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# THE PHONON SPECTRUM OF METALS IN A MAGNETIC FIELD

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It is shown that a series of singularities exists in the phonon spectrum owing to the quantization of the electron states in a magnetic field H. For a spherical Fermi surface the spectrum singularities are of a logarithmic nature. This means that they are more pronounced than in the case H = 0 considered by Migdal and Kohn. The anomalies of the phase velocities are related directly to the giant quantum oscillations of the phonon damping.[9]

1. The emission and absorption of Bose quasiparticles by the conduction electrons leads, as is well known, to threshold effects in the attenuation and to anomalies in the spectra of these quasiparticles. Migdal[1] and Kohn[2] showed that because of the interaction of the electrons with the phonons there appears in the phonon spectrum of metals a logarithmic singularity. Overhauser[3] pointed out an analogous effect in the spectrum of the spin waves in ferromagnetic metals. The anomaly in the phonon spectrum occurs for a phonon momentum q equal to twice the Fermi momentum of the electrons. A similar anomaly was observed experimentally by Brockhouse and his co-workers[4,5] in experiments on the scattering of slow neutrons. The nature of the singularity in the phonon spectrum is closely related to the topology of the Fermi surface. Migdal[1] and Kohn[2] showed that for a spherical Fermi surface the group velocity of the phonons tends to infinity as $\ln|q - 2p|$. Afanas'ev and Kagan[6] investigated the phonon spectrum for an arbitrary dependence of the energy of the electron on the momentum. They showed that the singularity becomes stronger for open Fermi surfaces of various types. Kaganov and Semenenko[7] and Taylor[8] also studied the phonon spectrum of metals in the case of an arbitrary Fermi surface.

The existence and experimental observation of an anomaly in the phonon spectrum makes it apparently possible to obtain direct information about the equal-energy surface of the electrons in a metal. It is of interest to elucidate the nature of the singularity in a magnetic field, since the available methods for studying Fermi surfaces are connected with effects in strong magnetic fields.

In this paper we show that the singularity of the phonon spectrum of a metal becomes stronger in a quantizing magnetic field, in that a logarithmic singularity appears in the phase velocity of the phonon even in the case of a spherical Fermi surface. In addition, instead of one anomaly, there appears a whole system of singular points due to the quantization of the electron states on the Fermi surface.

We note that the absorption of sound in a magnetic field has been investigated in numerous papers[1)]. Quantization of the energy levels of the electron leads to a series of oscillatory effects, in particular to giant oscillations in the absorption of sound.[9] The singularities of the phonon spectrum investigated in this paper are in essence giant quantum oscillations of the phase velocity of sound in metals.

2. In order to investigate the spectrum and attenuation of phonons, we calculate, as in the work of Migdal,[1] the double-time Green's function. We consider the simplest Froehlich model in which account is taken only of the interaction of the electrons with the longitudinal phonons. The interaction Hamiltonian is written in the standard form

$$H_{int} = \frac{g}{2} \sum_{\mathbf{p},\mathbf{q}} a_{\mathbf{p}}^+ a_{\mathbf{p}-\mathbf{q}} (b_{\mathbf{q}} + b_{-\mathbf{q}}^+), \tag{1}$$

where $a_{\mathbf{p}}^+$ and $a_{\mathbf{p}}$ are the production and annihilation operators of the electrons, $b_{\mathbf{q}}^+$ and $b_{\mathbf{q}}$ are the analogous operators for the phonons,

$$g = (2\pi^2 \zeta / m_0 p)^{1/2} \tag{2}$$

is the coupling constant of the electrons and the

[1)]In view of the large number of papers on the absorption of sound in metals in a magnetic field, we will confine ourselves to references dealing directly with problems considered in this article.

Here $D_0$ is a phonon Green's function in the absence of interaction ($g = 0$), and $\Gamma(x, x', x'')$ is the total vertex part. The Green's function $G(x, x')$ of the electrons in a magnetic field depends not only on the difference of the spatial coordinates.

It is well known that the interaction of the electrons with the phonons occurs only in a narrow layer with a width of the order of $\omega_D$ ($\omega_D$ is the limiting frequency of the phonons) near the Fermi phonons, $\hbar = 1$, $m_0$ is the mass of the electron, p is the Fermi momentum, and $\zeta$ is a dimensionless constant ($\zeta < 1/2$) connected with the deformation potential.

We shall not take into account Umklapp processes and the interaction of the electrons with the transverse phonons due to these processes. We restrict ourselves in this paper to an isotropic dispersion of the electrons.

As V. Gurevich has shown,[10] in a strong magnetic field there is generally speaking an inductive interaction in addition to the deformation interaction (1) This mechanism is related to the appearance of an induced electric field in the deformed conductor. However, our calculations show that in all the cases considered below the inductive interaction turns out to be insignificant.

