EQUATIONS OF MOTION OF A FLUID WITH HYDROMAGNETIC PROPERTIES

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The motion is considered of a non-conducting magnetic fluid with an internal angular momentum, due to self rotation of the molecules. It is assumed that the magnetization of the fluid and its internal angular momentum are proportional to each other. Because of the correlation of the translational (hydrodynamic) and rotational motions of the molecules on the one hand, and between the external magnetic field and the magnetization on the other, a complex interaction between the magnetic and hydrodynamic phenomena appears in liquids with the indicated magnetomechanical coupling. A complete set of hydrodynamic equations, including the field equation and the equations of motion for magnetization, can be obtained on the basis of the conservation laws. The magnetization equation of motion is a generalization of the Bloch-Bloembergen and the Landau-Lifshitz equations. Stationary flow of fluid in a hydrodynamic tube with a circular cross section is considered.

THE presence in the molecules of a fluid of rotational degrees of freedom leads to the result that, under the action of any type of orientation factor, an internal angular momentum is produced whose density K is connected with the magnetization M by the relation

$$\mathbf{M} = \lambda \mathbf{K}. \tag{1}$$

A similar type of dependence should hold for all fluid molecules; however, the specific nature of the coupling and the magnitude of the magnetomechanical ratio λ will change from substance to substance. Evidently a strong effect should be expected in diatomic fluids, in the molecules of which b-type coupling exists (according to the classification of Hund).

For the indicated coupling between the magnetization of the fluid and the internal rotation, the external magnetic field is the reason for the orientation of the molecules. Another reason for the orientation is the interaction between the self rotation of the molecules and the hydrodynamic motion (with velocity v) of their centers of mass.[1)] It is then clear that, when a fluid (even a nonconducting one) moves in the magnetic field H, the latter changes the character of the motion.

Account of the interaction between v, M, and H is connected with the significant change in the equations of hydrodynamics. Here, together with the field equations and the Navier-Stokes equation, which expresses the law of momentum conservation, we consider the equation of conservation of the total angular momentum, the volume density of which is composed of the ordinary angular momentum $\mathbf{L} = \rho\mathbf{r} \times \mathbf{v}$ and the internal angular momentum K. The last equation reduces, by means of (1), to the equation of motion for magnetization.

In this research, the hydrodynamic equations are derived by a phenomenological method, on the basis of the conservation laws, for a nonconducting fluid with coupled magnetic and characteristic mechanical moments. As an example, we consider stationary flow of the Poiseuille type.

1. THERMODYNAMIC RELATIONS

We introduce the local set of coordinates S′ in which the velocity of the given fluid element is zero. This system rotates relative to the laboratory system S with an angular velocity $\Omega = \frac{1}{2}$ curl v. The volume densities of the total energy in the systems of coordinates considered are connected by the well-known relation[2]

$$E = E' + (\mathbf{L} + \mathbf{K})\Omega. \tag{2}$$

To find the thermodynamic quantities, we make use of the equations

$$-\frac{\partial E'}{\partial \Omega} = \mathbf{L} + \mathbf{K}, \quad -\frac{\partial E'}{\partial \mathbf{H}} = \frac{\mathbf{B}}{4\pi} = \frac{\mathbf{H} + 4\pi\mathbf{M}}{4\pi}. \tag{3}$$

The derivatives here are taken at fixed values of the internal parameters K and M, the entropy, and the fluid density. In connection with the last formula, we make two observations. In carrying out the transition from one set of coordinates to the other, we transformed the energy but let H and M remain unchanged. Under the condition that an external electric field is absent, the transformation of the given quantities should have led to corrections of the order of $(v/c)^2$. Furthermore, no distinction is made here between the magnetic field in which a given molecule is located and the applied field H. Such an approach is a valid one, inasmuch as the reorientation and diffusion of the molecules in the fluid are so great that the local field evens out to a very small mean value.

