CONTRIBUTION TO THE THEORY OF NONLINEAR INTERACTION OF WAVES IN A MAGNETOACTIVE ANISOTROPIC PLASMA

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Submitted to JETP editor February 12, 1965


General equations that describe nonlinear interaction of waves in a magnetoactive anisotropic plasma are obtained. Explicit expressions are derived for the probabilities of scattering of normal waves by plasma electrons and ions and for the probabilities of the decay processes.

1. INTRODUCTION

The problems of nonlinear interaction of waves in a plasma have recently attracted much attention, both in connection with the extensive expansion of experiments in which intense oscillations are excited in a plasma by various means (such as noise\(^1\)), and in connection with the theoretical problems of finding the stationary or quasi-stationary spectra of the oscillations of a turbulent plasma.\(^2\) Nonlinear interactions of waves in a plasma situated in a magnetic field were considered for some particular cases in \([5,6]\) (see \([3-14]\) concerning wave interaction in an isotropic plasma). In the present paper we attempt to use a previously developed procedure\(^{13,14}\) to obtain general equations describing nonlinear interaction of waves in an isotropic plasma both in the presence and in the absence of external magnetic fields. Unlike in earlier papers\(^{13,12}\), the results are not confined to the assumption that the plasma is isotropic\(^1\) even when \(H = 0\), i.e., they are suitable for a description of the interaction of waves in a system of interpenetrating plasmas, in the presence of beams, etc. An important factor is that the results can be used for an analysis of the interaction and nonlinear conversion of non-potential oscillations and waves in a plasma, a fact of interest for the problem of interaction between waves that satisfy the dispersion relations of the linear theory.

Let us expand the linear fields in normal waves\(^{18-20}\), introducing for the wave \(\sigma\) polarization vectors \(a_\sigma\):

\[
(k^2\delta_{ij} - k_i k_j - \omega^2 \epsilon_{ij}) a_{\sigma j}(k) = 0, \quad a_{\sigma^*}(k) a_\sigma(k) = 1. \quad (1.1)
\]

We neglect the antihermitian part of \(\epsilon_{ij}\). We introduce \(\epsilon^0\):

\[
\epsilon^0(\omega, k) = \epsilon_{ij}(\omega, k) a_{\sigma j}(k) a_\sigma^*(k) + \frac{\omega^2}{(k a_\sigma(k)) (k a_{\sigma^*}(k))}. \quad (1.2)
\]

It is convenient in what follows to use a gauge \(a^4 = 0\).

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\(^{1}\)Many problems involving the interaction of potential waves in an anisotropic plasma are considered in the cited literature, especially in\(^{14}\).

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THEORY OF NONLINEAR INTERACTION OF WAVES

The number of quanta for the normal wave $N_k^\sigma$ is best introduced by comparing the quantum and classical expressions for the energy of the electromagnetic field:

$$|E_{\sigma\omega}|^2 = \frac{1}{2\pi^2} \omega^2 \epsilon_0 c^2 N_k^\sigma$$

The introduction of the number of quanta enables us to interrelate the induced and spontaneous processes and makes it possible to obtain, by means of a simple semiquantum analysis, general expressions describing nonlinear wave interaction (Sec. 2). We also show that the sought probabilities, which are contained in these equations, can be expressed in general form, in the small-linearity approximation, in terms of the components of the nonlinear plasma current (Sec. 3) and a tensor describing the oscillations of the charge in the field of the normal waves.