Initially we will consider zero temperature and we will neglect the scattering of the electrons; subsequently we shall allow for the effect of dissipation.

We write Dyson's equation for the phonon Green's function:

$$D(x-x') = D_0(x-x') + 2ig\int D_0(x-x_1)G(x_1,x_2)G(x_3,x_1)$$
$$\times\Gamma(x_2,x_3;x_4)D(x_4-x')\,d^4x_1\,d^4x_2\,d^4x_3\,d^4x_4 \qquad (x = \mathbf{x}, t). \tag{3}$$

level. For this reason one can, as Migdal[1] showed, replace the electron Green's functions G in Eq. (3), with an accuracy up to terms of the order $(m_0/M)^{1/2}$ (M is the mass of the atom), by the functions $G_0$, and $\Gamma$ can be replaced by the simple vertex part. This assertion is obviously also true in the presence of a magnetic field.

In the **p**-representation Dyson's equation (3) can be written in the form[11]

$$D^{-1}(q) = D_0^{-1}(q) - \Pi(q), \qquad q = (\mathbf{q}, \omega),$$
$$\omega_0^2(\mathbf{q})D_0^{-1}(q) = \omega^2 - \omega_0^2(\mathbf{q}), \tag{4}$$

where $\omega_0(q) = s\,|\mathbf{q}|$ and s is the unrenormalized speed of sound.

The polarization operator $\Pi$ is defined by the relation

$$\Pi(q)\,\delta(\mathbf{q}-\mathbf{q}') = -\frac{2ig^2}{(2\pi)^4}\int d\mathbf{p}_1\,d\mathbf{p}_2\,d\omega_2\,G_0(\mathbf{p}_1+\mathbf{q},\mathbf{p}_2;\omega_2+\omega)$$
$$\times G_0(\mathbf{p}_2-\mathbf{q}',\mathbf{p}_1;\omega_2). \tag{5}$$

The zeroth-order Green's function of the electron in the magnetic field has in the **p**-representation the following form:

$$G_0(\mathbf{p},\mathbf{p}';\omega) = \int d\pi_x\,d\pi_z \sum_{n=0}^{\infty}\frac{\Phi_{n\pi_x\pi_z}(\mathbf{p})\,\Phi^*_{n\pi_x\pi_z}(\mathbf{p}')}{\omega-\varepsilon_n(p_z)+\mu+i\delta\,\mathrm{sgn}\,\omega}$$
$$= \gamma^{1/2}\delta(p_x-p_x')\,\delta(p_z-p_z')$$
$$\times\sum_{n=0}^{\infty}\frac{\psi_n(\xi)\,\psi_n(\xi')\exp[i\gamma^{1/2}p_x(\xi-\xi')]}{\omega-(n+1/2)\Omega-p_z^2/2m_0+\mu+i\delta\,\mathrm{sgn}[\varepsilon_n(p_z)-\mu]} \tag{6}$$

The wave function of the electron is given by the expression

$$\Phi_{n\pi_x\pi_z}(\mathbf{p}) = \gamma^{1/4}\exp(i\gamma p_xp_y)\psi_n(\xi)\,\delta(\pi_x-p_x)\,\delta(\pi_z-p_z),$$

$\gamma = c/eH$ is the square of the magnetic length, $\xi = p_y\gamma^{1/2}$

$$\varepsilon_n(p_z) = (n+1/2)\Omega + p_z^2/2m_0$$

is the energy of the electron in the magnetic field, $\Omega$ is the cyclotron frequency, $\mu$ is the Fermi energy, $\delta \to +0$

$$\psi_n(\xi) = H_n(\xi)\exp(-\xi^2/2)$$

is the Hermite function normalized to unity, $H_n(\xi)$ is the Hermite polynomial. The z axis is chosen along the direction of the constant magnetic field H. It is shown in the Appendix how one can in expression (6) make the transition to the limiting case $H = 0$.

Substituting (6) in (5), we integrate over the momenta and over the frequency. The integrals over $\xi$ and $\xi'$ are obtained with the use of the formula

$$\int_{-\infty}^{\infty} dx\exp(-x^2)H_n(x+y)H_m(x+z)\exp(-iux)$$
$$= 2^{(m-n)/2}(z-y-iu)^{n-m}L_m^{n-m}\left[\frac{(y-z)^2+u^2}{2}\right]$$
$$\times\exp\left[-\frac{(y-z)^2+u^2}{2}+i\frac{u(y+z)}{2}\right]\ (n>m)$$

Here $L_n^{\alpha}(x)$ is the generalized Laguerre polynomial normalized to unity. Integration over $\omega_2$ is carried out with the aid of the relation

$$(x+i\delta)^{-1} = \mathrm{P}\left(\frac{1}{x}\right) - \pi i\delta(x), \tag{7}$$

where P denotes the principal value.