Integrating (3) and taking (1) into account, we get

$$E' = U_0(M) - \mathbf{M}\left(\mathbf{H} + \frac{\Omega}{\lambda}\right) - \frac{H^2}{8\pi} - \int \mathbf{L}\, d\Omega. \tag{4}$$

Equation (2) is conveniently rewritten in the form

$$E = \left(\frac{\rho v^2}{2} - \frac{H^2}{8\pi}\right) + \mathbf{K}\Omega + U, \tag{5}$$

by introducing into consideration the internal energy of the fluid, U, which includes the energy of hidden rotation and of magnetization. For K = M = 0, the latter expression transforms into the usual

$$E = \frac{\rho v^2}{2} - \frac{H^2}{8\pi} + U_{00}(\rho, s).$$

[1)]For $\lambda = 0$, this orienting factor is shown to be unique. Such a situation was considered in [1].

The minus sign in front of the second term is due to the fact that the independent variable in the thermodynamic potential E in our consideration is the field intensity H and not the induction B [see (3)]. Furthermore, we compute the derivatives $\partial E/\partial \mathbf{M}$ from (2) and (5) and equate them. With account of (1) and (4), we have

$$\frac{\partial U}{\partial \mathbf{M}}=\frac{\partial U_0(M)}{\partial \mathbf{M}}-\left(\mathbf{H}+\frac{\Omega}{\lambda}\right)$$

Integrating this equation and eliminating U_{00} in $U_0(M)$, we get

$$U=U_0(M)-\mathbf{M}(\mathbf{H}+\Omega/\lambda).$$

The magnetization of the fluid is always small. Therefore, we can limit ourselves to the quadratic term in the expansion of the isotropic function $U_0(M)$ in even powers of the vector M:

$$U=U_{00}(\rho, s)+\frac{M^2}{2\chi}-\mathbf{M}\left(\mathbf{H}+\frac{\Omega}{\lambda}\right). \tag{6}$$

From the condition $\partial U/\partial \mathbf{M} = 0$, we determine the equilibrium value of the magnetization to be

$$\mathbf{M}=\chi(\mathbf{H}+\Omega/\lambda). \tag{7}$$

This value should correspond to the minimum energy of U. It then follows that $\chi > 0$, i.e., we have to deal with the unusual case of paramagnetism of the rotating molecules ("hydroparamagnetism"). As is seen from the formula (7), the role of the effective magnetic field in such a medium is played by

$$\mathbf{H}'=\mathbf{H}+\Omega/\lambda. \tag{8}$$

For a motionless fluid, found in thermodynamic equilibrium, Eqs. (5) and (6) give

$$E=U_{00}(\rho, s)-\mu H^2/8\pi \qquad (\mu=1+4\pi\chi).$$

It follows from the thermodynamic identity for the internal energy,

$$dU=\rho T ds+w d\rho+\chi^{-1}(\mathbf{M}-\chi\mathbf{H}')d\mathbf{M}-\mathbf{M}d\mathbf{H}' \tag{9}$$

(s and w are the entropy and enthalpy per unit mass, T the absolute temperature), that the expression for the differential pressure $p = \rho w - U$ is

$$dp=-\rho T ds+\rho dw-\chi^{-1}(\mathbf{M}-\chi\mathbf{H}')d\mathbf{M}+\mathbf{M}d\mathbf{H}'. \tag{10}$$

2. CONSERVATION LAWS

For a phenomenological derivation of the equation of motion of a liquid in a constant magnetic field, we use the conservation laws for mass, energy, linear and angular momenta:

$$\partial\rho/\partial t+\operatorname{div}(\rho\mathbf{v})=0, \tag{11}$$

$$\partial E/\partial t+\operatorname{div}\mathbf{Q}=0, \tag{12}$$

$$\frac{\partial}{\partial t}(\rho v_i)+\frac{\partial \Pi_{ik}}{\partial x_k}=0, \tag{13}$$

$$\frac{\partial}{\partial t}(L_{ik}+K_{ik})+\frac{\partial G_{ikl}}{\partial x_l}=0. \tag{14}$$

