper half plane when $s > s_0$. Then

$$\frac{s^{2N-p}}{\pi^2}\int\frac{dt'}{t'^{p+1}}\int\frac{\rho_{st}ds'}{s'^{2N-p}(s'-s)} = \frac{s^{2N-p}f(s)}{\pi^2}\int t'^{l(s)-p-1}dt' + \frac{s^{2N-p}}{\pi^2}\int_{s_0}^{\infty}\frac{\sigma(s')ds'}{s'^{2N-p}(s'-s)}, \quad (36)$$

where the first integral includes terms that grow at infinity, and the second includes the converging expressions. Integrating, we obtain from the first term

$$\frac{s^{2N-p}}{\pi^2}f(s)\,t'^{l(s)-p}\Big|_{t_0}^{\infty}\frac{1}{l(s)-p} = \frac{f'(s)}{l(s)-p}, \quad (37)$$

where $f'(s)$ is some function of s.

When we move in the s plane to the right along the real axis, $l(s)$ increases, and if it passes through an integer point at some value $s < s_0$, the amplitude has a pole at this value of s. When $s > s_0$ the function $l(s)$ goes into the upper half-plane, and at the values of s for which $\mathrm{Re}\, l(s)$ is a positive integer we have resonance. In the amplitude this pole (resonance) is associated with $t^p(z_s^p)$, and consequently has a momentum p.

The inequality (30) establishes the lower bound of the mass of the particle with momentum $p \ge 2$, if the latter exists:

$$m_p \geqslant s_0 \left(\frac{p-1}{N-1}\right)^2. \qquad (38)$$

If we advance the very likely hypothesis [1,2] that a more stringent limitation than indicated by Froissart exists for the amplitudes of the inelastic processes and for the elastic backward-scattering amplitude, for example $|A| < Ct^{1-\epsilon}$, and if such a situation obtains in the t and in the u channels, then the single first-degree integral in the s channel should also be expressed in terms of double spectral functions ρ_{st} and ρ_{su} in accordance with (34). There are grounds for assuming that the single first-degree integrals are always expressed in terms of the double functions, i.e., even if there is elastic forward scattering in the t and u channels. To this end it is merely necessary that a more stringent limitation exist for the real part of the amplitude than for the imaginary part, namely, $|\mathrm{Re}\, A| < \epsilon u$ for all $\epsilon > 0$, so long as $u > u_0(\epsilon)$. In order to obtain this expression, it is sufficient then to take the real part of (21) with $s \leq 0$, divide it by t, and let t go to infinity; we use here the reality of the single integral when $s < 0$.

9. The integral (34) determines the single integrals in the regions $s < s_t$; analytic continuation is needed to find their values in other regions. The function represented by the integral (34) is, in accordance with the Mandelstam hypothesis, analytic in the entire plane except on the real positive axis, where it has a cut and may have poles. Obviously, if we find the jump on the cut (the poles give a jump equal to the δ-function), then the Cauchy integral will provide the analytic continuation in the entire plane.

We introduce an auxiliary variable $y = 1 - 2s_{min}/s$, where $0 < s_{min} < s_p$, which varies from −1 to +1 when s varies from s_{min} to infinity. We then represent ρ_{ps}/s^{2N-p-1} in the form of a sum in Legendre polynomials (or some other orthogonal polynomials):

$$\rho_{ps}/s^{2N-p-1} = \varphi(y) = \sum (2l+1)\alpha_l P_l(y). \qquad (39)$$

Then α_l can be expressed in the form of a finite sum of derivatives of the integral (34) with respect to s in the region $s < s_p$, for example at the point $s = 0$. Indeed,

$$\alpha_l = \frac{1}{2}\int_{-1}^{1} \varphi(y) P_l(y)\, dy;$$

we substitute here

$$P_l(y) = \sum_{k=0}^{l} \beta_{lk}/s'^k$$

and go over to integration with respect to s':

$$\alpha_l = \frac{1}{2}\int_{s_{min}}^{\infty} \frac{\rho_{ps}}{s'^{2N-p-1}} \sum_{k=0}^{l} \frac{\beta_{lk}}{s'^k}\frac{2s_{min}}{s'^2}ds' = s_{min}\sum_{k=0}^{l}\beta_{lk}\int_{s_{min}}^{\infty}\frac{\rho_{ps}ds'}{s'^{2N-p+1+k}}$$

