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BOSONS IN THE FIELD OF A PLANE ELECTROMAGNETIC WAVE

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The solution of the Kemmer equation in the field of a plane wave is found. The problem is solved by the use of reducible representations of the Kemmer algebra and a projection-operator technique.

FOR the relativistic electron this problem was first solved by Volkov,[1] and was later investigated by Schwinger in a more general form; the analogous problem for bosons has been considered by F. I. Fedorov (private communication) in the framework of irreducible representations of the Kemmer algebra. In the present note a solution in reducible representations is obtained which contains the result for all types of Kemmer bosons (vector, pseudoscalar, axial vector, and scalar). Besides the well known rules of the Kemmer algebra, the matrices of the reducible representations $\beta_\mu(s)$, where $s = \pm 1$, satisfy the relations[3]

$$\beta_\mu(s)\beta_\nu(s') = -\beta_\nu(-s)\beta_\mu(-s') + {}^1\!/_2(1+ss')\delta_{\mu\nu}I. \quad (1)$$

The separating out of the irreducible cells from the $\beta_\mu(s)$ ($s = +1$ corresponds to the vector and pseudoscalar fields, and $s = -1$ to the axial-vector and scalar fields) is accomplished with projectors presented in a paper by Karpenko and Yaroshenko.[4]

The Kemmer equation ($\hbar = c = 1$)

$$(\beta_\lambda(s)D_\lambda + m)\psi = 0, \qquad D_\mu = \partial_\mu + ieA_\mu,$$

where the external magnetic field A_μ depends on the phase θ

$$A_\mu = a_\mu f(\theta), \quad \theta = n_\lambda x_\lambda, \quad (2)$$

$$a_\lambda n_\lambda = n_\lambda^2 = 0 \qquad (a_4 \equiv 0) \quad (3)$$

(a_μ is a constant unit vector), leads to the supplementary condition

$$\left(\beta_\lambda(-s)D_\lambda + \frac{ie}{2m}\beta_\lambda(-s)\beta_\sigma(s)F_{\lambda\sigma}\right)\psi = 0. \quad (4)$$

By raising the order of the wave equation and the supplementary condition by repeated application of the operator $\beta_\lambda(\pm s)D_\lambda$, we get the second-order equation

$$\Big(D_\lambda^2 - m^2 + \frac{ie}{2}[\beta_\lambda\beta_\sigma]F_{\lambda\sigma} + \frac{ie}{2m}\beta_\lambda(-s)D_\lambda\beta_\rho(-s)\beta_\sigma(s)F_{\rho\sigma}\Big)\varphi = 0. \quad (5)$$

Following[1], we introduce a new function $F(\theta)$:

$$\varphi(x) = \exp(ik_\lambda x_\lambda)F(\theta), \qquad k_\lambda^2 + m^2 = 0 \quad (6)$$

and then apply the following obvious relations:

$$(\Box^2 - m^2)\exp(ik_\lambda x_\lambda)F(\theta) = 2ik_\lambda n_\lambda \exp(ik_\rho n_\rho)\,dF/d\theta, \quad (7)$$

$$a_\lambda D_\lambda\varphi = ib(a,\theta)\varphi, \qquad b(a,\theta) = \mathbf{ka} + ef(\theta), \quad (8)$$

$$n_\lambda\partial_\lambda\varphi = ik_\lambda n_\lambda\varphi. \quad (9)$$

Equation (5) then reduces to the ordinary differential equation

$$\frac{dF}{d\theta} + \frac{e}{2k_\lambda n_\lambda}\Big(ib(f - im^{-1}\beta_n(s)f') + [\beta_n(s)\beta_a(s)]f' + \frac{ie}{m^2}\beta_n^2(s)\beta_a^2(s)f'^2 + m^{-1}\beta_n^2(s)\beta_a(s)f''\Big)F + \frac{ie}{2m}\beta_a(s)f'F = 0, \quad (10)$$

where

$$\beta_n(s) = \beta_\lambda(s)n_\lambda, \qquad \beta_a(s) = \boldsymbol{\beta}(s)\mathbf{a}, \qquad f' = df/d\theta. \quad (11)$$

To get rid of the matrices, we introduce the projection operators $\beta_a^2(s)$:

$$F = F_s + F_{-s}, \qquad F_s = \beta_a^2(s)F, \quad (12)$$

so that we have the system of equations

$$\frac{dF_s}{d\theta} + \frac{e}{2k_\lambda n_\lambda}\Big(ibf + m^{-1}\beta_n^2(s)(2igf'^2 - \beta_a(s)f'') + ig\beta_a(s)f'\Big)F_s = \frac{e}{2k_\lambda n_\lambda}B_a^{(+)}(s)\beta_n(s)f'F_{-s}, \quad (13)$$

$$\frac{dF_{-s}}{d\theta} + \frac{ie}{2k_\lambda n_\lambda}bfF_{-s} = -\frac{e}{2k_\lambda n_\lambda}\beta_n(s)f'B_a^{(-)}(s)F_s, \quad (14)$$

where

$$g = e/2m, \qquad B_a^{(\pm)}(s,\theta) = \beta_a(s) \pm im^{-1}b(\theta)f'. \quad (15)$$