As a result we obtain for the polarization operator the following expression:

$$\Pi(q) = \frac{g^2}{2\pi^2\gamma}\sum_{n,\,m=0}^{\infty}M^2_{nm}(\rho)$$
$$\times\int_{-\infty}^{\infty}dp_z\frac{f_0[\varepsilon_m(p_z-q_z)]-f_0[\varepsilon_n(p_z)]}{\varepsilon_n(p_z)-\varepsilon_m(p_z-q_z)-\omega+i\delta\,\mathrm{sgn}\,\omega}. \tag{8}$$

Here

$$M_{nm}(\rho)=\exp\left(-\frac{\rho}{2}\right)\rho^{|m-n|/2}L_{min(m,n)}^{|m-n|}(\rho),\qquad(9)$$

$\rho = \frac{1}{2}\gamma(q_x^2+q_y^2)$; $f_0(\epsilon)$ is the Fermi function at zero temperature. Carrying out the integration over $p_z$ in (8), using relation (7), we obtain

$$\operatorname{Re}\Pi(q)=\frac{\zeta}{2k}\frac{\Omega}{\mu}\sum_{n,m=0}^{N}M_{nm}^2\ln\left|\frac{[(k-x_m)^2-x_n^2]^2-(\omega/\mu)^2}{[(k+x_m)^2-x_n^2]^2-(\omega/\mu)^2}\right|\qquad(10)$$

$$\operatorname{Im}\Pi(q)=-\pi\frac{\zeta}{2|k|}\frac{\Omega}{\mu}\sum_{n,m=0}^{\infty}M_{nm}^2|f_0[\varepsilon_m(p_z^{(0)}-q_z)]-f_0[\varepsilon_n(p_z^{(0)})]|,\qquad(11)$$

where

$$x_n=[1-(n+1/2)\Omega/\mu]^{1/2},\qquad(12)$$

N is the largest integer for which $x_n$ is real; $k = q_z/p$; $p_z^{(0)}$ is determined from the condition

$$\varepsilon_n(p_z^{(0)})-\varepsilon_m(p_z^{(0)}-q_z)=\omega.\qquad(13)$$

The general equations (10) and (11) describe the change of the spectrum and the attenuation of the phonons resulting from their interaction with the electrons. From the expression (10) for $\operatorname{Re}\Pi(q)$ it is obvious that in the isotropic model being considered logarithmic-type singularities exist in the phonon spectrum. These anomalies occur (neglecting the phonon frequency $\omega$) at values of the projection of the phonon momentum

$$q_z\approx\pm p_{zn}\pm p_{zm}\quad(p_{zn}=px_n).\qquad(14)$$

The appearance of the system of singular points (14) is connected with the quantization of the electron momentum projected on the direction of the magnetic field on the Fermi surface $\epsilon_n(p_z)=\mu$. The phonon attenuation exhibits jumps (giant quantum oscillations[9]) depending on whether $p_z^{(0)}$ falls into the "allowed" intervals of values of $p_z$ determined by the conditions

$$0<\varepsilon_n(p_z)-\mu<|\omega|.\qquad(15)$$

3. It is convenient to consider the nature of the singularity in the phonon spectrum in the simplest case when the vectors **q** and H are parallel. In this case $\rho = 0$, $M_{nm}=\delta_{nm}$, and expression (10) goes over into

$$\operatorname{Re}\Pi=\frac{\zeta}{2k}\frac{\Omega}{\mu}\sum_{n=0}^{N}\ln\left|\frac{(k-2x_n)^2-(\omega/k\mu)^2}{(k+2x_n)^2-(\omega/k\mu)^2}\right|.\qquad(16)$$

We separate in (16) the term with the singularity, and, keeping in mind that in the quasi-classical approximation $N\gg1$, we transform the remaining sum with the aid of Poisson's equation

$$\operatorname{Re}\Pi=\frac{\zeta}{2k}\frac{\Omega}{\mu}\ln\left|\frac{(k-2x_n)^2-(\omega/k\mu)^2}{(k+2x_n)^2-(\omega/k\mu)^2}\right|+\frac{\zeta}{2k}\frac{\Omega}{\mu}\int_0^N dn\ln\left|\frac{(k-2x_n)^2-(\omega/k\mu)^2}{(k+2x_n)^2-(\omega/k\mu)^2}\right|+2\frac{\zeta}{k}\frac{\Omega}{\mu}\sum_{j=1}^{\infty}\int_0^N dn\cos2\pi jn\ln\left|\frac{k-2x_n}{k+2x_n}\right|.\qquad(17)$$

The second term $J_2$ in (17) defines a renormalization of the sound velocity which does not depend on the magnetic field. Completing the elementary integration, we obtain Migdal's result[1]

$$J_2=-\zeta\left[1+\frac{4-k^2}{4k}\ln\left|\frac{k+2}{k-2}\right|\right].\qquad(18)$$

We do not consider small corrections of the order $(\omega/k\mu^2)\sim(s/v)^2\sim m_0/M$ ($v$ is the velocity of the electrons on the Fermi surface) which generally speaking exceed the accuracy, since no account was taken of the interaction of the electrons with the electric fields appearing in the deformation of the lattice. This interaction leads, as the calculations show, to terms of the same order in $(s/v)^2$ (cf. also[12]).

