ON THE COMPLETENESS OF THE "COMPLETE SCATTERING EXPERIMENT"

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The complete experiment on the scattering of particles with spins (0, 0), (0, ½), and (½, ½) is consistent with only two scattering amplitudes satisfying the unitarity condition and analytic in $\cos\theta$ in some vicinity of the segment $-1 \leq \cos\theta \leq 1$ if this experiment is invariant with respect to helicity inversion, and with only one amplitude if this invariance is violated.

1. FORMULATION OF THE PROBLEM

THE analysis of the scattering of particles with different spins has met with considerable success in recent years. Among the processes investigated are the scattering of pions by protons and the interaction between protons and protons or neutrons. It is quite probable that the obtained results are unique although this has not yet been investigated.

The purpose of an analysis of particle scattering at a fixed energy is to determine the scattering amplitudes from the results of an experiment equivalent to the "complete experiment." The concept of the complete experiment was introduced by Puzikov, Ryndin, and Smorodinskiĭ [1]. This, still leaves open the question of the conditions for the existence and uniqueness of the solution of the unitarity integral equations, and also the possibility of obtaining their solution by some numerical means, if the observed quantities are subject to experimental errors. It is natural to regard as complete only an experiment that yields a unique result for the measured parameters or, more accurately, a probability distribution with a single vertex. In light of this definition, the completeness of the "complete experiments" mentioned in [1] calls for verification.

We consider here only the elastic scattering of particles with spins (0, 0), (0, ½), and (½, ½). For the first of these the complete experiment consists in measuring the cross section $\sigma(\theta)$. For particles with spins (0, ½) we consider only the set $\sigma(\theta)$, $P_n(\theta)$, for although measurements of the components of the tensor that relates the initial and final polarizations of a beam (or target) of spin ½ particles are feasible, they are most inconvenient experimentally. For the scattering of identical spin ½ particles we shall investigate only the set σ, P_n, D_{nn}, K_{nn}, C_{nn}. Each of the sets indicated here will be called for brevity the normal complete experiment. The uniqueness of the analysis of some other sets that are also equivalent to the complete experiment will be considered separately.

The complete experiment, as noted by MacGregor et al [2], cannot be carried out in pure form, since no angular distribution can be measured at all angles and with absolute accuracy. Nonetheless, it is quite convenient to use the concept of the complete experiment to prove the uniqueness of an analysis of real experiments, since it indicates the situation for sufficiently subtle experiments.

2. COMPATIBILITY AND UNIQUENESS

The existence and uniqueness of the solution of the problem of determining the scattering amplitude from the results of a normal complete experiment will be examined under two supplementary assumptions: either the cross sections are sufficiently small, or else the scattering amplitude can be uniformly expanded in spherical functions.

The conditions for the applicability of the first case are apparently quite rarely satisfied so that we shall consider it only for spinless particles. The integral equation for the phase shift $\alpha(\theta)$ in terms of the cross section $\sigma(\theta)$ is of the form [1]

$$\alpha(\theta) = \arcsin \frac{k}{4\pi}\int d\Omega \left(\frac{\sigma(\theta')\,\sigma(\theta'')}{\sigma(\theta)}\right)^{1/2} \cos[\alpha(\theta') - \alpha(\theta'')]. \quad (2.1)$$

A simple generalization of Newton's method of solving equations of the type $\alpha = F(\alpha)$ shows that Eq. (2.1) can be solved by successive approximations (starting with $\alpha = c$), and has a unique solution in the interval $-\pi/2 < \alpha < \pi/2$, if

$$\left| \int_0^{\pi} \frac{\delta F[\alpha(\theta)]}{\delta\alpha(\theta)} \sin\theta d\theta \right| < 1$$