$$= s_{min}\sum_{k=0}^{l}\frac{\beta_{lk}}{(2N-p+k)!}\frac{d^{2N-p+k}}{ds^{2N-p+k}}s^{2N-p}$$

$$\times \int_{s_{min}}^{\infty}\frac{\rho_{ps}ds'}{s'^{2N-p}(s'-s)}\Big|_{s=0}. \qquad (40)$$

Substituting (34), we get

$$\alpha_l = s_{min}\sum_{k=0}^{l}\beta_{lk}\left[\frac{1}{\pi}\int\frac{dt'}{t'^{p+1}}\int\frac{\rho_{st}ds'}{s'^{2N-p+1+k}} + \frac{(-1)^p}{\pi}\sum_{l=p}^{N-1}\binom{l}{p}\right.$$

$$\left.\times\int\frac{du'}{u'^{l+1}}\int\frac{(1-s')^{l-p}\rho_{us}ds'}{s'^{2N-p+1+k}}\right]$$

$$= \frac{s_{min}}{\pi}\int\frac{dt'}{t'^{p+1}}\int\frac{\rho_{st}}{s'^{2N-p+1}}P_l\left(1-\frac{2s_{min}}{s}\right)ds'$$

$$+\frac{(-1)^p s_{min}}{\pi}\sum_{l=p}^{N-1}\binom{l}{p}\int\frac{du'}{u'^{l+1}}\int\frac{(1-s')^{l-p}}{s'^{2N-p+1}}$$

$$\times P_l\left(1-\frac{2s_{min}}{s}\right)ds'. \qquad (41)$$

10. We now calculate the coefficients of the subtraction polynomial. Formulas (32) and (35) give expressions for ρ_{pq} with $p \geq 2$ in terms of the double and single spectral functions. With the aid of a procedure analogous to that used in the calculation, we can determine all the ρ_{pq} except ρ_{01}, ρ_{10}, and ρ_{00}. To determine ρ_{01} and ρ_{10} it is not sufficient to use the limitation on the asymptotic value at fixed s, t, or u ($|A| < s\ln^2 s$), but it is necessary to use also the limitation for the fixed angle ($|A| < \ln^{3/2} s$).

We now obtain an expression for the coefficients of the subtraction polynomial, suitable for all the ρ_{pq} (except ρ_{00}). To this end we rewrite (21) in the form

$$A = A_1 + \sum_{p,q}(1-z)^p(1+z)^q(-2q_u^2)^{p+q}\rho_{pq} + A_2. \qquad (42)$$

We write this formula for $(2N+1)N$ values of the angle z_u and solve the obtained system of equations for ρ_{pq}:

$$\rho_{pq} = \frac{1}{(-2q_u^2)^{p+q}\Delta}$$

$$\times\sum_{i=1}^{(2N+1)N}[A(q_u, z_i) - A_1(q_u, z_i) - A_2(q_u, z_i)\Delta_{pq,i}]. \qquad (43)$$

For the function A_2 we have the obvious limitation $|A_2| < Cu^{\epsilon}$ for $z_u = \text{const}$. Therefore if we now let q_u go to infinity in this formula, then the terms with A and A_2 will tend to zero, and we obtain

$$\rho_{pq} = -\frac{1}{\Delta}\lim_{q_u\to\infty}\frac{1}{(-2q_u^2)^{p+q}}\sum_i A_1(q_u, z_i)\Delta_{pq,i} \tag{44}$$

for all p and q except p = q = 0.

From formulas (34) for the single integrals and (44) for the subtraction-polynomial coefficients we see that they are linearly related with the double spectral functions.

In conclusion I wish to thank I. M. Shmushkevich for valuable remarks.