2. GENERAL EQUATIONS

To obtain general equations for the nonlinear interaction of plasma particles and normal waves, we introduce the probabilities of the different processes (the index $\sigma$ pertains to the type of wave, and $\alpha$ to the particle species): $w^{\alpha\sigma}(p_\alpha, k_\sigma)$ are the probability that a particle $\alpha$ with momentum $p_\alpha$ will emit a wave $\sigma$ with momentum $k_\sigma$; $w^{\sigma\alpha}(p_\alpha, k_\sigma, k_\sigma')$ is the probability that a wave $\sigma$ with momentum $k_\sigma$ will be scattered by a particle $\alpha$ with momentum $p_\alpha$ and be transformed into a wave $\sigma'$ with momentum $k_\sigma'$; $w^{\sigma\alpha''}(k_\sigma, k_\sigma', k_\sigma'')$ is the probability of decay of a wave $\sigma$ into $\sigma'$ and $\sigma''$ with corresponding momenta $k_\sigma, k_\sigma', k_\sigma''$ and frequencies $\omega_\sigma, \omega_\sigma', \omega_\sigma''$; and $w^{\sigma\alpha''}(p_\alpha, k_\sigma, k_\sigma')$ is the probability of emission of two waves. In the quantum approach, the motion of the particle must be characterized by quantum numbers $p_z$ and $n$. It must be borne in mind here that the emission probability depends on $\nu = n - n'$. In considering the scattering process it is convenient to write $n' = n = \nu - \nu'$, assuming that $w^{\sigma\alpha''}$ depends on $\nu$ and $\nu'$. The equations of nonlinear interaction of the waves are balance equations for the number of particles and the number of waves. The derivation of such equations is a simple generalization of \cite{20}. We present expressions for the variation of the number of quanta $N_k^\sigma$, assuming that the wave function of the particles $f(p_\alpha)$ depends only on $p_1$ and $p_2$, which characterize the parameters of particle motion along a helical line. We have

$$\frac{\partial N_k^\sigma}{\partial t} = \sum_{\alpha'\nu'} w^{\alpha\sigma}(p_\alpha, k_\sigma) N_k^\sigma$$

$$+ \sum_{\sigma'\nu'} \int dp_{\alpha'}(p_{\alpha'}) dk_{\sigma'} N_{k_{\sigma'}}^\sigma \left( w^{\sigma\alpha''}(p_{\alpha'}, k_{\sigma}, k_{\sigma'}) d\sigma + w^{\sigma\alpha''}(k_{\sigma}, k_{\sigma'}, k_{\sigma''}) \right)$$

$$+ \sum_{\sigma\alpha''} \int d\sigma' dk_{\sigma'} N_{k_{\sigma}}^\sigma \left( w^{\sigma\alpha''}(k_{\sigma}, k_{\sigma'}, k_{\sigma''}) d\sigma' + w^{\sigma\alpha''}(k_{\sigma}, -k_{\sigma'}, k_{\sigma''}) \right)$$

$$+ w^{\sigma\alpha''}(k_{\sigma}, -k_{\sigma'}, k_{\sigma''}).$$

(2.1)

where

$$k_\sigma = (k_\omega, \omega_\sigma), \quad \omega_{H\sigma} = |eH|/\epsilon_\sigma.$$
Let us find the change in the energy of the normal wave \( \sigma \), due to decay and scattering. This change has, owing to (2.2), the form

\[
Q^\sigma = \sum_\alpha \int dp_\alpha f(p_\alpha) Q^\sigma (p_\alpha)
\]

\[
+ \sum_\alpha \int \frac{dk \cdot d^2k' \cdot d^2k'' \cdot d^2k''' \cdot d^\sigma'}{\omega_{\sigma'}(p_\alpha, k_\alpha, k_{\alpha'})}{\partial_\sigma'} \left| \frac{\partial}{\partial \omega} \right| E^{\sigma'}(p_\alpha, k_\alpha, k_{\alpha'})
\]

\[
+ \sum_\alpha \int \frac{dk \cdot d^2k' \cdot d^2k'' \cdot d^2k''' \cdot d^\sigma'}{\omega_{\sigma'}(p_\alpha, k_\alpha, k_{\alpha'})}{\partial_\sigma'} \left| \frac{\partial}{\partial \omega} \right| E^{\sigma'}(p_\alpha, k_\alpha, k_{\alpha'})
\]

(2.3)

Here \( Q^\sigma (p_\alpha) \) is the change in energy upon scattering of a single test charge of the plasma in the field of the wave \( \sigma \):

\[
Q^\sigma (p_\alpha) = \int \omega (k_\alpha) \cdot d^2k \cdot d^2k' \cdot d^2k'' \cdot d^2k''' \cdot d^\sigma' \left| \frac{\partial}{\partial \omega} \right| E^{\sigma'}(p_\alpha, k_\alpha, k_{\alpha'})
\]

\[
+ \omega_{\sigma'}(p_\alpha, k_\alpha, k_{\alpha'}) \cdot d^2k \cdot d^2k' \cdot d^2k'' \cdot d^2k''' \cdot d^\sigma'
\]

(2.3)

The last relation makes it possible to determine the probabilities of scattering by considering the emission of the wave \( \sigma \) by a test charge in the field of the normal waves \( \sigma' \).