Here Q, Π_{ik}, and G_{ikl} correspond to the definition of the flux density of energy, linear and angular momenta, $L_{ik} = e_{ikl}L_l$, $K_{ik} = e_{ikl}K_l$. Another equation for the rate of change of the internal angular momentum and entropy must be added to the set (11)–(14). One also needs the field equation in a nonconducting medium:

$$\frac{\partial K_{ik}}{\partial t}+\frac{\partial}{\partial x_l}(v_l K_{ik})=f_{ik}, \tag{15}$$

$$\rho T\left(\frac{\partial s}{\partial t}+\mathbf{v}\nabla s\right)=F, \tag{16}$$

$$\operatorname{rot}\mathbf{H}=0, \qquad \operatorname{div}(\mathbf{H}+4\pi\mathbf{M})=0. \tag{17}$$

The unknowns in these equations are the dissipation function F and the antisymmetric tensor of the angular momentum density of the internal forces f_{ik}. It is convenient to express the latter in terms of the stress tensor σ_{ik}, defined by the equality

$$\Pi_{ik}=\rho v_i v_k+p\delta_{ik}-\sigma_{ik}. \tag{18}$$

From the definition of the tensor $L_{ik} = \rho(x_i v_k - x_k v_i)$, Eqs. (14) and (15), and also the equation of motion

$$\rho\left(\frac{\partial v_i}{\partial t}+v_k\frac{\partial v_i}{\partial x_k}\right)=\frac{\partial\sigma_{ik}}{\partial x_k}-\frac{\partial p}{\partial x_i}, \tag{19}$$

we find, after simple calculations,[1]

$$f_{ik}=\sigma_{ki}-\sigma_{ik}-\partial g_{ikl}/\partial x_l. \tag{20}$$

Here g_{ikl} is the flux density tensor of the internal angular momentum. It is connected with the previously introduced tensor G_{ikl} by the relation

$$g_{ikl}=G_{ikl}-v_l(L_{ik}+K_{ik})+x_i\sigma_{kl}-x_k\sigma_{il}.$$

Equation (15), with account of (1) and (20), takes the form

$$\frac{\partial M_{ik}}{\partial t}+\frac{\partial}{\partial x_l}(v_l M_{ik})=-\lambda\left(\sigma_{ik}-\sigma_{ki}+\frac{\partial g_{ikl}}{\partial x_l}\right). \tag{21}$$

The equations constructed from this scheme take on meaning if the form of the quantities Q, σ_{ik}, g_{ikl}, and F entering into them is made clear.

3. EQUATIONS OF MOTION

We carry out the standard procedure[1,4] of the determination of the unknown terms of the hydrodynamic equations. For this purpose, we find $\partial E/\partial t$ from Eq. (5) and use the identity (9). Substituting the time derivatives of ρ and **v** from (11) and (19), we find

$$\frac{\partial E}{\partial t}=-\left(\frac{v^2}{2}+w\right)\operatorname{div}(\rho\mathbf{v})-\rho\mathbf{v}\nabla\frac{v^2}{2}-\mathbf{v}\nabla p$$
$$+v_i\frac{\partial\sigma_{ik}}{\partial x_k}+\rho T\frac{\partial s}{\partial t}+\chi^{-1}(\mathbf{M}-\chi\mathbf{H})\frac{\partial\mathbf{M}}{\partial t}. \tag{22}$$

On the basis of the identity (10), the third term on the right side of (22) can be written in the form

$$-\mathbf{v}\nabla p=\rho T\mathbf{v}\nabla s-\rho\mathbf{v}\nabla w-\lambda^{-1}\mathbf{v}\nabla(\mathbf{M}\Omega)$$
$$+\chi^{-1}(\mathbf{M}-\chi\mathbf{H})(\mathbf{v}\nabla)\mathbf{M}-\mathbf{M}(\mathbf{v}\nabla)\mathbf{H}.$$

The last component in this expression is transformed by means of the field equation (17):*

$$-\mathbf{M}(\mathbf{v}\nabla)\mathbf{H}=\operatorname{div}\left\{\frac{1}{4\pi}[\mathbf{H}[\mathbf{vB}]]-\mathbf{v}\frac{H^2}{8\pi}-\mathbf{v}(\mathbf{MH})\right\}$$
$$+\frac{1}{4\pi}\left(H_iB_k-\frac{H^2}{2}\delta_{ik}\right)\frac{\partial v_i}{\partial x_k}.$$