The last term in (17) describes quantum oscillations of the phonon spectrum of the same type as the de Haas-van Alphen oscillations. For values of k on the order of unity $\ln|(k-2x_n)/(k+2x_n)|$ can be expanded in a series in $k^{-1}$, since the main role in the integral over n is played by large values of $n \lesssim N$ for which $x_n \ll 1$. Allowing for this circumstance, we find that

$$J_3=-\frac{8\zeta}{\pi k^2}\left(\frac{\Omega}{\mu}\right)^{3/2}\sum_{j=1}^{\infty}\frac{(-1)^j}{(2\pi j)^{3/2}}\sin\left(2\pi j\frac{\mu}{\Omega}-\frac{\pi}{4}\right).\qquad(19)$$

From a comparison of (17) and (19) it follows that the oscillatory part of the spectrum is considerably smaller than the term with the singularity (for $k\sim1$).

Far from the singularity the first term in (17) disappears. For small values of $k$ ($k\ll N^{-1/2}$) it follows from (16) that

$$\operatorname{Re}\Pi=-\zeta\frac{\Omega}{\mu}\sum_{n=0}^{N}x_n^{-1},\qquad(20)$$

i.e., the change in the phonon spectrum is proportional to the density of electron states in the magnetic field. This proportionality has been investigated repeatedly by a number of authors (see, for example,[13]). It follows from (20) that

$$\operatorname{Re}\Pi=-2\zeta\left\{1+\left(\frac{\Omega}{2\mu}\right)^{1/2}\sum_{j=1}^{\infty}\frac{(-1)^j}{j^{1/2}}\cos\left(2\pi j\frac{\mu}{\Omega}-\frac{\pi}{4}\right)\right\}.\qquad(21)$$

Equation (21) is valid far from singularities of the

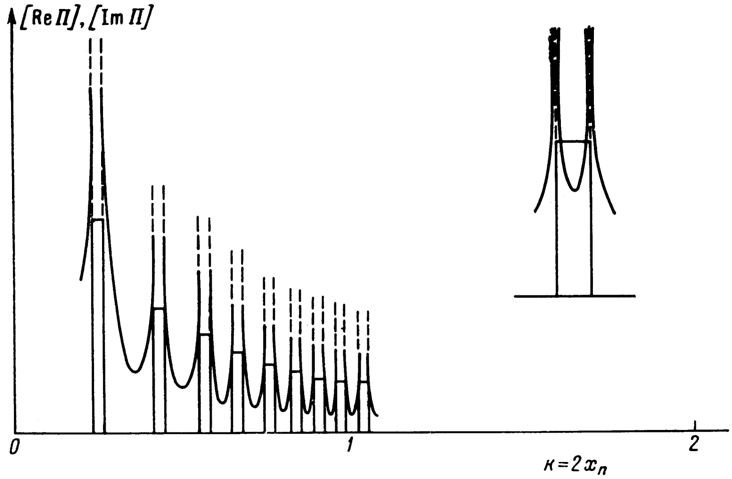


FIG. 1

density of states in a magnetic field[13]

$$x_n \gg 1/N. \quad (22)$$

Let us now discuss the attenuation of the phonons. For q ∥ H the expression in the sum in (11) differs from zero only for a single value n = m, given by the inequalities

$$\frac{\mu}{\Omega}\left(1-\frac{\omega_+^2}{4k^2\mu^2}\right) < n+\frac{1}{2} < \frac{\mu}{\Omega}\left(1-\frac{\omega_-^2}{4k^2\mu^2}\right), \quad (23)$$

where $\omega_\pm = |\omega| \pm k^2\mu$, the width of the interval in which the attenuation differs from zero being $|\omega|/\Omega$. Consequently the phonon attenuation is an aggregate of narrow rectangular maxima. The value of the imaginary part of $\Pi(\mathbf{q}, \omega)$ at the maximum is

$$\mathrm{Im}\,\Pi = -\pi\frac{\zeta}{k}\frac{\Omega}{\mu}. \quad (24)$$

Outside the regions (23) the attenuation vanishes, since we have taken no account of dissipative mechanisms. Equation (24) coincides in essence with the result of Gurevich, Skobov, and Firsov[9] and describes giant quantum oscillations in the sound absorption.