3. CONNECTION BETWEEN DECAY PROBABILITIES AND THE NONLINEAR PLASMA CURRENT COMPONENTS

Let us consider a certain current \( j(k, \omega) \) in the plasma. It is easy to find the intensity of its radiation from (see [13])

\[
Q^\sigma = \lim_{\tau \to \infty} \int_0^\ell \int_0^\ell \frac{d^2k \cdot d^2k'}{T} \int d^2k' \cdot d^2k'' \cdot d^2k''' \cdot d^\sigma' \left| \frac{\partial}{\partial \omega} \right| E^{\sigma'}(k, \omega, \omega_{\sigma'})
\]

\[
\times \delta (\omega - \omega_{\sigma'}(k)) \cdot \left| \frac{\partial}{\partial \omega} \right| E^{\sigma'}(\omega, k, k_{\sigma'})
\]

(3.1)

where \( E(k, \omega) \) is the field produced by the current \( j(k, \omega) \). Expanding \( E(k, \omega) \) in normal waves and using Maxwell’s equations to express \( E(k, \omega) \) in terms of \( j(k, \omega) \), we obtain for a weakly absorbing medium

\[
Q^\sigma = \lim_{\tau \to \infty} \int_0^\ell \int_0^\ell \frac{d^2k \cdot d^2k'}{T} \int d^2k' \cdot d^2k'' \cdot d^2k''' \cdot d^\sigma' \left| \frac{\partial}{\partial \omega} \right| E^{\sigma'}(k, \omega, \omega_{\sigma'})
\]

\[
\times \delta (\omega - \omega_{\sigma'}(k)) \cdot \left| \frac{\partial}{\partial \omega} \right| E^{\sigma'}(\omega, k, k_{\sigma'}) \cdot \delta (\omega - \omega_{\sigma'}(k))
\]

(3.2)

When the nonlinearity is weak the plasma current can be regarded as a quadratic function of the intensity of the external field:

\[
j^{(2)}(k, \omega)
\]

\[
= \int dk \cdot d^2k \cdot d\omega \cdot d\omega_{\sigma'} \delta (k - k_{\sigma'}) \cdot \delta (\omega - \omega_{\sigma'}(k))
\]

(3.3)

It is important that in the employed approach it is sufficient, for the calculation of the probabilities, to confine oneself to the calculation of the nonlinear plasma current \( j^{(2)} \).

The result (3.2) can be used to calculate both the probabilities of the decay processes and the scattering probabilities. Starting from (3.2) and (3.3), assuming that \( j^{(2)}(k, \omega) \) is the nonlinear current which is produced in the plasma when normal waves are present in it, we can relate the decay probabilities with \( S_{ij} \). Expanding the fields in normal waves (1.3), substituting (3.3) in (3.2), and using the fact that the average values of the products of four \( E_{\sigma} \) can be broken up with the required accuracy into paired products, we obtain

\[
Q^\sigma = \int_0^\ell \int d^2k \cdot d^2k' \cdot d^2k'' \cdot d^2k''' \cdot d^\sigma' \left| \frac{\partial}{\partial \omega} \right| E^{\sigma'}(k, \omega, \omega_{\sigma'})
\]

\[
\times \delta (k - k_{\sigma'}) \cdot \delta (\omega - \omega_{\sigma'}(k)) \cdot \delta (k - k_{\sigma'}) \cdot \delta (\omega - \omega_{\sigma'}(k))
\]

\[
\times S_{ij} \cdot S_{i'j'} \cdot S_{i''j''} \cdot S_{i'''j''''} \cdot S_{i'''}j'''' \cdot S_{i''''j'''''} \cdot S_{j'''''} \cdot S_{j''''''}
\]

(3.4)

In the derivation of (3.4) we used the fact that

\[
S_{ij} = S_{ij} \cdot S_{i'j'} \cdot S_{i''j''} \cdot S_{i'''j'''} \cdot S_{i'''}j'''' \cdot S_{i''''j'''''} \cdot S_{j''''''}
\]

(3.5)

This symmetry condition can always be satisfied, since (3.3) contains only the part of \( S_{ij} \) satisfying this condition.