We substitute all this in Eq. (22). Collecting in them the terms of the form div, and taking into consideration the equation of entropy growth (16), we find

*$[\mathbf{vB}] \equiv \mathbf{v}\times\mathbf{B}$

$$\frac{\partial E}{\partial t} = -\operatorname{div}\left\{\rho\mathbf{v}\left(\frac{v^2}{2}+w\right) - \frac{1}{4\pi}[\mathbf{H}[\mathbf{vB}]] + \mathbf{v}\frac{H^2}{8\pi} + \mathbf{v}(\mathbf{MH}') - (\mathbf{v}\sigma)\right\}$$
$$+ F - \left[\sigma_{ik} + \frac{\mathbf{M}}{\chi}(\mathbf{M}-\chi\mathbf{H}')\delta_{ik} - \frac{1}{4\pi}\left(H_iB_k - \frac{H^2}{2}\delta_{ik}\right)\right]\frac{\partial v_i}{\partial x_k} \quad (23)$$
$$+ \chi^{-1}(M_i - \chi H_i)\left[\frac{\partial M_i}{\partial t} + \frac{\partial}{\partial x_k}(v_k M_i)\right].$$

To write out the last term on the right side of (23), we must make use of Eq. (21), which gives

$$\chi^{-1}(M_i - \chi H_i)\left[\frac{\partial M_i}{\partial t} + \frac{\partial}{\partial x_k}(v_k M_i)\right]$$
$$= -\frac{\lambda}{\chi}(M_{ik} - \chi H_{ik})\left(\sigma_{ik} + \frac{1}{2}\frac{\partial g_{ikl}}{\partial x_l}\right)$$

Using the obvious relations

$$g_{ikm} = e_{ikl}g_{lm}, \quad H_iB_k(M_{ik} - \chi H_{ik}) = e_{ikl}H_iB_k(M_l - \chi H_l) = 0,$$

we obtain as a result

$$\frac{\partial E}{\partial t} + \operatorname{div}\left\{\rho\mathbf{v}\left(\frac{v^2}{2}+w\right) - \frac{1}{4\pi}[\mathbf{H}[\mathbf{vB}]] + \mathbf{v}\frac{H^2}{8\pi} + \mathbf{v}(\mathbf{MH}') - (\mathbf{v}\sigma)\right.$$
$$\left. + \frac{\lambda}{\chi}(\mathbf{M}-\chi\mathbf{H}, g)\right\} = F + \frac{\lambda}{\chi}g_{ik}\frac{\partial}{\partial x_k}(M_i - \chi H_i)$$
$$- \left[\sigma_{ik} + \frac{\mathbf{M}}{\chi}(\mathbf{M}-\chi\mathbf{H}')\delta_{ik} - \frac{1}{4\pi}\left(H_iB_k - \frac{H^2}{2}\delta_{ik}\right)\right] \quad (24)$$
$$\cdot\left[\frac{\partial v_i}{\partial x_k} + \frac{\lambda}{\chi}(M_{ik} - \chi H_{ik})\right],$$

where

$$(\mathbf{v}\sigma) = v_i\sigma_{ik}, \quad (\mathbf{M}-\chi\mathbf{H}, g) = (M_i - \chi H_i)g_{ik}.$$

Comparing Eq. (24) with the equation of energy conservation (12), we have

$$\mathbf{Q} = \rho\mathbf{v}\left(\frac{v^2}{2}+w\right) - \frac{1}{4\pi}[\mathbf{H}[\mathbf{vB}]] + \mathbf{v}\frac{H^2}{8\pi} + \mathbf{v}(\mathbf{MH}')$$
$$- (\mathbf{v}\sigma) + \frac{\lambda}{\chi}(\mathbf{M}-\chi\mathbf{H}, g), \quad (25)$$

$$F = -\frac{\lambda}{\chi}g_{ik}\frac{\partial}{\partial x_k}(M_i - \chi H_i) + \left[\sigma_{ik} + \frac{\mathbf{M}}{\chi}(\mathbf{M}-\chi\mathbf{H}')\delta_{ik}\right.$$
$$\left. - \frac{1}{4\pi}\left(H_iB_k - \frac{H^2}{2}\delta_{ik}\right)\right]\left[\frac{\partial v_i}{\partial x_k} + \frac{\lambda}{\chi}(M_{ik} - \chi H_{ik})\right]. \quad (26)$$