(and a second solution $\alpha_2 = \pi - \alpha_1$ in the interval $\pi/2 < \alpha < 3\pi/2$). Calculating this derivative and using the identity

$$\left| \int a(\theta) \sin f(\theta) d\theta \right| < \left\{ 1 - \left[\int a(\theta) \cos f(\theta) d\theta \right]^2 \right\}^{1/2},$$

which is valid when $\int |a(\theta)| d\theta < 1$, we find that in order for the successive approximations to converge it is sufficient to have

$$\frac{k}{4\pi} \int d\Omega \left(\frac{\sigma(\theta')\sigma(\theta'')}{\sigma(\theta)} \right)^{1/2} < \frac{1}{2}. \tag{2.2}$$

On the other hand, if $|\sin\alpha(\theta)| = 1$ for any angle in the interval $0 < \theta < \pi$, then the curves $\alpha(\theta)$ and $\pi - \alpha(\theta)$ cross and the successive approximations for the corresponding $\sigma(\theta)$ cannot converge. The latter case is possible, although not obligatory, if the left half of (2.2) is larger than unity. On the other hand, if this quantity lies between $1/2$ and 1, the approximations converge for some $\sigma(\theta)$ and diverge for others. This is confirmed by calculations with synthetic examples[3].

The condition (2.2) for the convergence of the successive approximations can be obtained also from the general theory of nonlinear integral equations[4]. On the other hand, in the more general case this theory does not yield a proof for the presence of only two solutions of (2.1).

We note that we had no need for any conditions for the existence of the solution. Apparently the arbitrary complete-experiment curves remain compatible (if $\sigma(\theta) \geq 0$) also when condition (2.2) is violated and also for particles with spin. The situation changes appreciably however, if we stipulate in addition that the scattering amplitude be representable by a sum of a known function and a uniformly and absolutely converging series of spherical functions, particularly a finite sum (to this end it is sufficient that the amplitude be analytic in $\cos\theta$ in a certain vicinity of the interval $-1 \leq \cos\theta \leq 1$). Uniform and absolute convergence of this series is essential for the products of such series to be integrable term by term in the unitarity equations. On the other hand, we arrive at such series if the forces between the particles have a finite radius of action.

Since

$$|P_l^m(\cos\theta)| \leqslant \sin^m\theta \frac{(l+m)!}{(l-m)!\, 2^m m!}, \tag{2.3}$$

it is sufficient, for uniform and absolute convergence of the series in the scattering amplitude, that the limit of the product l^{2m+1} by the coefficient of $P_l^m(\cos\theta)$ vanish as $l \to \infty$.

The elements of the complete experiment, regarded as a uniformly convergent series in $P_l(\cos\theta)$ or $P_l^1(\cos\theta)$, are in general not compatible if the observed quantities are chosen by guesswork or do not correspond to the assumed particle spins. Correct interpretation of the particles guarantees this compatibility automatically within the limits of experimental error. Unfortunately, the conditions for analytic compatibility of the total-experiment curves cannot be expressed as yet in a readily conceivable form, except in the simplest cases. Such conditions would allow us to predict a part of the most complicated elements of the complete experiment, which are the most difficult to measure, from elements making up the necessary experiment[5,6], accurate to within one or several continuous parameters.

We present the following examples of compatibility conditions:

1) The curve

$$\sigma(\theta) = k^{-2}[A_0 P_0 + 6A_1 P_1(\cos\theta) + 6A_2 P_2(\cos\theta)]$$

can be regarded as the cross section for S, P scattering of spinless particles, if

$$0 \leqslant A_0 \leqslant 1, \quad 0 \leqslant A_2 \leqslant 1,$$

$$\begin{vmatrix} A_0 - 3A_2 & A_1 & A_0 - 3A_2 \\ A_1 & A_2 & A_2 \\ A_0 - 3A_2 & A_2 & 1 \end{vmatrix} = 0;$$

2) the curves

$$\sigma(\theta) = k^{-2}[A_0 + A_1 \cos\theta]$$

and

$$\sigma(\theta) P(\theta) = B_1 k^{-2} \sin\theta$$

can correspond to $S_{1/2}$, $P_{1/2}$ scattering of particles with spins $(0, 1/2)$ if

$$-2 \leqslant A_1 \leqslant 2, \quad -2 \leqslant B_1 \leqslant 2,$$

$$A_0 = B_1^2 / (A_1^2 + B_1^2) + A_1.$$

3) The curve

$$\sigma(\theta) = k^{-2}[A_0 P_0 + A_1 P_1(\cos\theta) + A_2 P_2(\cos\theta)]$$

can be the cross section of S, P scattering of particles with spins $(0, 1/2)$ if[7]

$$A_0^2 \leqslant A_0 + A_1 + A_2, \quad (A_0 - A_1)^2 \leqslant A_0 - A_1 + A_2.$$