It follows from (16) and (24) that near a singularity the change of the phonon spectrum is logarithmically large compared with the attenuation. The position of the singularities is, as can be seen from Eq. (16), determined by the condition

$$x_n^2 = \left(\frac{k}{2} \pm \frac{|\omega|}{2k\mu}\right)^2. \quad (25)$$

One can readily show that the position of the singularities (25) coincides with the boundaries of the interval (23) in which the attenuation differs from zero. Figure 1 is a schematic diagram of the dependence of the real and imaginary parts of Π on k in the absence of dissipation. The figure also shows the fine structure of the resonance peak corresponding to (25).

4. The above calculation referred to the case T = 0. Here we shall consider the effect of temperature on the singularities of the phonon spectrum. The polarization operator is expressed in terms of temperature Green's functions as follows:

$$\mathcal{P}(\mathbf{q}, \mathbf{q}', \omega_s) = \frac{g^2T}{(2\pi)^4}\sum_{l=-\infty}^{\infty}\int d\mathbf{p}_1\, d\mathbf{p}_2 \mathfrak{G}_0(\mathbf{p}_1+\mathbf{q}, \mathbf{p}_2, \omega_s+\omega_l) \times \mathfrak{G}_0(\mathbf{p}_2-\mathbf{q}', \mathbf{p}_1, \omega_l). \quad (26)$$

Here $\mathfrak{G}_0(p, \omega_j)$ is the temperature Green's function of the electron, $\omega_j$ takes on discrete values $(2j+1)\pi T$, and j is an integer.

Summing over the frequencies and integrating over the momenta, as was done for T = 0, we obtain

$$\mathcal{P}(\mathbf{q}, \omega_s) = -\frac{g^2}{(2\pi)^2\gamma}\sum_{n,m=0}^{\infty} M_{nm}^2(\rho) \times \int_{-\infty}^{\infty} dp_z \frac{\mathrm{th}\,[(\varepsilon_m(p_z-q_z)-\mu)/2T] - \mathrm{th}\,[(\varepsilon_n(p_z)-\mu)/2T]}{i\omega_s+\varepsilon_m(p_z-q_z)-\varepsilon_n(p_z)}. \quad (27)^*$$

The spectrum of the system can be obtained with the aid of a retarded Green's function $D_R$. To this end one must continue $\mathcal{P}(q, \omega_s)$ analytically into the upper half-plane of the complex variable ω. Setting $i\omega_s = \omega + i\delta$, we obtain

$$\mathcal{P}(\mathbf{q}, \omega) = \frac{g^2}{(2\pi)^2\gamma}\sum_{n,m} M_{nm}^2 \int_{-\infty}^{\infty} dp_z \times \frac{\mathrm{th}\,[(\varepsilon_m(p_z-q_z)-\mu)/2T] - \mathrm{th}\,[(\varepsilon_n(p_z)-\mu)/2T]}{\varepsilon_n(p_z)-\varepsilon_m(p_z-q_z)-\omega-i\delta}. \quad (28)$$

*th ≡ tanh.

In the case when $\mathbf{q} \parallel \mathbf{H}$ it follows hence that (after integrating by parts and a change of variable $p_z = px$)

$$\operatorname{Re}\mathcal{H}(\mathbf{q},\omega) = \frac{\zeta}{4k}\frac{\Omega}{T}\sum_{n=0}^{\infty}\int_0^{\infty}\frac{xdx}{\operatorname{ch}^2\mu(x^2-x_n^2)/2T} \times \ln\left|\frac{(x-k/2)^2-(\omega/2k\mu)^2}{(x+k/2)^2-(\omega/2k\mu)^2}\right|. \quad (29)^*$$

Equation (29) describes the temperature smearing of the resonance peak (17). Account of the temperature obviously leads to a finite correction to the spectrum. At the singular point

$$\operatorname{Re}\mathcal{H} = -\frac{\zeta}{k}\frac{\Omega}{\mu}\ln\left(\frac{2\gamma_c}{\pi}\frac{\mu k^2}{T}\right), \quad (30)$$

where $\ln\gamma_c = C = 0.577$, the Euler constant.

The logarithm in (30) is large when $\mu k^2 \gg T$, i.e.,

$$\omega^2 \gg 2m_0 s^2 T. \quad (31)$$

In this case we need retain in the sum (29) only one resonance term. The resonance in this case is, as can be seen from (28), as previously a series of narrow maxima (of width $\omega/\Omega$) with a flat "top" and a characteristic height $\zeta\Omega/k\mu$ whose boundaries are washed out exponentially by the temperature.

For typical materials with $m \sim m_0$ inequality (31) is fulfilled at temperatures on the order of one degree for frequencies $\omega$ larger than $10^{10}\ \mathrm{sec}^{-1}$. We note that condition (31) can be satisfied for smaller frequencies for anomalously small electron groups with a small effective mass.

