CURRENT AND MAGNETIC FIELD OF A ROTATING NUCLEUS

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The current corresponding to vortical and potential motions of a nucleus and its magnetic field are calculated. The correction to the moment of inertia of the nucleus due to the magnetic field was found to be 0.3%.

1. DEFORMED atomic nuclei have a well pronounced rotational spectrum. It is of interest to investigate the macroscopic motion of the nuclear matter during rotation. We shall show that this motion can be represented as a sum of a vortical and potential motions. To this end, following Migdal,[1] we write Gor'kov's equation in first-order perturbation theory with respect to the angular velocity of rotation:

$$\left(i\frac{\partial}{\partial t}-H\right)G^1-(\mathbf{M}\Omega)G=i\Delta^1F+i\Delta F^1,$$
$$\left(i\frac{\partial}{\partial t}+H\right)F-(\mathbf{M}\Omega)F=-i\Delta^{1*}G-i\Delta G^1. \qquad (1)$$

Here G and F are Green's functions in the zeroth approximation, G^1 and F^1 Green's functions in first-order perturbation theory, H the Hamiltonian, and **M** the angular-momentum operator; Δ and Δ^1 describe the pair correlation in the nucleus.

The correction to the density matrix of the rotating nucleus is

$$\rho^{(1)}(\mathbf{r})=\sum_{\lambda\lambda'}\frac{(E_\lambda E_{\lambda'}-\varepsilon_\lambda\varepsilon_{\lambda'}-\Delta^2)M_{\lambda\lambda'}+i\Delta(\varepsilon_\lambda-\varepsilon_{\lambda'})f_{\lambda\lambda'}}{2E_\lambda E_{\lambda'}(E_\lambda+E_{\lambda'})}\Omega\varphi_{\lambda'}^*(\mathbf{r})\varphi_\lambda(\mathbf{r}), \qquad (2)$$

where φ_λ and ϵ_λ are the eigenfunctions and eigenvalues of the Hamiltonian H, $f_{\lambda\lambda'}$ is the matrix element of the function f(r), $E_\lambda=(\epsilon_\lambda^2+\Delta^2)^{1/2}$, $\Delta^1 = if(\mathbf{r})$, and $f(\mathbf{r})$ can be obtained from an integral equation.[1] The current can then be written in the form

$$\mathbf{j}=\frac{\mathbf{p}}{m}\rho(\mathbf{r})=\frac{\Omega}{m}\sum_{\lambda\lambda'\lambda''}\frac{(E_\lambda E_{\lambda'}-\varepsilon_\lambda\varepsilon_{\lambda'}-\Delta^2)M_{\lambda\lambda'}+i\Delta(\varepsilon_\lambda-\varepsilon_{\lambda'})f_{\lambda\lambda'}}{2E_\lambda E_{\lambda'}(E_\lambda+E_{\lambda'})}\mathbf{p}_{\lambda''\lambda'}\varphi_{\lambda''}^*\varphi_{\lambda'}, \qquad (3)$$

m—nucleon mass.

Since many levels fit inside the width Δ, expression (3) can be transformed to

$$\mathbf{j}=\frac{\Omega}{m}\sum_{\lambda\lambda'\lambda''}\left\{\left[1-g\left(\frac{\varepsilon_\lambda-\varepsilon_{\lambda'}}{2\Delta}\right)\right]M_{\lambda\lambda'}+\frac{i(\varepsilon_\lambda-\varepsilon_{\lambda'})}{2\Delta}g\left(\frac{\varepsilon_\lambda-\varepsilon_{\lambda'}}{2\Delta}\right)f_{\lambda\lambda'}\right\}\times\mathbf{p}_{\lambda''\lambda'}\varphi_{\lambda''}^*\varphi_\lambda\delta(\varepsilon_\lambda),$$
$$g(x)=\frac{\ln(x+\sqrt{1+x^2})}{x\sqrt{1+x^2}}. \qquad (4)$$