To obtain the decay probabilities it is sufficient to express \( E_{\sigma} \) in terms of the number of quanta and to compare (3.5) with (2.4) 4:

\[
\omega_{\sigma'}(k_{\sigma'}, \nu_{\sigma'}, \nu_{\sigma''}) = 16 \pi \delta (k_{\sigma'} - k_{\nu_{\sigma'}}) \delta (\omega_{\sigma'} - \omega_{\nu_{\sigma'}})
\]

\[
\times S_{ij} \cdot S_{i'j'} \cdot S_{i''j''} \cdot S_{i'''j'''} \cdot S_{i'''}j'''' \cdot S_{i''''j'''''} \cdot S_{j''''''}
\]

(3.7)

4In the derivation of (3.7) we used the following relations, which follow from the fact that the nonlinear current is real:

\[
S_{ij} = S_{i'j'} \cdot S_{i''j''} \cdot S_{i'''j'''} \cdot S_{i'''}j'''' \cdot S_{i''''j'''''} \cdot S_{j''''''}
\]

\[
S_{i'''}j'''' \cdot S_{i''''j'''''} \cdot S_{j''''''}
\]

(3.7)

\[
\omega_{\sigma} = |\omega_{\sigma}|, \quad \omega_{\sigma'} = |\omega_{\sigma'}|, \quad \omega_{\sigma''} = |\omega_{\sigma''}|
\]

(3.7)
Further, in order to write out the probabilities of the decays \( u_{\sigma,-\sigma'} \) \( (k_{\sigma'}, -k_{\sigma'}) \) and \( u_{\sigma'',-\sigma'} \) \( (k_{\sigma'}, -k_{\sigma''}) \), which correspond either to absorption of \( k_{\sigma'} \) or to the absorption of \( k_{\sigma''} \), it is necessary to interchange in (3.7) the signs of the corresponding four-momenta of the quanta, which reduces to a reversal of the signs in the conservation laws and in the arguments of \( S_{ij} \); it is also necessary to recognize that

\[
S_{\sigma,-\sigma'} = S_{\sigma+}, \sigma'-, \quad S_{\sigma,-\sigma'} = S_{\sigma+}, \sigma'+., (3.8)
\]

The nonlinear current in a plasma situated in a magnetic field can be determined by solving the system of nonlinear equations obtained by expanding the kinetic equation in the field amplitudes:

\[
S^{(\alpha)}_{ij}(k, \omega_1, \omega_2, k_1, \omega_1, k_2, \omega_2) = -\sum_{\mu \nu} 2\pi e^2 \int p_1 dp_1 dp_2 \delta_{\mu+\nu, \mu+\nu} \times \exp \{i(\mu - \mu_0)q_0 + i(\nu - \nu_0)q_2(\omega - k_1v_2 - \mu_0\omega_2)^{-1} \times A_j B_i (\omega_2 - k_2v_2 - \nu_0\omega_2)^{-1} \Gamma_{\mu\nu}(p_1, p_2), (3.9)
\]

where

\[
\hat{B}_1 = \frac{1}{2} J_{\mu+1}\exp(i\nu_{1j}(D + L) + \frac{1}{2} J_{\mu+1}\exp(i\nu_{1j}(D - L)) - i \frac{1}{2} J_{\mu+1}\exp(i\nu_{1j}(D - L) - \frac{1}{2} J_{\mu+1}\exp(i\nu_{1j}(D - L) + \frac{1}{2} J_{\mu+1}\exp(i\nu_{1j}(D - L)) - (\nu - \nu_0)k_1 ) J_{\mu+1},
\]

\[
\hat{B}_3 = \frac{k_{22} - ik_{22} \nu_{1j} L'}{2e_{10}} \nu_{1j \nu_0} + \frac{k_{22} + ik_{22} \nu_{1j} L'}{2e_{10}} \nu_{1j \nu_0} + \frac{1}{2} J_{\mu+1}\exp(i\nu_{1j}(D - L)) - (\nu - \nu_0)k_1 ) J_{\mu+1},
\]