The "temperature width" $\Delta x_n$ of the resonance peak is of the order of $T/\mu x_n$, or, in terms of the magnetic field,

$$\Delta H/H \sim T/\Omega. \quad (32)$$

The requirement that the width be small compared with unity coincides thus with the condition of "strong" quantization

$$T \ll \Omega. \quad (33)$$

In the case of low frequencies when the inequality (31) is replaced by its inverse, the term with the singularity disappears and one can use in the expression (29) Poisson's summation formula

$$\operatorname{Re}\mathcal{H}(\mathbf{q},\omega) = \zeta\frac{\mu}{2T}\int_0^{\infty}x^2dx\,\operatorname{ch}^{-2}\frac{\mu}{2T}(x^2-1) \times\left\{1-\frac{x^2-k^2/4}{kx}\ln\left|\frac{x-k/2}{x+k/2}\right|\right\} + \frac{\zeta}{k}\frac{\Omega}{T}\sum_{j=1}^{\infty}\int_0^{\infty}\cos 2\pi jndn \int_0^{\infty}xdx\,\operatorname{ch}^{-2}\frac{\mu}{2T}(x^2-x_n^2)\ln\left|\frac{x-k/2}{x+k/2}\right|. \quad (34)$$

*ch ≡ cosh.

We have integrated the first term of (34) over n, and then integrated by parts. The first term in (34) describes the temperature smearing of Migdal's singularity in the absence of a magnetic field (the transition to the limiting case $T = 0$ is obvious), and the second term describes the de Haas-van Alphen type oscillations of the sound velocity at a finite temperature. For the latter we obtain

$$\operatorname{Re}\mathcal{H}_{DHVA}(\mathbf{q},\omega) = 4\frac{\zeta}{k}\sum_j(-1)^jA_j\int_{(0)}xdx\cos 2\pi j\frac{\mu}{\Omega}(1-x^2)\ln\left|\frac{x-k/2}{x+k/2}\right|, \quad (35)$$

where*

$$A_j = \frac{2\pi^2 jT/\Omega}{\operatorname{sh}(2\pi^2 jT/\Omega)}$$

For large and small values of k we arrive respectively at Eqs. (18) and (21) with account of the temperature factor $A_j$ in the summation.

The attenuation of phonons in the low-frequency case under consideration was investigated in [9]. One readily obtains from (28)

$$\operatorname{Im}\mathcal{H} = -\frac{\pi\zeta}{4k}\frac{\Omega}{\mu}\frac{\omega}{T}\operatorname{ch}^{-2}\left[\frac{\mu}{2T}\left(x_n^2-\left(\frac{k}{2}\right)^2\right)\right]. \quad (36)$$

From a comparison of (24) and (36) it is seen that the attenuation differs considerably, depending on the relation between the energy of the phonon $\omega$ and the temperature T (see Fig. 2). At low frequencies (curve a) $\omega \ll T$ the attenuation maxima become lower and broader, whereas the case of high frequencies (curve b) corresponds essentially to the case of zero temperature.

5. In order to take into account the scattering of the electrons in the relaxation-time approximation, the adiabatic parameter $\delta$ in (28) must be replaced by the collision frequency $\nu$. As a result the frequency $\omega$ in Eq. (29) is replaced by $\omega + i\nu$. Let us consider the conditions under which one can separate the resonance term in (29)

$$\frac{\zeta}{4k}\frac{\Omega}{\mu}\ln\left[\left(x_n-\frac{k}{2}-\frac{\omega}{2k\mu}\right)^2+\left(\frac{\nu}{2k\mu}\right)^2\right] \times\left[\left(x_n-\frac{k}{2}+\frac{\omega}{2k\mu}\right)^2+\left(\frac{\nu}{2k\mu}\right)^2\right]. \quad (37)$$

Two close maxima (25) with the same n (see also Fig. 1) are allowed under the obvious condition

$$\omega \gg \nu. \quad (38)$$

A singularity in the spectrum will not smear out as a result of electron scattering if the dis-

*sh = sinh.