For an oscillator potential, the integral equation for $f(\mathbf{r})$ can be easily solved[1] and we obtain

$$f(\mathbf{r})=-\frac{(g_++g_-)(\omega_y^2-\omega_z^2)}{g_-\nu_-^2+g_+\nu_+^2}yz,$$
$$g_+=g(\nu_+),\quad g_-=g(\nu_-),$$
$$\nu_+=\frac{\omega_y+\omega_z}{2\Delta},\quad \nu_-=\frac{\omega_y-\omega_z}{2\Delta}. \qquad (5)$$

For an oscillator potential formula (4) can be written as $\mathbf{j}=\mathbf{j}_1+\mathbf{j}_2$, where

$$\mathbf{j}_1=\left(1-\frac{g_-\nu_+^2-g_+\nu_-^2}{\nu_+^2-\nu_-^2}\right)\sum_{\lambda\lambda'}[(yp_z-zp_y)\mathbf{p}]_{\lambda\lambda'}\varphi_\lambda\varphi_{\lambda'}\delta(\varepsilon_\lambda)+\frac{\nu_+\nu_-(g_+-g_-)}{\nu_+^2-\nu_-^2}\sum_{\lambda\lambda'}[(yp_z+zp_y)\mathbf{p}]_{\lambda\lambda'}\varphi_\lambda\varphi_{\lambda'}\delta(\varepsilon_\lambda), \qquad (6)$$

$$\mathbf{j}_2=\frac{(g_++g_-)(\nu_+\nu_-)^2}{(g_+\nu_+^2+g_-\nu_-^2)(\nu_+^2-\nu_-^2)}\times\left\{(g_--g_+)\sum_{\lambda\lambda'}[(yp_z-zp_y)\mathbf{p}]_{\lambda\lambda'}\varphi_\lambda\varphi_{\lambda'}\delta(\varepsilon_\lambda)+\frac{g_+\nu_+^2-g_-\nu_-^2}{\nu_+\nu_-}\sum_{\lambda\lambda'}[(yp_z+zp_y)\mathbf{p}]_{\lambda\lambda'}\varphi_\lambda\varphi_{\lambda'}\delta(\varepsilon_\lambda)\right\} \qquad (7)$$

If we sum over λ' and put

$$\rho(\varepsilon_0,\mathbf{r})=\sum_\lambda\varphi_\lambda^*(\mathbf{r})\varphi_\lambda(\mathbf{r})\delta(\varepsilon_\lambda), \qquad (8)$$

where $\rho(\epsilon_0, \mathbf{r})$ is the level density on the Fermi surface, we obtain after averaging over the directions **p**

$$\rho(\varepsilon_0,\mathbf{r})\langle(yp_z-zp_y)\mathbf{p}\rangle=(y\mathbf{k}-z\mathbf{j})n(r),$$

$$\rho(\varepsilon_0, \mathbf{r})\langle (yp_z + zp_y)\mathbf{p}\rangle = (y\mathbf{k} + z\mathbf{j})n(r).$$

Since*

$$(y\mathbf{k} - z\mathbf{j})\Omega_x \to [\Omega\mathbf{r}], \quad (y\mathbf{k} + z\mathbf{j})\Omega \to \Omega\nabla(yz),$$

the vortical current is

$$\mathbf{j}_v = [\Omega\mathbf{r}]n(r), \tag{9}$$

and the potential current is

$$\mathbf{j}_p = A\Omega_x\nabla(yz)n(r), \tag{9'}$$

where A is a constant.