The need for satisfying the compatibility condition casts doubts on any successful use of numerical methods to find the analytic solutions of the unitarity conditions in the presence of experimental errors in the complete-experiment curves, unless we resort to finite parametrization.

Under the analyticity condition, the solution of the integral unitarity equations is equivalent to the solution of an infinite system of equations for the phase shifts and the mixing parameters in terms of the coefficients of the expansion of the experimental complete-experiment curves in $P_l(\cos\theta)$ or $P_l^1(\cos\theta)$. To prove the uniqueness or, in the worst case, the duality of the solution of this problem we shall consider in Secs. 4—6, for each separate spin combination, auxiliary problems in which the number n of nonvanishing parameters of the scattering matrix (first phase shifts and mixing parameters in natural [5] order) will be assumed finite and specified. The compatibility of the systems will be ensured by the fact that we express the coefficients of expansion of the observed curves in terms of one of the solutions (initial solution).

The number of parameters n cannot be perfectly arbitrary, since the auxiliary problems should have the same symmetry as the complete problem, in particular—symmetry with respect to the helicity inversion operation. To this end it is necessary to regard as different from zero all the parameters which go over into each other under this operation. Thus, for particles with spins $(0, \frac{1}{2})$ the number of phase shifts should be even, and for the scattering of two identical particles with spin $\frac{1}{2}$ the parameters $\delta_{j-1,j}$, δ_j, and $\delta_{j+1,j}$ should be included simultaneously.

We shall show that in each of the three cases under consideration the auxiliary problem, for arbitrary n satisfying the limitation indicated above, has exactly two solutions, which go over into each other under helicity inversion. From this it follows, in particular, that the more realistic "necessary experiment" [5,6] for arbitrary n has actually not more than a two-valued interpretation. We shall then proceed to the limit as $n \to \infty$.

3. METHODS OF PROVING TWO-VALUEDNESS

For particles without spin, the proof of the two-valuedness of the solution of the system of $2n-1$ equations with n unknown phases can be obtained by considering the algorithm of its successive solution, starting with the equation containing the coefficient of the highest polynomial of degree $L = n-1$. Solving successively the last n equations, we obtain 2^n solutions. Using the remaining $n-1$ equations, we can show that each of these decreases the number of solutions by not less than one-half, after which there remain only two solutions with $\delta_{l1} = \delta_{l0}$ and $\delta_{l2} = -\delta_{l0}$. For particles with spins $(0, \frac{1}{2})$, an analogous algorithm gives $2^{n/2}$ solutions, compatible with the last n/2 pairs of equations, and the use of the remaining equations decreases the number of solutions to 2, with $\delta_{j1}^{\epsilon} = \delta_{j0}^{\epsilon}$ and $\delta_{j2}^{\epsilon} = -\delta_{j0}^{\epsilon}$.

We shall not derive this proof in detail because, first, we shall present below a much simpler proof (which, to be sure, does not give the algorithm for solving the system); second, the algorithm for the successive solution of the system, starting with large L, is not convenient from the computation point of view when experimental errors are taken into account; third, for spin $\frac{1}{2}$ particles such an analysis leads not to quadratic equations, but to equations of the eighth degree, the investigation of which is a very complicated matter.

To prove the two-valuedness of the solution of the systems of equations under consideration, we can use previously obtained results [5], where the number of solutions of a system of n equations of the type in question, with n unknowns, coincides with the number of solutions of a simplified system, obtained from the initial system by assuming all the parameters contained in it to be very small, and by retaining only the lowest nonvanishing powers of these parameters. More accurately, the solutions of the simplified system correspond to the solutions of the complete system, which vanish when the initial solution vanishes. However, if the unknown quantities include the total cross section expressed in the form of the sum of the square of the moduli of such functions of all the parameters, which vanish together with them, then the complete system cannot have any other solutions, and the investigation of the simplified system yields the total number of solutions.