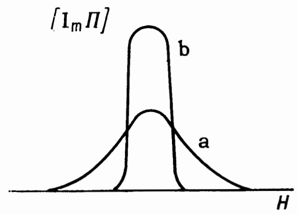


FIG. 2

tances between two neighboring peaks are considerably larger than their widths. In the low-frequency case this requirement leads to the inequality

$$(ql)^2 \gg \mu / \Omega, \qquad (39)$$

which coincides with the known condition for the existence of giant quantum oscillations in the attenuation of sound.[9]

6. Singularities in the phonon spectrum appear only when the projection of the wave vector of the phonon $q_z$ on the direction of the magnetic field differs from zero. For $\mathbf{q} \perp \mathbf{H}$ transitions with a change in $p_z$ are forbidden by the conservation laws. Going in (10) to the limit $k \to 0$, we obtain

$$\operatorname{Re}\Pi(q) = -2\zeta\frac{\Omega}{\mu}\sum_{n,m=0}^{N}\frac{M_{nm}^2(\gamma q^2/2)}{x_n + x_m}\left\{1 + \frac{\omega}{(n-m)\Omega - \omega}\right\}. \qquad (40)$$

Inasmuch as there are in this case no quantum singularities in the phonon dispersion, one can make use of a quasiclassical approximation. Thus

$$M_{nm}^2(x) = J_{n-m}^2(\sqrt{2(n+m+1)x}) \qquad (41)$$

and

$$\operatorname{Re}\Pi(q) = -\zeta\frac{\Omega}{\mu}\sum_{n=0}^{N} x_n^{-1}\left\{1 + \sum_{\alpha=-\infty}^{\infty} J_\alpha^2(qR_n)\frac{\omega}{\alpha\Omega - \omega}\right\}, \qquad (42)$$

where $J_\alpha(x)$ is the Bessel functions, and $R_n \approx [(2n+1)\gamma]^{1/2}$ is the radius of the electron orbit with the principal quantum number $n$.

As can be seen from expression (42), the polarization operator is proportional to the density of states of the electrons in the magnetic field [cf. (20)]. Therefore the quantum effects which occur in this case are described by Eq. (21). Going over in (42) to integration over $n$, we obtain the classical result in the absence of scattering:

$$\operatorname{Re}\Pi(q) = -2\zeta\left\{1 + \sum_{0}^{\pi/2}\sin\vartheta\, d\vartheta \int_{\alpha=-\infty}^{\infty}\frac{J_\alpha^2(qR\sin\vartheta)\,\omega}{\alpha\Omega - \omega}\right\}, \qquad (43)$$

where $R = cp/eH$ is the maximum radius of the electron orbit.

The term $-2\zeta$ is the electron renormalization of the phonon spectrum for $H = 0$. The second term in (38) for $qR \gg 1$ describes the oscillations of the geometrical resonance (see Pippard's paper[14]) and also the acoustic cyclotron resonance.[15] In a strong magnetic field ($qR \ll 1$) there remains in the sum of (43) only one term with $\alpha = 0$ which corresponds to the effect of a sharp change in the velocity of longitudinal sound in the high-frequency case ($\omega \gg \nu$) considered in[16].

7. The appearance of singularities in the phonon spectrum in a quantizing magnetic field is a rather obvious result of the Kramers-Kronig type dispersion relations. Therefore all the formulas cited above for the phonon spectrum can of course be obtained with the aid of these relations, starting from the known expressions for the absorption coefficient.

As has been shown above, the change in the spectrum near a singularity is relatively large compared with the attenuation. Therefore the experimental observation of the anomalous phonon spectrum in a magnetic field can serve as an additional method of obtaining information about the Fermi surface and about the electron-phonon interaction in metals. The effects considered can be observed experimentally in measurements of the phase velocity of ultrasound under the same conditions as the giant quantum oscillations of the absorption,[17,18] and also in experiments on the scattering of slow electrons in metals at low temperatures. The ultrasound experiments correspond to the low-frequency case considered above, whereas in the neutron scattering experiments one can realize the high-frequency case.

Bismuth, in which giant quantum oscillations of the absorption of low-frequency sound were first observed,[17] is of special interest from the point of view of observing the above effects experimentally. It is well known[19] that bismuth has a relatively low carrier density ($\sim 10^{17}$ cm$^{-3}$) and a strong anisotropy of the Fermi surface. The electron Fermi surface constitutes an aggregate of ellipsoids strongly elongated in directions almost perpendicular to the binary axes. They go over into each other in 120° rotations about the trigonal axis. The Fermi energy in bismuth is $2.75 \times 10^{-14}$ erg $= 2.6 \times 10^{13}$ sec$^{-1}$.

We present a table of values of the main parameters along the principal directions of one of the ellipsoids

The axis 1 coincides with the binary axis, the axis 2 is inclined at a small angle ($\sim 6°$) to the basal plane, and the axis 3 is almost parallel to the trigonal axis. Here $2p_i$ is the maximum

| Direction | $2p_i$, cm$^{-1}$ | $s_l$, cm/sec | $\frac{\omega_{max}}{2\pi}$, cps | $\frac{e}{mc}$, $\frac{1}{\text{cps-Oe}}$ |
|---|---|---|---|---|
| 1 | $10^6$ | $2.6\cdot10^5$ | $4.1\cdot10^{10}$ | $1.5\cdot10^8$ |
| 2 | $15\cdot10^6$ | $2.7\cdot10^5$ | $6.5\cdot10^{11}$ | $2.2\cdot10^9$ |
| 3 | $1.4\cdot10^6$ | $2\cdot10^5$ | $4.5\cdot10^{10}$ | $2\cdot10^8$ |

diameter of the ellipsoid along the i-th axis, $s_l$ is the speed of the longitudinal sound, $\omega_{max} = 2p_i s_l$ is the maximum phonon frequency corresponding to $p_z = p_{z\,max}$ ($n = 0$). The values of the gyromagnetic ratio for a magnetic field orientation along the corresponding axis are indicated in the last column.