In the limit as $\Delta \to 0$ we should obtain the current corresponding to the rigid-body rotation of the nucleus. Indeed, in this limit

$$\mathbf{j}_1 \to [\Omega\mathbf{r}]n(r), \ \mathbf{j}_2 \to 0.$$

In the limit as $\Delta \to \infty$ we have

$$\mathbf{j}_1 \to 0, \quad \mathbf{j}_2 \to 2\left(\frac{\omega_-\omega_+}{\omega_+^2 + \omega_-^2}\right)(z\mathbf{j} + y\mathbf{k})n(r),$$

which corresponds to potential flow of an ideal liquid. Thus

$$A = 2\omega_-\omega_+ / (\omega_+^2 + \omega_-^2). \tag{10}$$

The total current due to rotation is equal to

$$\mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2 = (1 - \alpha)\mathbf{j}_v + \alpha\mathbf{j}_p, \tag{11}$$

where

$$\alpha = g_-g_+(\omega_+^2 + \omega_-^2) / (g_-\omega_-^2 + g_+\omega_+^2).$$

Formula (11) is very interesting, since it shows that the current in the nucleus can be represented in the form of the motion of a normal liquid with mass $(1 - \alpha)$ and an ideal vortex-free liquid with mass α, so that the total mass is equal to the mass of the nucleus. When $\Delta = 0$, the entire mass of the nucleus rotates like a normal liquid; when $\Delta \to \infty$, to the contrary, the entire motion is potential.

Calculating the moment of inertia the aid of (11), we obtain the well known expression[1]

$$I = \int [\mathbf{j}\mathbf{r}]_x \, dv = \left[1 - \frac{g_+g_-(\omega_+^2 - \omega_-^2)^2}{(g_+\omega_+^2 + g_-\omega_-^2)(\omega_+^2 + \omega_-^2)}\right] I_0, \tag{12}$$

where I_0 is the rigid-body moment of inertia.

In the derivation of (12) we used the easily-verified relation

$$I_p = \frac{2\omega_+\omega_-}{\omega_+^2 + \omega_-^2} m \int n(r)(z^2 - y^2)\,dv = \frac{2\omega_-\omega_+}{\omega_+^2 + \omega_-^2} I_0. \tag{13}$$

*$[\Omega\mathbf{r}] \equiv \Omega \times \mathbf{r}$.

2. Usually in calculating the moment of inertia[1, 2] one neglects the influence of the magnetic field induced by the proton current during rotation. It is of interest to determine more accurately the correction to the moment of inertia, connected with the magnetic field of the proton. To this end we first calculate the magnetic field produced by the rotating nucleus, which is proportional to the angular velocity.

The vector potential of the magnetic field $\mathbf{A}$ satisfies the equation

$$\nabla^2\mathbf{A} = -4\pi\mathbf{j}/c, \tag{14}$$

where $\mathbf{j}$ is the proton current of the rotating nucleus. Neglecting the corrections connected with the reaction of the field on the mass current, ($\sim (r_0/\delta_L)^2 \ll 1$, where δ_L—London depth of penetration, r_0—radius of nucleus) we can take the current in the form (11).

In the right side of (14) we can neglect the potential current, since its presence leads to small corrections proportional to the square of the quadrupole deformation of the nucleus β^2. In addition, when solving (14) we can assume that the nucleus is spherical, since allowance for the deformation leads to corrections $\sim\beta$.

For the oscillator model of the nucleus, the nucleon density is

$$n(r) = n_0[1 - (r/r_0)^2]^{3/2},$$

where n_0 is the density of the center of the nucleus, $r_0 = (2\epsilon_0/m\omega^2)^{1/2}$, ϵ_0 is the Fermi energy, and ω is the oscillator frequency. Then the equation for the field takes the form

$$\nabla^2\mathbf{A} = -\frac{4\pi e}{mc}(1 - \alpha)n_0[1 - (r/r_0)^2]^{3/2}[\Omega\mathbf{r}], \quad r \leqslant r_0,$$
$$\nabla^2\mathbf{A} = 0, \quad r \geqslant r_0, \tag{15}$$