Thus, for each spin combination, it is sufficient to consider a successive solution of the simplified system, and to obtain the number of solutions that vanish simultaneously with the initial solutions. In addition, the coefficient of the square of each of the parameters of the initial solution in the expression for the total cross section, obtained during each stage of the solution of the system, cannot exceed the corresponding coefficient for the initial solution, since all the terms of the total cross section are positive and too large a coefficient cannot be offset in any way during the succeeding stages of the system solution.

4. SPINLESS PARTICLES

The expression for the scattering amplitude is well known [see, for example, the book by Landau and Lifshitz [8], formula (122.10)]. The system of

$2n-1$ equations with n phases, the number of solutions of which is to be determined, is of the form

$$\sum_{l,\,l'=0}^{n-1}(2l+1)(2l'+1)\begin{pmatrix} l & l' & L\\ 0 & 0 & 0\end{pmatrix}^2$$
$$\times \sin\delta_l \sin\delta_{l'}\cos(\delta_l-\delta_{l'}) = A_L, \qquad (4.1)$$

where $(2L+1)k^{-2}A_L$ are the coefficients of the expansion of $\sigma(\theta)$ in terms of $P_l(\cos\theta)$.

Applying the method of simplified equations to the system (4.1), we find that the last equation ($L = 2n-2$) is quadratic in δ_{n-1} and has two solutions, $\delta_{n-1,1} = \delta_{n-1,0}$ and $\delta_{n-1,2} = -\delta_{n-1,0}$, and that the succeeding $n-1$ equations are linear with respect to the new phase shifts. Consequently, we conclude immediately that the system (4.1) has only two solutions.

5. PARTICLES WITH SPINS $(0, 1/2)$

The scattering matrix is given in the cited book[8] [formulas (138.4) and (138.5)].[1)] The investigated system of equations with n phase shifts (n even) is written in the form

$$\sum_{j,\,j'=1/2}^{(n-1)/2}(2j+1)(2j'+1)\sum_{\varepsilon,\,\varepsilon'=\pm1}(2l+1)(2l'+1)$$
$$\times\begin{pmatrix} l & l' & L\\ 0 & 0 & 0\end{pmatrix}^2\begin{Bmatrix} j & j' & L\\ l' & l & 1/2\end{Bmatrix}^2\{\sin\delta_j^{\varepsilon}\sin\delta_{j'}^{\varepsilon'}\cos(\delta_j^{\varepsilon}-\delta_{j'}^{\varepsilon'})$$
$$+\sin\delta_j^{-\varepsilon}\sin\delta_{j'}^{-\varepsilon'}\cos(\delta_j^{-\varepsilon}-\delta_{j'}^{-\varepsilon'})\} = A_L, \qquad (5.1)$$

$$\sum_{j,\,j'=1/2}^{(n-1)/2}(2j+1)(2j'+1)\sum_{\varepsilon,\,\varepsilon'=\pm1}(2l+1)(2l'+1)$$
$$\times\begin{pmatrix} l & l' & L\\ 0 & 0 & 0\end{pmatrix}^2\begin{Bmatrix} j & j' & L\\ l' & l & 1/2\end{Bmatrix}\begin{Bmatrix} L & l' & l\\ 1 & 1/2 & 1/2\\ L & j' & j\end{Bmatrix}$$
$$\times\{\sin\delta_j^{\varepsilon}\sin\delta_{j'}^{\varepsilon'}\sin(\delta_j^{\varepsilon}-\delta_{j'}^{\varepsilon'})$$
$$-\sin\delta_j^{-\varepsilon}\sin\delta_{j'}^{-\varepsilon'}\sin(\delta_j^{-\varepsilon}-\delta_{j'}^{-\varepsilon'})\} = B_L, \qquad (5.2)$$

where $\frac{1}{4}(2L+1)k^{-2}A_L$ and $k^{-2}[3(2L+1)^3/2L(L+1)]^{1/2}B_L$ are the coefficients of expansion of $\sigma(\theta)$ in terms of $P_L(\cos\theta)$ and of $\sigma(\theta)P(\theta)$ in terms of $P_L^1(\cos\theta)$, while the phase shifts δ_j^{ϵ} correspond to states with total angular momentum j and parity ϵ.