It is clear from the presented data that owing to the large value of the gyromagnetic ratio it is comparatively simple to satisfy the quantum conditions (33) and (39) in magnetic fields of $10^3$–$10^4$ Oe. This apparently explains the fact that giant quantum oscillations are observed in bismuth in the region of comparatively low frequencies. To observe high-frequency singularities in the phonon spectrum, use must be made, as can be seen from the table, of hypersound frequencies (of the order of some tens of Gcs). Recently experimental papers[20] reported the production of hypersound with frequencies $\omega/2\pi = (1 \text{ to } 4)\times 10^{10}$ sec$^{-1}$; this makes it apparently possible to observe experimentally the singularities in the phonon spectrum also in the high-frequency case. Analogous conclusions are of course also valid for other metals which include anomalously small groups with small effective masses (for example, tin, aluminum, indium, etc.).

High-frequency singularities in the phonon spectrum (near $k = 2p$) for the fundamental electron groups can probably also be observed in typical metals by means of neutron scattering. Here neighboring peaks will be resolved if the relative indeterminacy of the neutron momentum $\lesssim (pR)^{-1}$.

## APPENDIX

We shall show how the transition to the limit $H = 0$ is accomplished in expression (6) for the zeroth-order Green's function in the magnetic field. We write (6) in the form

$$G_0(p,p') = -i\gamma^{1/2}\delta(p_x-p_x')\delta(p_z-p_z') \times \exp\left[i\gamma p_x(p_y-p_y') - \frac{\xi^2+\xi'^2}{2}\right] \times \int_0^\infty d\tau \exp\left\{i\tau\left[\omega - \frac{\Omega}{2} - \frac{p_z^2}{2m_0} + \mu + i\delta\,\mathrm{sgn}\,\omega\right]\right\} \times \sum_n e^{-in\Omega\tau} H_n(\xi) H_n(\xi') \quad \text{(A.1)}$$

and make use of the generating function of the Hermite polynomials

$$\sqrt{\pi}\sum_n z^n H_n(x) H_n(y) = (1-z^2)^{-1/2} \exp\left\{\frac{2xyz-(x^2+y^2)z^2}{1-z^2}\right\}. \quad \text{(A.2)}$$

The zeroth-order Green's function of the electron takes on the form

$$G_0(p,p') = -i\gamma^{1/2}\delta(p_x-p_x')\delta(p_z-p_z')\exp[i\gamma p_x(p_y-p_y')] \times \int_0^\infty d\tau (2\pi i \sin\Omega\tau)^{-1/2} \exp\left\{i\tau\left[\omega - \frac{p_z^2}{2m} + \mu + i\delta\,\mathrm{sgn}\,\omega\right] + \frac{i}{2\sin\Omega\tau}[(\xi^2+\xi'^2)\cos\Omega\tau - 2\xi\xi']\right\}. \quad \text{(A.3)}$$

Using the definition

$$\delta(x) = \lim_{\varepsilon\to\infty}(\varepsilon/2\pi)^{1/2}\exp(-\varepsilon x^2/2),$$

we obtain

$$\lim_{H\to 0}\gamma^{1/2}(2\pi i\Omega\tau)^{-1/2}\exp\left\{i\gamma^{1/2}p_x(\xi-\xi') + \frac{i}{2\sin\Omega\tau}[(\xi^2+\xi'^2)\cos\Omega\tau - 2\xi\xi']\right\}$$
$$= \gamma^{1/2}(2\pi i\,\Omega\tau)^{-1/2}\exp\left\{\frac{i}{2\Omega\tau}[\xi-\xi'+\Omega\tau\gamma^{1/2}p_x]^2 + \frac{i\tau}{2m_0}\left(p_x^2 + \frac{p_y^2+p_y'^2}{2}\right)\right\} = \delta(p_y-p_y')\exp(i\tau p_\perp^2/2m_0).$$

Hence and from (A.3)

$$\lim_{H\to 0} G_0(p,p') = \delta(\mathbf{p}-\mathbf{p}')\left[\omega - \frac{p^2}{2m_0} + \mu + i\delta\,\mathrm{sgn}\,\omega\right]^{-1}.$$


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