Introduction of the function

$$G_s(\theta) = \exp\left(\frac{ie}{2k_\lambda n_\lambda}\int_{\theta_0}^{\theta} b(a,\theta')f(\theta')d\theta'\right) F_s(\theta) \quad (16)$$

enables us to express $G_{-s}(\theta)$ in terms of an integral containing $G_s(\theta)$:

$$G_{-s}(\theta) = -\frac{e}{2k_\lambda n_\lambda}\beta_n(s)\int_{\theta_0}^{\theta} B_a^{(-)}(s,\theta')f'(\theta')G_s(\theta')d\theta' \quad (17)$$

and to get as the equation for $G_s(\theta)$

$$\frac{dG_s}{d\theta} + \frac{g}{k_\lambda n_\lambda}(2ig\beta_n^2(s)f'^2 - \beta_n^2(s)\beta_a(s)f'' + 2ik_\lambda n_\lambda\beta_a(s)f')G_s$$

$$= -\left(\frac{e}{k_\lambda n_\lambda}\right)^2 B_a^{(+)}(s,\theta)f'\beta_n^2(s)$$

$$\times \int_{\theta_0}^{\theta} B_a^{(-)}(s,\theta')f'(\theta')G_s(\theta')d\theta'. \quad (18)$$

Because $\beta_a^2(s)$ is present as a factor in G_s, the removal of the matrices $\beta_a(s)$ is accomplished with the projection operators

$$P_s^{r|a|} = {}^1/_2(I + r\beta_a(s)), \quad P_s^{r|a|} + P_s^{-r|a|} = I, \quad r = \pm 1;$$

if there were no factor $\beta_a^2(s)$, more complicated projection operators would be required. These operators have the properties

$${}^1/_2(I + r\beta_a(s))\beta_n^2(s') = \beta_n^2(s')\,{}^1/_2(I - r\beta_a(s)) \quad (19)$$

[here use must be made of the fact that $\beta_n^2(s) + \beta_n^2(-s) = 0$].

The splitting of $G_s(\theta)$ into components $G_{sr}(\theta)$

$$G_{sr}(\theta) = {}^1/_2(I + r\beta_a(s))G_s(\theta) \quad (20)$$

leads to a system in which only $\beta_n^2(s)$ remains ($r = \pm 1$):

$$e^{-irgf}\frac{d(e^{irgf}G_{sr}(\theta))}{d\theta} = -\frac{e}{2k_\lambda n_\lambda}\beta_n^2(s)[m^{-1}(2igf'^2 + rf'')G_{sr}(\theta)$$

$$-\frac{e}{2k_\lambda n_\lambda}[1 + irm^{-1}b(\theta)f'(\theta)]f'(\theta)\int_{\theta_0}^{\theta}[1 + irm^{-1}b(\theta')f'(\theta')]$$

$$\times f'(\theta')G_{s,-r}(\theta')d\theta'\Big]. \quad (21)$$

In eliminating one of the functions, for example $G_{s,-r}(\theta)$, from this system, use must be made of the fact that $\beta_n^3(s) = 0$, and then the system (21) reduces to the form

$$\frac{d(e^{irgf}G_{sr}(\theta))}{d\theta} = -\frac{e}{2k_\lambda n_\lambda}\beta_n^2(s)M^r(\theta)G_0^{sr}. \quad (22)$$

Here

$$M^r(\theta) = m^{-1}(2igf'^2 + rf'') - \frac{e}{2k_\lambda n_\lambda}[1 + irm^{-1}b(\theta)f'(\theta)]$$

$$\times f'(\theta)\int_{\theta_0}^{\theta}[1 + irm^{-1}b(\theta')f'(\theta')]d\theta', \quad (23)$$

and the G_0^{sr} are arbitrary constant matrices; if they are chosen orthogonal to $\beta_n^2(s)$, the right member of (22) vanishes and we can take

$$G_s(\theta) = \cos(gf(\theta))G_0, \quad (24)$$

$$G_{-s}(\theta) = -\frac{e}{2k_\lambda n_\lambda}\beta_n(s)\int_{\theta_0}^{\theta} B_a^{(-)}(s,\theta')\cos(gf(\theta'))d\theta' G_0. \quad (25)$$

The solution of Eq. (5) is of the form

$$\varphi = \exp\left\{ik_\lambda x_\lambda - \frac{e}{2k_\lambda n_\lambda}\int_{\theta_0}^{\theta} b(\theta')f(\theta')d\theta'\right\}$$

$$\times\left[\cos(gf(\theta)) - \frac{e}{2k_\lambda n_\lambda}\beta_n(s)\int_{f_0}^{f(\theta)} B_a^{(-)}(s)\cos(g\xi)d\xi\right]\varphi_0. \quad (26)$$

For the Kemmer equation we get the solution

$$\psi = (\beta_\lambda(s)D_\lambda - m)\varphi. \quad (27)$$

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142