For what follows, it is convenient to symmetrize all the terms in Eq. (26). Taking into account the definition of the angular velocity vector

$$\Omega_l = \frac{1}{2}e_{ikl}\Omega_{ik} = \frac{1}{4}e_{ikl}\left(\frac{\partial v_k}{\partial x_i} - \frac{\partial v_i}{\partial x_k}\right)$$

and introducing the notation

$$\sigma_{ik} = S_{ik} + e_{ikl}A_l \quad (27)$$

we get for the symmetric and antisymmetric parts of the stress tensor

$$F = \frac{2\lambda}{\chi}\left(\mathbf{A} + \frac{1}{2}[\mathbf{MH}]\right)(\mathbf{M}-\chi\mathbf{H}')$$
$$+ \frac{1}{2}\left[S_{ik} + \frac{\mathbf{M}}{\chi}(\mathbf{M}-\chi\mathbf{H}')\delta_{ik} - \frac{1}{4\pi}\left(H_iH_k - \frac{H^2}{2}\delta_{ik}\right)\right.$$
$$\left. - \frac{1}{2}(M_iH_k + M_kH_i)\right]\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) \quad (28)$$
$$- \frac{\lambda}{2\chi}g_{ik}\left[\frac{\partial(M_i - \chi H_i)}{\partial x_k} + \frac{\partial(M_k - \chi H_k)}{\partial x_i} - \left(\frac{\partial M_k}{\partial x_i} - \frac{\partial M_i}{\partial x_k}\right)\right].$$

By virtue of the law of entropy growth, the dissipation function should be positive. The most general expression for $\mathbf{A}$, S_{ik}, and g_{ik} then are as follows:

$$2\lambda\mathbf{A} = -\lambda[\mathbf{MH}] - \lambda\nu[\mathbf{MH}'] + \frac{1}{\tau}(\mathbf{M}-\chi\mathbf{H}')$$
$$+ \frac{\alpha\chi}{M^2}[\mathbf{M}[\mathbf{MH}']] + \frac{\beta}{M^2}\mathbf{M}(\mathbf{M}, \mathbf{M}-\chi\mathbf{H}'), \quad (29)$$

$$S_{ik} = -\frac{\mathbf{M}}{\chi}(\mathbf{M}-\chi\mathbf{H}')\delta_{ik} + \frac{1}{4\pi}\left(H_iH_k - \frac{H^2}{2}\delta_{ik}\right)$$
$$+ {}^1/_2(M_iH_k + M_kH_i) + \varkappa_{ik}, \quad (30)$$

$$-\lambda g_{ik} = \gamma_1\left[\frac{\partial(M_i - \chi H_i)}{\partial x_k} + \frac{\partial(M_k - \chi H_k)}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial(M_l - \chi H_l)}{\partial x_l}\right]$$
$$+ \gamma_2\delta_{ik}\frac{\partial(M_l - \chi H_l)}{\partial x_l} - \gamma_3\left(\frac{\partial M_k}{\partial x_i} - \frac{\partial M_i}{\partial x_k}\right). \quad (31)$$

From the symmetric part of the stress tensor, we separate the tensor

$$\varkappa_{ik} = \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right) + \zeta\delta_{ik}\frac{\partial v_l}{\partial x_l},$$

which characterizes the viscous stresses in the ordinary fluid. It is easy to establish the fact that for the given choice of unknown quantities the dissipation function will be a quadratic form, to guarantee the positive nature of which we must satisfy the conditions

$$\alpha, \beta, \tau, \gamma_1, \gamma_2, \gamma_3, \eta, \zeta \geqslant 0. \quad (32)$$

The term with the coefficient ν does not make a contribution to the dissipation function and therefore the sign of ν remains undetermined.