The last two equations of (5.1) and (5.2) (with $L = n-1$) assume for small phase shifts with angular momentum $J = (n-1)/2$ the form

$$\delta_J^+\delta_J^- = \delta_{J0}^+\delta_{J0}^-,\quad \delta_J^+\delta_J^-(\delta_J^+-\delta_J^-)$$
$$= \delta_{J0}^+\delta_{J0}^-(\delta_{J0}^+-\delta_{J0}^-) \qquad (5.3)$$

and have two solutions: $\delta_{J1}^{\epsilon} = \delta_{J0}^{\epsilon}$ and $\delta_{J2}^{\epsilon} = -\delta_{J0}^{\epsilon}$.

The pair of equations determining the phase shifts of the state with angular momentum j from the phase shifts with angular momentum J are of the form

$$\delta_j^+\delta_J^+ + \delta_j^-\delta_J^- = \delta_{j0}^+\delta_J^+ + \delta_{j0}^-\delta_J^-,$$
$$\delta_j^+\delta_J^+(\delta_j^+-\delta_J^+) - \delta_j^-\delta_J^-(\delta_j^--\delta_J^-)$$
$$= \delta_{j0}^+\delta_J^+(\delta_{j0}^+-\delta_J^+) - \delta_{j0}^-\delta_J^-(\delta_{j0}^--\delta_J^-). \qquad (5.4)$$

One of the solutions of the system (5.4) coincides with the initial solution, while the second is useless since it does not vanish when $\delta_{j0}^+ = \delta_{j0}^- = 0$, except for the case $\delta_J^- = -\delta_J^+$, when the two solutions of the system (5.3) coincide; in the latter case $\delta_{j2}^+ = -\delta_{j0}^-$ and $\delta_{j2}^- = -\delta_{j0}^+$. Therefore during each stage of the successive solution of the system the number of its solutions does not increase, and we conclude that the system (5.1) and (5.2) has two solutions. The case of odd n is obtained from that considered here by imposing the condition $\delta_J^+ = 0$ (or $\delta_J^- = 0$); consequently, in this case the solution is unique (except for $n = 1$, when $P \equiv 0$).

6. IDENTICAL PARTICLES WITH SPIN $1/2$

The scattering amplitude of two identical particles with spin $1/2$ is given in several papers (see, for example, [6,9]). The complete system of equations for the determination of the scattering-matrix parameters when the angular distributions of the normal complete experiment are known is quite cumbersome. We therefore proceed immediately to the simplified equations. The system that contains the parameters of the state with the highest total angular momentum J participating in the scattering can be written in the form

$$(u_J^2 - v_Jw_J)(v_J - w_J + u_J/(J(J+1))^{1/2})$$
$$= (u_{J0}^2 - v_{J0}w_{J0})(v_{J0} - w_{J0} + u_{J0}/(J(J+1))^{1/2}), \qquad (6.1)$$

$$(w_J + 2(J(J+1))^{1/2}u_J)^2 = (w_{J0} + 2(J(J+1))^{1/2}u_{J0})^2, \qquad (6.2)$$

$$v_Jw_J - u_J^2 = v_{J0}w_{J0} - u_{J0}^2, \qquad (6.3)$$

$$y_J^2 = y_{J0}^2, \qquad (6.4)$$

$$y_J[(J-1)x_{J-1} + v_J + (J/J+1))w_J - 2u_J(J/(J+1))^{1/2}]$$
$$= y_{J0}[(J-1)x_{J-1,0} + v_{J0} + (J/(J+1))w_{J0}$$
$$- 2u_{J0}(J/(J+1))^{1/2}], \qquad (6.5)$$

where the unknown quantities are expressed in terms of the elements of the unitary matrix $S_{\lambda l}^j$ in the following fashion:

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1) There is a misprint in [8]: there is no need for i in the denominator for the coefficient B.

$$2ix_{J-1} = S^{J-1,1}_{J-1,J-1} - 1, \quad 2iy_J = S^{J0}_{JJ} - 1,$$

$$2iv_J = S^{J1}_{J-1,J-1} - 1, \quad 2iw_J = S^{J1}_{J+1,J+1} - 1, \quad 2iu_J = S^{J1}_{J+1,J-1}.$$

The system (6.1)–(6.3) has, in addition to the initial solution, three solutions, in which v_J, w_J, and u_J are expressed linearly in terms of v_{J0}, w_{J0}, and u_{J0} with the aid of the matrices

$$U_2 = \frac{1}{1+4t}\begin{pmatrix} -1 & -(1+2t)^2/t & -2(1+2t)/\sqrt{t} \\ -4t & -1 & -4\sqrt{t} \\ 2\sqrt{t} & (1+2t)/\sqrt{t} & 3+4t \end{pmatrix}, \tag{6.6}$$

$$U_3 = \frac{1}{1+4t}\begin{pmatrix} -1 & -4t & 4\sqrt{t} \\ -4t & -1 & -4\sqrt{t} \\ 2\sqrt{t} & -2\sqrt{t} & 1-4t \end{pmatrix}, \tag{6.7}$$

$$U_4 = \begin{pmatrix} 1 & 1/t & 2/\sqrt{t} \\ 0 & 1 & 0 \\ 0 & -1 & -1/\sqrt{t} \end{pmatrix}, \tag{6.8}$$

where $t = J(J+1)$. On the other hand, v_J, w_J, and u_J are contained in the total scattering cross section in the form of the combination $v_J^2 + w_J^2 + 2u_J^2$. From (6.7) we find that

$$v_{J3}^2 + w_{J3}^2 + 2u_{J3}^2 = v_{J0}^2 + w_{J0}^2 + 2u_{J0}^2,$$

whereas for the second and fourth solution w_{J0}^2 enters into this sum with a factor $(1 + 1/J(J+1))^2 > 1$.

Equation (6.4) has two solutions, each of which leads, in combination with the initial solution or with (6.7) with the aid of (6.5), to $x_{J-1,1} = x_{J-1,0}$ and $x_{J-1,3} = -x_{J-1,0}$, respectively. The other two solutions drop out, since they contain in the expressions for x_{J-1} terms that are proportional to v_{J0}, w_{J0}, and u_{J0}, in contradiction to the total cross section. If $u_{J0}^2 - v_{J0}w_{J0} = 0$, then (6.1) is replaced by

$$(v_J + w_J)^4(v_J - w_J + (v_Jw_J)^{1/2}/(J(J+1))^{1/2}) = (v_{J0} + w_{J0})^4(v_{J0} - w_{J0} + (v_{J0}w_{J0})^{1/2}/(J(J+1))^{1/2}),$$

which together with (6.2) and (6.3) leads likewise to only two solutions: the initial one and (6.7), compatible with the requirement $v_J^2 + w_J^2 + 2u_J^2 \le (v_{J0} + w_{J0})^2$. If $y_{J0} = 0$, then it is necessary to consider in place of (6.5) the equation

$$\left[(J-1)x_{J-1} + v_J + \frac{J}{J+1}w_J - 2u_J\left(\frac{J}{J+1}\right)^{1/2}\right]^2 = \left[(J-1)x_{J-1,0} + v_{J0} + \frac{J}{J+1}w_{J0} - 2u_{J0}\left(\frac{J}{J+1}\right)^{1/2}\right]^2,$$

which leads to the same results for x_{J-1}. We see that for the parameters of the state with the highest angular momentum there are only two solutions and, as indicated earlier[10], the initial solution goes over into (6.7) under helicity inversion.