\[
A_{1,2} = v_{1j}(J_{\mu+1}\nu_0 J_{\mu+1}\nu_0 + J_{\mu+1}\nu_0 J_{\mu+1}\nu_0), \quad A_3 = v_{1j} J_{\mu+1}, \quad v_{1j} = \frac{p_{1j}}{e}, \quad \nu_{1j} = \nu_{1j \nu_0}, \quad v_{1j} = \frac{p_{1j}}{e}, \quad \nu_{1j} = \nu_{1j \nu_0},
\]

\[
L = \frac{\nu - \nu_0}{p_{1j}} \left( 1 - \frac{k_{11}v_1}{e_{10}} \right), \quad L' = \frac{\nu - \nu_0}{p_{1j}} \left( 1 - \frac{k_{11}v_1}{e_{10}} \right),
\]

\[
\Gamma_1 = \frac{v_{1j} J_{\mu+1}}{p_{1j}} \nu_{1j}, \quad \Gamma_2 = -i J_{\mu+1} \nu_{1j}, \quad \Gamma_3 = \frac{v_{1j} J_{\mu+1}}{p_{1j}} \nu_{1j}, \quad \nu_{1j} = \frac{k_{11}}{|k_{11}|}, \quad \nu_{1j} = \frac{k_{11}}{|k_{11}|}.
\]

The decay of a high frequency extraordinary wave into a high frequency extraordinary and magnetic-sound wave with propagation transverse to the field is described by the formula

\[
\nu^{(\alpha)}_{1j} = \frac{e^{2} \omega_0^2}{8 \pi m_e e_0^2} \frac{k_2^2}{\omega_2} \left[ 1 + \sum_{\alpha} \frac{\omega_0^2}{\omega_0^2 (1 - \omega_0^2/\omega_0^2)^2} \right] \delta(\omega - \omega_0 - \omega_2) \delta(k_m - k_1 + k_3).
\]

\[
\nu^{(\alpha)}_{1j} = \sum_{\alpha} \frac{e^{2} \omega_0^2 \nu_{1j} (m^2)}{8 \pi m_e e_0^2} \delta(\omega_0 - \omega_1 + \omega_2) \delta(k_m - k_1 + k_3).
\]

4. SCATTERING PROBABILITIES

We now proceed to the calculation of the scattering cross sections. To obtain these cross sections we must regard \( j(k, \omega) \) in (3.2) as the current produced in the plasma by a charge \( p_q \) with account of the perturbations of the motion of the charge by the normal waves of the plasma. The part of \( j(k, \omega) \) independent of the wave field gives the probabilities \( w^{(\alpha)}_0(p_q) \) for the emission of waves by a charge. On the other hand, the probabilities of scattering are described by the part of the current \( j(k, \omega) \) which is proportional to the first power of the electric field of the waves:

\[
j^{(\alpha)}_0(k, \omega) = \int dk_1 d\omega_1 \Lambda^{(\alpha)}_{ij}(k, \omega, k_1, \omega_1) E_j(k_1, \omega_1). (4.1)
\]

It is necessary to take into account here the fact that \( \Lambda^{(\alpha)}_ij \) consists of two parts, which describe two physically different scattering mechanisms:

\[
\Lambda^{(\alpha)}_{ij}(k, \omega, k_1, \omega_1) = \Lambda^{(\alpha)}_{ij}(k, \omega, k_1, \omega_1) + \Lambda^{(\alpha)}_{ij}(k, \omega, k_1, \omega_1).
\]

The first, \( \Lambda^{(1)}_ij \), is connected with the oscillations
of the charge in the field of the waves, while the second, $\Lambda^{(2)}$ is connected with polarization effects which arise in the plasma and are interpreted in particular cases as emissions of the type of transition radiation from the plasma inhomogeneities produced by the plasma waves. An important factor is that in case of electrons these currents can cancel each other, greatly reducing the scattering cross section. For ions, owing to their large mass, only $\Lambda^{(2)}$ is of importance for high frequency waves; therefore scattering by ions, if allowed by the conservation laws, yields nonlinear interaction effects which exceed greatly the effects of induced scattering by the electrons.