With good accuracy $1 - \alpha = I_z/I_{0z}$ (I_z—proton moment of inertia and I_{0z} its rigid-body value). We seek a solution in the form

$$A = (A_r, A_\theta, A_\varphi) = (0, 0, A_\varphi). \tag{16}$$

Then the gauge condition div $\mathbf{A} = 0$ yields $\partial A_\varphi/\partial\varphi = 0$, and Eq. (15) in spherical coordinates takes the form

$$(\Delta\mathbf{A})_\varphi = \frac{1}{r}\frac{\partial^2(rA_\varphi)}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 A_\varphi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial A_\varphi}{\partial\theta} - \frac{A_\varphi}{r^2\sin^2\theta}$$
$$= \begin{cases} -k\Omega r_0^{-3}[1 - (r/r_0)^2]^{3/2} r\sin\theta, & r \leqslant r_0 \\ 0, & r \geqslant r_0 \end{cases}$$

$$k = 4\pi e n_0 \frac{I_z}{I_{0z}} r_0^3. \tag{17}$$

It is easy to verify that a solution of this equation, satisfying the conditions for the continuity of the component A_φ and its normal derivative, is

$$A_\varphi = \begin{cases} \frac{k \sin\theta}{128}\left[\frac{\arcsin R}{R^2} + \frac{(1-R^2)^{1/2}(48R^6-136R^4+118R^2-15)}{15R}\right], & R \geqslant 1 \\ \frac{\pi k \sin\theta}{256R^2}, & R \leqslant 1 \end{cases}$$

$$R = r/r_0, \quad A_r = A_\theta = 0. \tag{18}$$

Thus, the total energy of the protons of the rotating nucleus, with allowance for the magnetic field, is

$$E_{\mathrm{rot}} = \frac{I_z\Omega^2}{2} + \frac{1}{2c}\int \mathbf{j}_{\mathrm{rot}}\mathbf{A}_{\mathrm{rot}}\,dV. \tag{19}$$

With the aid of (11) and (18) we get

$$\frac{1}{2c}\int \mathbf{j}_{\mathrm{rot}}\mathbf{A}_{\mathrm{rot}}\,dV = \frac{3\pi^2 2^{12}}{15!!}\left(\frac{e\Omega I_z n_0}{cI_{0z}}\right)^2 r_0^7.$$

Since

$$I_{0z} = \pi^2 n_0 m r_0^5/32, \qquad n_0 = 8Z/\pi^2 r_0^3, \tag{20}$$

we obtain

$$E_{\mathrm{rot}} = \frac{I_z\Omega^2}{2}\left[1 + \frac{3\cdot 2^{21}}{\pi^2\cdot 15!!}\left(\frac{I_z}{I_{0z}}\right)\frac{e^2}{mc^2}\frac{Z}{r_0}\right]$$
$$= \frac{I_z\Omega^2}{2}\left[1 + \frac{2^{21}}{\pi^2\cdot 15!!}\left(\frac{I_z}{I_{0z}}\right)\left(\frac{r_0}{\delta_L}\right)^2\right] \tag{21}$$

where

$$\delta_L = (mc^2 V/4\pi Z e^2)^{1/2} = (mc^2 r_0^3/3Ze^2)^{1/2} \tag{22}$$

is the London depth of penetration. The moment of inertia takes the form

$$\tilde{I}_z = I_z\left[1 + \frac{2^{21}}{\pi^2\cdot 15!!}\frac{I_z}{I_{0z}}\left(\frac{r_0}{\delta_L}\right)^2\right]. \tag{23}$$

A similar calculation for a well with constant density leads to the following expression for the moment of inertia:

$$\tilde{I}_z = I_z\left[1 + \frac{2}{21}\frac{I_z}{I_{0z}}\left(\frac{r_0}{\delta_L}\right)^2\right]. \tag{24}$$

In rare-earth atomic nuclei $(r_0/\delta_L)^2 \sim 1/20$, $I_z/I_{0z} \sim 1/2$, and the correction to the moment of inertia does not exceed 0.3%.

The theoretical calculations[1,2] of the moments of inertia include the quantities Δ and β, and also sums containing matrix elements and single-particle energies. The total accuracy of such theoretical calculations is apparently ~20%, and errors of ~0.3% can hardly be noticed at the present level of the theory.[1)]

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[1)]We note that Kadmienskii's estimate of this effect [3] is highly exaggerated.