The system of equations for the determination of the parameters of the state with angular momentum j from the parameters with the maximum angular momentum J can be written in the form

$$\frac{J(2j+1)^2}{J+1}\left(v_J - w_J + \frac{u_J}{(J(J+1))^{1/2}}\right) \times [u_j^2 + w_j(w_j - v_j - w_J) - u_{j0}^2 - w_{j0}(w_{j0} - v_J - w_J)]$$
$$+ \frac{j(2J+1)^2}{j+1}\left[\left(v_j - w_j + \frac{u_j}{(j(j+1))^{1/2}}\right) \times \left(u_J^2 + w_J(w_J - v_j - w_j)\right) - \left(v_{j0} - w_{j0} + \frac{u_{j0}}{(j(j+1))^{1/2}}\right) \times \left(u_J^2 + w_J(w_J - v_{j0} - w_{j0})\right)\right] = 0, \tag{6.9}$$

$$(w_J + 2(J(J+1))^{1/2}u_J)(w_j + 2(j(j+1))^{1/2}u_j - w_{j0} + 2(j(j+1))^{1/2}u_{j0}) = 0, \tag{6.10}$$

$$\frac{j(2J+1)^2}{J+1}w_J(v_j - v_{j0}) + \frac{J(2j+1)^2}{j+1}v_J(w_j - w_{j0}) + \frac{2j(2J+1)(J-j)}{(J+1)(j(j+1))^{1/2}}w_J(u_j - u_{j0})$$
$$+ \frac{2J(2j+1)(j-J)}{(j+1)(J(J+1))^{1/2}}u_J(w_j - w_{j0}) - \frac{2(2j+1)(2J+1)\sqrt{jJ}}{[(j+1)(J+1)]^{1/2}}u_J(u_j - u_{j0}) + \frac{(J-j)^2}{(j+1)(J+1)}w_J(w_j - w_{j0}) = 0, \tag{6.11}$$

$$y_J(y_j - y_{j0}) = 0, \tag{6.12}$$

$$(2J+1)y_J\left[(j-1)(x_{j-1} - x_{j-1,0}) + v_j - v_{j0} + \frac{j}{j+1}(w_j - w_{j0}) - 2\left(\frac{j}{j+1}\right)^{1/2}(u_j - u_{j0})\right]$$
$$+ (2j+1)(y_j - y_{j0})\left[(J-1)x_{J-1} + v_J + \frac{J}{J+1}w_J - 2\left(\frac{J}{J+1}\right)^{1/2}u_J\right] = 0. \tag{6.13}$$

The system (6.9)–(6.11) has two solutions: the initial solution and a solution that does not vanish for zero initial solution. Then the equations (6.12) and (6.13) have likewise only the initial solution. When $w_J + 2(J(J+1))^{1/2}u_J = 0$ there appears in place of (6.10) the equation

$$(w_j + 2(j(j+1))^{1/2}u_j)^2 = (w_{j0} + 2(j(j+1))^{1/2}u_{j0})^2,$$

which leads in conjunction with (6.9) and (6.11) to four solutions, two of which were mentioned earlier. The third vanishes together with the initial solution, but leads in the sum $v_j^2 + w_j^2 + 2u_j^2$ to a coefficient $(1+1/j(j+1))^2 > 1$ for w_{j0}^2. The fourth solution vanishes when the initial solution vanishes only under the supplementary condition $v_J = 2(J(J+1))^{1/2} u_J$, when (6.7) coincides with the initial solution; in this case the fourth solution of the system is written in the form (6.7) in which J is replaced by j, and coincides with one of the solutions of this system, obtained after substituting (6.7) in it in place of the initial solution.

Consequently, an analysis of the scattering of spin $\frac{1}{2}$ particles has for arbitrary $n = 5m+2$ only two solutions, of which one goes over into the other under helicity inversion. When $n = 5m+3$ (when $v_{J0} = w_{J0} = u_{J0} = y_{J0} = 0$), there are also two solutions, and when $n = 5m+4$, $n = 5m+5$, and $n = 5m+1$ there is only one solution (the states are taken in natural order).