Expanding in terms of normal waves in (4.1), and then substituting (4.1) in (2.4), we obtain after averaging over the phases

$$\sum_{\pm \pm} \rho_{\pm \pm}(p_{\pm}) = \lim_{T \to 0} \frac{T}{2} \sum_{\pm \pm} \int dk1 \omega_{\pm}(k) \omega_{\pm}(k_{1})$$

$$\times \left[ \frac{\partial}{\partial \omega} \omega_{\pm} \left|_{\omega_{\pm}(k_{1})} \right] \frac{\partial}{\partial \omega} \omega_{\pm} \left|_{\omega_{\pm}(k)} \right] \right] \Lambda_{\pm \pm, \pm \pm}(k,\omega_{\pm}(k),k_{1},\omega_{\pm}(k_{1}))$$

$$\Lambda_{\pm \pm, \pm \pm}(k,\omega_{\pm}(k),k_{1},\omega_{\pm}(k_{1}))
\Rightarrow a_{\pm \pm}(k) A_{\pm \pm}(k,\omega_{\pm}(k),\omega_{\pm}(k_{1})). \tag{4.3}$$

Substituting in (3.3) $E$ in the form of the sum of the wave field and the field produced by the charge moving on a helical line, we can readily express $\Lambda^{(2)}$ in terms of $S_{ij}$'s:

$$\Lambda_{ij}^{(2)}(k,\omega_{ij},\omega_{ij}) = [S_{ij}(k,\omega,\omega_{ij},\omega_{ij},\omega_{ij})]$$

$$+ S_{ij}(k,\omega_{ij},\omega_{ij},\omega_{ij},\omega_{ij})]$$

$$\times E_{ij}^{(2)}(k,\omega_{ij},\omega_{ij}). \tag{4.4}$$

We see from (4.3) and (4.4) that the frequency of the charge field $\omega - \omega_{ij} = \omega_{ij}^{\pm} - \omega_{ij}^{\pm}$ is generally speaking not close to the frequency of the normal field oscillations. It is clear, however, that the inverse Maxwellian operator has poles when $\omega = \omega_{ij}^{\pm}(k)$

$$\Pi_{ij}(\omega,k) = \sum_{\omega^{\prime}} \frac{\Pi_{ij}^{(\omega)}(\omega,k)}{k^{2}/\omega^{2} - \omega^{\prime 2}(\omega,k)},$$

$$(k^{2} \delta_{ij} - \nu \delta_{ij}) \Pi_{ij} = \delta_{ij},$$

$$k^{2}/\omega_{\pm}^{2} - \omega_{\pm}^{\prime 2}(\omega_{\pm}(k),\omega_{\pm}(k_{1})) = 0. \tag{4.5}$$

The individual terms of this series describe the process of scattering via a virtual polarization wave $\omega^{\prime}$, while the individual terms entering in the operator are the Green’s functions for the corresponding waves. We obtain

$$E_{ij}^{(2)} = \Pi_{ij}(\omega,k) \frac{\sigma}{(2\pi)^{3}} \delta(\omega - \omega - \omega_{ij}^{\prime}), \tag{4.6}$$

$$\Gamma_{ij} = \frac{\nu_{\perp} v_{\perp} J_{\perp}(z)}{z}, \Gamma_{ij}^{\prime} = -i v_{\perp} \frac{\partial J_{\perp}(z)}{\partial z},$$

$$\Gamma_{ij} = v_{\perp} J_{\perp}(z), z = \frac{k_{\perp} v_{\perp} a_{\perp}}{\omega_{ij}^{2}}. \tag{4.7}$$

Substituting (4.4)-(4.7) in (4.3) and comparing with (2.5), we obtain the scattering probability in the form

$$\omega_{ij}^{\prime}(p_{\pm},k_{\pm},\omega_{\pm}) = 2 \delta[\omega_{\pm}(k_{\pm}) - \omega_{\pm}^{\prime}(k_{\pm})]$$

$$- (k_{\pm} - k^{\prime}_{\pm}) v_{\pm} - \omega_{\pm}^{\prime 2} \left[ \frac{\partial}{\partial \omega} \omega_{\pm}^{\prime 2} \right]_{\omega_{\pm}(k_{\pm})}$$

$$\times \left[ \frac{\partial}{\partial \omega} \omega_{\pm}^{\prime 2} \right]_{\omega_{\pm}(k_{\pm})} \times \left| \Lambda_{ij}^{(2)}(k_{\pm},\omega_{\pm}(k_{\pm}),\omega_{\pm}^{\prime}(k_{\pm})) \right|^{2}$$

$$= a_{\pm \pm}(k) A_{ij}(k,\omega_{ij},\omega_{ij}). \tag{4.8}$$

The tensor $\Lambda_{ij}$ can be obtained by solving by perturbation theory the equation of motion of the charge in the field of the normal waves and in an external magnetic field.