We now write down the equations of hydrodynamics with internal rotation and magnetic moment. Substituting the resultant values of σ_{ik}, $-2A_{ik} = \sigma_{ki} - \sigma_{ik}$, and g_{ik} in (19) and (21), we obtain the equation of motion of the fluid:

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v}\right] = -\nabla\left[p + \frac{\mathbf{M}}{\chi}(\mathbf{M}-\chi\mathbf{H}')\right] + (\mathbf{M}\nabla)\mathbf{H}$$
$$+ \left[\eta + \frac{\chi(1+\alpha\tau)}{4\lambda^2\tau}\right]\Delta\mathbf{v} + \left[\zeta + \frac{\eta}{3} - \frac{\chi(1+\alpha\tau)}{4\lambda^2\tau}\right]\nabla\operatorname{div}\mathbf{v}$$
$$+ \frac{1}{2\lambda\tau}\left[1 + \beta\tau + (\alpha-\beta)\tau\chi\frac{(\mathbf{MH}')}{M^2}\right]\operatorname{rot}\mathbf{M} \quad (33)$$
$$- \frac{\chi(\alpha-\beta)}{2\lambda}\left[\mathbf{M}, \nabla\frac{(\mathbf{MH}')}{M^2}\right] - \frac{\nu}{2}\operatorname{rot}[\mathbf{MH}']$$

and the equation of motion for the magnetization

$$\frac{\partial\mathbf{M}}{\partial t} + (\mathbf{v}\nabla)\mathbf{M} = \lambda(1+\nu)[\mathbf{MH}'] - [\mathbf{M}\Omega]$$
$$- \frac{1}{\tau}(\mathbf{M}-\chi\mathbf{H}') - \frac{\alpha\chi}{M^2}[\mathbf{M}[\mathbf{MH}']] - \frac{\beta}{M^2}\mathbf{M}(\mathbf{M}, \mathbf{M}-\chi\mathbf{H}') \quad (34)$$
$$- D_1\operatorname{rot}\operatorname{rot}\mathbf{M} + D_2\nabla\operatorname{div}\mathbf{M} - \mathbf{M}\operatorname{div}\mathbf{v},$$

where we have used the notation

$$D_1 = \gamma_1 + \gamma_3, \quad D_2 = \mu(\gamma_2 + 4\gamma_1/3) \quad (35)$$

for the coefficients of the "transverse" (D_1) and the "longitudinal" (D_2) diffusion of the magnetic moment. The complete set of hydrodynamic equations also includes the field equation (17), the continuity equation (11) and the entropy growth equation (16).

In the determination of the equations of motion, we assumed the magnetic field to be constant in time. The same is not assumed in the calculation of the electric field which would arise as the result of induction from the variable magnetic field. Furthermore, the equations used previously are also valid for the variable H for not too high rates of their change. In fact, as shown in[3], Sec. 60, the induction effects are unimportant if

$$\omega^2 \ll \chi(c/l)^2, \quad (36)$$

where ω is the rate of change of the magnetic field and l is a characteristic dimension of the plane in which the fluid moves.

The dissipation terms in Eqs. (33) and (34) contain seven independent kinetic coefficients: besides the usual coefficients of shear (η) and bulk (ζ) viscosities, there are also three coefficients α, β, and τ of magnetization relaxation, and two (D_1, D_2) of diffusion of the magnetization. The coefficient of rotational viscosity $\chi(1 + \alpha\tau)/4\lambda^2\tau$ in Eq. (33) is expressed in terms of the kinetic coefficients α and τ and the thermodynamic parameters χ and λ). The coefficient $\lambda(1 + \nu)$ in (34) has the meaning of the effective magnetomechanical ratio. However, it must be expected that ν is close to zero, inasmuch as no direct physical mechanism is evident which could lead to the renormalization of the magnetomechanical ratio λ.

