CALCULATION OF THE KINETIC COEFFICIENTS OF MEDIA WITH RANDOM INHOMOGENEITIES

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The kinetic coefficients of media with random inhomogeneities that characterize the connection between the mean forces and currents are calculated on the basis of a variational principle of minimum energy dissipation. Upper and lower estimates of the quantities under consideration can be obtained simultaneously by means of this approach.

We consider a conducting medium with inhomogeneous properties, placed in an external electric field. We assume the dimensions of the inhomogeneities to be macroscopic (much larger than the path length of the carriers) but significantly less than the dimensions of the system. The local value of the current density in such a medium is determined by means of Ohm's law

$$\mathbf{j} = \sigma(\mathbf{r})\mathbf{E}. \tag{1}$$

All the quantities entering into (1) are point functions. We are interested in the relation between the mean current density in the medium and the mean field in the medium. By virtue of the linearity of (1), this relation will also be linear:

$$\langle \mathbf{j} \rangle = \sigma_{eff} \langle \mathbf{E} \rangle, \tag{2}$$

σ_{eff} is defined to be the effective conductivity of the medium.[1)] The brackets $\langle\ \rangle$ here and below denote spatial averaging. The problem of the determination of σ_{eff} in terms of $\sigma(\mathbf{r})$ can be solved in explicit fashion only for the case of small parameters. Thus, if the change in $\sigma(\mathbf{r})$ is small, the solution has the form (see [1]):

$$\sigma_{eff} = \langle\sigma\rangle\left(1 - \frac{1}{3}\frac{\langle\sigma^2\rangle - \langle\sigma\rangle^2}{\langle\sigma\rangle^2}\right), \tag{3}$$

where it is assumed that $\langle\sigma^2\rangle - \langle\sigma\rangle^2 \ll \langle\sigma\rangle^2$. If the fluctuations of σ are not small, one cannot obtain a closed solution.

In the present communication we call attention to the existence of a variational principle, by means of which we can obtain estimates for the effective conductivity. It is important that the principle allows us to obtain upper and lower bounds.

We start out from the principle of the minimum rate of entropy growth

$$\frac{dS}{dt} = \int \frac{(\mathbf{Ej})}{T} dv \geqslant 0. \tag{4}$$

If the temperature of the medium is constant, then the principle can be written in the form

$$\delta \int \sigma E^2 dv = 0, \tag{5}$$

where the integral in (5) is minimal for a real distribution of the currents and fields. This minimum value of the integral is equal to the Joule heat dissipated in the conductor. The variation in (5) must be carried out under the additional condition curl $\mathbf{E} = 0$ and the corresponding boundary conditions. As the variable we introduce a potential φ such that

$$\mathbf{E} = \langle\mathbf{E}\rangle - \nabla\varphi. \tag{6}$$

The only condition which this potential φ must satisfy is that $\langle \nabla\varphi \rangle = 0$. For the satisfaction of this condition, it suffices to assume φ to be bounded.

For what follows, it is necessary for us to express the Joule heat in terms of the mean field and the current. The heat dissipated per unit volume of the conductor is equal to

$$q = \langle\mathbf{Ej}\rangle = (\langle\mathbf{E}\rangle\langle\mathbf{j}\rangle). \tag{7}$$

The relation (7) is easily obtained upon neglect of surface effects, if we assume that curl $\mathbf{E} = \operatorname{div} \mathbf{j} = 0$. With the help of (7) and (2) we get

$$q = \sigma_{eff}\langle E\rangle^2. \tag{8}$$

From the principle of the minimum and from expression (8) it follows that $\sigma_{eff} \leq \langle\sigma E^2\rangle / \langle E\rangle^2$. This inequality can be rewritten in the form $\sigma_{eff} \leq \langle\sigma e^2\rangle$, where $\mathbf{e}$ is a vector satisfying the conditions $\nabla \times \mathbf{e} = 0$, $|\langle \mathbf{e}\rangle| = 1$. Finally, using (6), we get

1)All the results are obviously valid also for other kinetic coefficients.

$$\sigma_{eff} \leqslant \langle \sigma (\mathbf{n} - \nabla \varphi)^2 \rangle, \quad (9)$$

where $\mathbf{n}$ is a unit vector directed along $\langle \mathbf{E} \rangle$, and φ is an arbitrary bounded function.

To get a lower bound for σ_{eff}, we note that the variational principle (5) can be written in the equivalent form

$$\delta \int \frac{j^2}{\sigma} dv = 0,$$

and the expression for the Joule heat in the form

$$q = \langle j \rangle^2 / \sigma_{eff}.$$

The principle of minimum dissipation leads to the inequality $1/\sigma_{eff} \leq \langle j^2/\sigma \rangle / \langle j \rangle^2$, and in analogy with the foregoing we get

$$\sigma_{eff} \geqslant \left\langle \frac{1}{\sigma} (\mathbf{n} + [\nabla \mathbf{A}])^2 \right\rangle^{-1}, \quad (10)^*$$

where $\mathbf{n}$ is a unit vector of the mean field and $\mathbf{A}$ is an arbitrary bounded vector.

The relations (9) and (10) permit us to estimate the effective conductivity above and below. By improving the test functions φ and $\mathbf{A}$, we can judge the limits of the estimates. It is essential here that, along with the computed conductivity, one can state the maximum error possible. The given relations were obtained under the assumption that the medium as a whole is isotropic. Generalization to the case of an anisotropic distribution of the inhomogeneities is elementary. The effective conductivity here is a tensor, and the estimates (9) and (10) are valid for the principal values if the vector $\mathbf{n}$ is directed along the corresponding principal axes.

In conclusion, we consider the simple estimates which can be obtained by writing for the test functions $\varphi = \mathbf{A} = 0$. We have

$$\langle 1/\sigma \rangle^{-1} \leqslant \sigma_{eff} \leqslant \langle \sigma \rangle. \quad (11)$$

These relations have a simple physical meaning, that the real conductivity always lies between two values, one of which corresponds to a "series," and the other to a "parallel" connection of all the inhomogeneities.

*$[\nabla \mathbf{A}] \equiv \nabla \times \mathbf{A}$.

[1] L. D. Landau and E. M. Lifshitz, Élektrodinamika sploshnykh sred (Electrodynamics of Continuous Media), Gostekhizdat, 1957.

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