For arbitrary charge velocities we obtain

$$\Lambda_{ij}(\omega,k,\omega_{ij},\omega_{ij}) = \frac{e}{m} \frac{1}{\gamma^{1} - \nu^{2} e^{\nu_{\perp}(\gamma - \nu_{\perp})} R_{ij}},$$

$$\times \delta[\omega(\omega,k) - (\omega_{ij} - \nu_{ij}) v_{\perp} - (\mu - \nu) \omega_{ij}^{2}], \tag{4.9}$$

where

$$R_{ij} = P_{ij}^{1} + \Gamma_{ij} P_{ij}^{1}; P_{ij}^{1} = \frac{\omega_{\perp}^{2}}{\omega_{\perp}^{2} - \omega_{ij}^{2}} \left[ p_{ij} - q_{ij}^{\prime} \right]$$

$$\times \left[ B_{ij}^{1} + \frac{\Omega}{\omega_{ij}^{2} - B_{ij}^{1}} \right],$$

$$P_{ij}^{2} = \frac{\omega_{\perp}^{2}}{\omega_{ij}^{2} - B_{ij}^{1}} \left[ p_{ij} - q_{ij}^{\prime} \left( B_{ij}^{1} + \frac{\Omega}{\omega_{ij}^{2} - B_{ij}^{1}} \right) \right]$$

$$- \frac{\nu_{\perp}^{2}}{2} \left( 2 J_{\perp}^{1} + J_{\perp}^{1} e^{2\nu_{\perp}} + J_{\perp}^{1 e^{2\nu_{\perp}}} \right),$$

$$P_{ij}^{3} = \frac{\omega_{\perp}^{2}}{\omega_{ij}^{2} - B_{ij}^{1}} \left[ p_{ij} + i q_{ij}^{\prime} \left( B_{ij}^{1} + \frac{\Omega}{\omega_{ij}^{2} - B_{ij}^{1}} \right) \right]$$

$$- \frac{\nu_{\perp}^{2}}{2} \left( 2 J_{\perp}^{1} + J_{\perp}^{1 e^{2\nu_{\perp}}} + J_{\perp}^{1 e^{2\nu_{\perp}}} \right).$$
The scattering by electrons is usually small compared with the scattering by ions. It must be noted that in a strong magnetic field $k_1 v_{1e}^2 \ll \omega_H^2$, in scattering by electrons, the compensation of the two scattering mechanisms (4.2), unlike the isotropic case [13], can occur only in particular cases ($|k_Z - k'_Z|^2 / |k - k'|^2 = 1$, $T_e \gg T_i$). But even in the absence of compensation, the scattering by the electrons is usually small compared with scattering by ions. Thus, for example, for the interaction of two plasma waves with frequencies $\omega$, $\omega' \sim (\omega_H/2)^{3/2}$, we obtain when $\mu - \nu = 0$

$$dW_s(\omega, \theta)dt = \frac{\pi \sqrt{\tau} W_s(\omega, \theta)}{16 \gamma^2 nm v_T^2 r_D^2} \int d\omega' d\theta' \sin \theta' \frac{1}{\omega' - \omega} W_s'(\omega', \theta')$$

$$\times \left( \omega^2 + \omega'^2 \right) \frac{|k_z - k'_z| |v_{T1}| \left( \omega' - \omega \right)}{\omega' - \omega} \frac{|k| |k'|}{\left( k^2 - k'^2 \right)} \frac{1}{(k^2 - k'^2)^2}.$$  

(4.11)

In this plasma-wave frequency region the interaction turns out to be stronger for waves propagating transverse to the field. A detailed analysis of the equations which follow from (3.9) and (4.8) shows that in the frequency region $\omega \sim \omega_H$ the plasma waves interact most strongly in the angle region $\theta$, $\theta' \ll 1$.

As can be seen from these examples, the nonlinear wave interaction effects described above can find application in problems of turbulent heating of plasma, in astrophysics, etc.

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Translated by J. G. Adashko