7. TRANSITION TO THE LIMIT

Let us now lift the requirement that the number of parameters n be finite. We assume here that the unitary scattering matrices can be represented in the form of the product

$$S = S_\alpha S_\beta, \tag{7.1}$$

where the matrices S_β are known, and the elements of the matrices S_α satisfy the conditions of uniform and absolute convergence. In particular, for particles with spins (0, 0) and (0, $\frac{1}{2}$) this means that the phases are regarded as consisting of two terms:

$$\delta_l = \alpha_l + \beta_l, \tag{7.2}$$

where the phase shifts β_l are known, and the phase shifts β_l decrease with increasing l more rapidly than l^{-2}. The latter, as is well known, occurs if the corresponding potential decreases more rapidly than r^{-3} as $r \to \infty$. The phase shifts β_l can decrease slowly or may even increase logarithmically, if they are due to electromagnetic interaction, but the amplitudes corresponding to these slowly-decreasing terms can be summed exactly. After such a partial summation we can assume that the series contained in the amplitude converge absolutely and uniformly.

In the case of a transition to the limit as $n \to \infty$, depending on the behavior of β_l at large values of the angular momentum, two cases can be encountered. If the β_l decrease so rapidly that the already-mentioned conditions for them are satisfied, as is the case, for example, for the nucleon scattering parameters in the one-meson-exchange model, then both sequences of amplitudes obtained in the auxiliary problems converge uniformly to two different solutions of the initial problem. On the other hand, if the β_l decrease slowly or do not decrease, then the solution is made up by the limit of only that sequence for which the correct solution of the auxiliary problems is chosen, for example, the correct sign of the phase shifts, for only in this case does separation of the summed amplitude of the long-range forces ensure uniform convergence of the series. Therefore the problem of the scattering analysis has a unique solution in the second case.

Taking account now of the errors in the experimental data, we can state that when the measurements are sufficiently accurate the depth of one (or two) among all the minima of the χ^2 sum becomes considerably larger than the remaining ones, and approaches the number of degrees of freedom, and the probability that such a random sequence of measurements is compatible with any other (third) solution tends to zero.

The depths of the two principal minima can be different not only because of the influence of the long-range forces, but also when account is taken of relativistic corrections to the expressions for the observed effects in terms of the scattering amplitude. These corrections, as indicated in [10], violate the invariance of the expressions under helicity inversion (this pertains only to experiments outside the normal complete experiment). Of course, the equivalence of the two solutions of the problem is lost if in some of the experiments the magnetic field is used for spin flip, or the targets have polarizing or analyzing properties that are known beforehand (including the sign of the polarization).

Summarizing and recalling that a correctly interpreted complete experiment satisfies the compatibility conditions automatically, we can conclude that if the influence of the long-range forces is sufficiently strong and the properties of the polarizers and analyzers are known the normal complete experiment at a single energy is complete for all three spin combinations considered here. On the other hand, if knowledge of the electromagnetic forces and properties of the targets cannot be used, then only the observation of the scattering cross section of spinless particles is complete, for in the case of interactions of particles with spins the two-valuedness can be eliminated by introducing a magnetic field. No account is taken here of the additional information which can be obtained from the dispersion relations and from a study of the energy dependence of the scattering parameters if the volume of the experimental material is sufficiently large.

We have taken the limit as $n \rightarrow \infty$ under the assumption that all but the first n parameters are equal to zero, since such a method of taking the limit is convenient to prove duality at $n = \infty$. In practice, however, the analysis is frequently carried out with a finite number n of the varied parameters α_l and with an infinite number of parameters β_l determined from some likely physical model of the interaction. If the experiment under consideration does not include fewer points than the corresponding necessary experiment, then it follows from our analysis that the scattering matrix is determined for this case uniquely, as a result of which the depths of two principal minima for each sufficiently large n and for sufficiently small errors of the points are completely different, even if the series with the specified parameters converge uniformly and absolutely. In addition, the parameters obtained from the principal minimum are much less sensitive to the choice of n than for the remaining solutions.

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106