APPENDIX

The amplitude of the two-particle reaction for $s \leq 0$ should satisfy the one dimensional dispersion relation with two subtractions:

$$A = \frac{t^2}{\pi}\int\frac{A_t dt'}{t'^2(t'-t)} + \frac{u^2}{\pi}\int\frac{A_u du'}{u'^2(u'-u)} + a_0(s) + a_1(s)t$$
$$= I_t + I_u + a_0(s) + a_1(s)t, \tag{A.1}$$

where I_t and I_u denote the integrals along the right and left cuts.

In order to obtain this dispersion relation from (21), we shall transform the individual terms in (21), breaking them down into sums of parts which will enter in I_t or I_u, and a polynomial in t. For the double integral of ρ_{st} we have

$$\frac{s^N t^N}{\pi^2}\int\frac{\rho_{st}ds'dt'}{s'^N t'^N(s'-s)(t'-t)}$$
$$= \frac{1}{\pi^2}\int dt'\left[\frac{t^2}{t'^2(t'-t)} - \sum_{p=2}^{N-1}\frac{t^p}{t'^{p+1}}\right]s^N\int\frac{\rho_{st}ds'}{s'^N(s'-s)}$$
$$= -\sum_{p=2}^{N-1}t^p\frac{s^{2N-p}}{\pi^2}\int\frac{dt'}{t'^{p+1}}\int\frac{\rho_{st}ds'}{s'^{2N-p}(s'-s)}$$
$$+\frac{1}{\pi^2}\int dt'\left[\frac{t^2 s^N}{t'^2(t'-t)}\int\frac{\rho_{st}ds'}{s'^N(s'-s)}\right.$$
$$\left.-\sum_{p=2}^{N-1}\sum_{q=N}^{2N-p-1}t^p s^q\frac{1}{t'^{p+1}}\int\frac{\rho_{st}ds'}{s'^{q+1}}\right]. \tag{A.2}$$

The first integral converges by virtue of the derived inequality (26). In the second integral, the first term represents the part I_t, and the second is a polynomial in t and s.

In exactly the same manner, the double integral of ρ_{us} can be represented in the form

$$\frac{u^N s^N}{\pi^2}\int\frac{\rho_{us}du'ds'}{u'^N s'^N(u'-u)(s'-s)} = -\sum_{p=0}^{N-1}t^p\frac{s^{2N-p}}{\pi}(-1)^p\sum_{l=2}^{N-1}\binom{l}{p}$$
$$\times\int\frac{du'}{u'^{l+1}}\int\frac{(1-s')^{l-p}\rho_{us}ds'}{s'^{2N-p}\ (s'-s)}$$
$$+\frac{1}{\pi^2}\int du'\left[\frac{u^2 s^N}{u'^2(u'-u)}\int\frac{\rho_{us}ds'}{s'^N(s'-s)}\right.$$
$$-\sum_{p=0}^{N-1}\sum_{q=N}^{2N-p-1}t^p s^q(-1)^p\sum_{l=2}^{N-1}\binom{l}{p}\frac{1}{u'^{l+1}}\int\frac{\rho_{us}ds'}{s'^{q+1}}$$
$$\left.\times\sum_{m=0}^{q-N}\binom{l-p}{m}(-s')^m\right]. \tag{A.3}$$

The double integral of ρ_{tu} is first broken up into two by means of the equality

$$\frac{1}{(u'-u)(t'-t)} = \frac{-1}{(t'-t)(s'-s)} - \frac{1}{(u'-u)(s'-s)}, \tag{A.4}$$

and each of these is then broken up into a term representing the part $I_t(I_u)$ and a polynomial in t. As a result we get

$$\frac{t^N u^N}{\pi^2}\int\frac{\rho_{tu}dt'du'}{t'^N u'^N(t'-t)(u'-u)} = \frac{1}{\pi^2}\int dt'$$
$$\times\left[\frac{t^2(1-s-t')^N}{t'^2(t'-t)}\int\frac{\rho_{tu}du'}{u'^N(u'-1+s+t')}\right.$$
$$+\sum_{p=2}^{2N-1}\sum_{q=0}^{2N-p-1}t^p s^q\int\frac{\rho_{tu}du'}{t'^{p+1}s'^{q+1}u'^N}\sum_{k=0}^{q}\binom{N}{k}(-s')^k$$
$$\left.\times\sum_{m=0}^{2N-p-q-1}\binom{N-k}{m}(-t')^{N-k-m}\right]$$
$$+\frac{1}{\pi^2}\int du'\left[\frac{u^2(1-s-u')^N}{u'^2(u'-u)}\int\frac{\rho_{tu}dt'}{t'^N(t'-1+s+u')}\right.$$
$$-\sum_{p=2}^{2N-1}\sum_{q=0}^{2N-1-p}t^p s^q\int\frac{\rho_{tu}dt'}{t'^N u'^N}\sum_{k=q}^{2N-1-p}\sum_{n=0}^{q}\frac{(1-u')^{2N-1-p-n-k}}{s'^{n+1}}$$
$$\times\sum_{m=k-q}^{k-n}\binom{N}{m}\binom{m}{k-q}\binom{2N-p-m-1}{k-n-m}(-1)^{N+q-n+m}$$
$$+\sum_{p=0}^{1}\frac{t^p}{u'^2}\sum_{m=0}^{p}\binom{2}{m}(t-1)^{2-m}(1-s-u')^{N-1-p-m}$$
$$\left.\times\int\frac{\rho_{tu}dt'}{t'^N(t'-1+s+u')}\right]. \tag{A.5}$$

An analogous breakdown of the sum of single integrals into parts yields

$$\sum_{p=0}^{N-1}s^p\frac{u^{2N-p}}{\pi}\int\frac{\rho_{pu}du'}{u'^{2N-p}(u'-u)} = \frac{1}{\pi}\int du'\left[\frac{u^2}{u'^2(u'-u)}\sum_{p=0}^{N-1}s^p\rho_{pu}\right.$$
$$-\sum_{p=0}^{2N-1}\sum_{q=0}^{2N-1-p}t^p s^q(-1)^p$$
$$\left.\times\sum_{m=0}^{\min[q,\,N-1]}\sum_{n=\max[2,\,p]}^{2N-1-m}\frac{1}{u'^{n+1}}\binom{n-p}{q-m}\binom{n}{p}(-1)^{q-m}\rho_{mu}\right], \tag{A.6}$$

$$\sum_{p=0}^{N-1}u^p\frac{t^{2N-p}}{\pi}\int\frac{\rho_{pt}dt'}{t'^{2N-p}(t'-t)}$$
$$= \frac{1}{\pi}\int dt'\left[\frac{t^2}{t'^2(t'-t)}\sum_{p=0}^{N-1}(1-s-t')^p\rho_{pt}\right.$$

$$- \sum_{p=2}^{2N-1} \sum_{q=0}^{\min[N-1,\,2N-1-p]} t^p s^q \frac{(-1)^q}{t'^{p+1}} \sum_{l=q}^{\min[N-1,\,2N-1-p]}$$

$$\times \binom{l}{q} \sum_{n=l}^{N-1} \binom{n}{l} (-t')^{n-l} \rho_{nt} \Bigg]. \qquad \text{(A.7)}$$

If we now add all the resultant expressions, then the sum of the first term in the second integral of (A.2) with the first term in the first integral of (A.5) and the first term in (A.7) will yield I_t, while the sum of the first term in the second integral of (A.3) with the first term in the second integral (A.5) and the first term in (A.6) yields I_u. All that is left is a polynomial of degree $(2N-1)$ in t.

[1] G. Chew, S-Matrix Theory of Strong Interactions, N.Y., (1961).
[2] M. Froissart, Phys. Rev. **123**, 1053 (1961).
[3] F. E. Low and O. W. Grennberg, Phys. Rev. **124**, 2047 (1961).
[4] E. C. Titchmarsch, The Theory of Functions, Oxford Univ. Press, 1939 (Russ. Transl. Gostekhizdat, 1951, p. 205).
[5] Kinoshita, Loeffel, and Martin, Phys. Rev. Lett. **10**, 460 (1963).
[6] G. Szego, Orthogonal Polynomials, Am. Math. Soc. N.Y., 1939 (Russ. Transl. Fizmatgiz, 1962).
[7] A. Martin, Lecture Notes Prepared for the Scottish Summer School of Theor. Phys. (1963).

Translated by J. G. Adashko
30