The sum of the relaxation terms in the equation for the magnetization (34) is equal to

$$-\frac{\mathbf{M}}{\tau}\left[1+\beta\tau+(\alpha-\beta)\tau\chi\frac{(\mathbf{MH}')}{M^2}\right]+\frac{\chi\mathbf{H}'}{\tau}(1+\alpha\tau). \tag{37}$$

From the ratio between the coefficients α, β, and τ we find which terms in the last expression are important. If

$$\alpha\tau\ll 1, \quad \beta\tau\ll 1, \tag{38}$$

then it suffices to keep in the equations for the magnetization of the three relaxing terms only the first, which is proportional to τ^{-1}, setting $\alpha = \beta = 0$. In this case, Eq. (34) can be regarded as the hydrodynamic generalization of the modified Bloch equation (with a single relaxation time equal to τ). In the other limiting case,

$$\alpha\tau\gg 1, \quad \beta\tau\ll 1, \tag{39}$$

the important term is that with the coefficient α, i.e., we obtain the hydrodynamic analog of the Landau-Lifshitz equation.

The equation is materially simplified if the fluid can be regarded as incompressible, and Eqs. (38) and $\nu = 0$ are satisfied.

$$\operatorname{div}\mathbf{v}=0, \quad \operatorname{div}(\mathbf{H}+4\pi\mathbf{M})=0, \quad \operatorname{rot}\mathbf{H}=0, \tag{40}$$

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}\right]=-\nabla\left[p+\frac{\mathbf{M}}{\chi}(\mathbf{M}-\chi\mathbf{H}')\right]+\eta_e\Delta\mathbf{v}$$
$$+\frac{1}{2\lambda\tau}\operatorname{rot}\mathbf{M}+(\mathbf{M}\nabla)\mathbf{H}, \tag{41}$$

$$\frac{\partial\mathbf{M}}{\partial t}+(\mathbf{v}\nabla)\mathbf{M}=\lambda[\mathbf{MH}]-\frac{1}{\tau}(\mathbf{M}-\chi\mathbf{H}')-D_1\operatorname{rot}\operatorname{rot}\mathbf{M}+D_2\nabla\operatorname{div}\mathbf{M}. \tag{42}$$

Here we have introduced the notation

$$\eta_e=\eta+\chi/4\lambda^2\tau \tag{43}$$

for the sum of the coefficients of shear (η) and rotational ($\chi/4\lambda^2\tau$) viscosity.

The equations obtained in this section permit us to consider a broad range of problems, in which the magnetomechanical effects can be seen to be important. In a medium with the investigated properties, the greatest physical interest attaches to nonstationary processes of various sorts. Their study lies outside the framework of the present paper and will be treated elsewhere. A simple stationary solution is given below for the system of equations (40)—(42) corresponding to cylindrical Poiseuille flow of an ordinary fluid.

4. STATIONARY FLOW IN A CHANNEL

Let us consider the stationary motion of a fluid in a long, cylindrical channel of circular cross cut in an infinite solid. A constant and homogeneous magnetic field $\mathbf{H}_0$ is directed along the axis of the channel (the z axis). We transform the dimensionless variables, choosing as a unit of length the radius of the channel R, the unit of field H_0, the unit of velocity $\eta_e/\rho R$, and the unit of magnetization $2\lambda\tau\eta_e^2/\rho R^2$, and introduce the notation

$$\varepsilon=\frac{\rho R^3}{4\eta_e^2}\left(-\frac{\partial p}{\partial z}\right), \quad \omega_0=\lambda\tau H_0,$$
$$\gamma=\chi/4\lambda^2\tau\eta_e, \quad \delta=\tau D_1/R^2, \quad N=8\pi(\lambda\tau\eta_e)^2/\mu\rho R^2 \tag{44}$$

for the dimensionless parameters which characterize the properties of the fluid, the pressure gradient, and the value of the applied field (ω_0 is the dimensionless Larmor frequency).

At first, we shall consider the case in which the external magnetic field is absent ($\omega_0 = 0$). In this case, we can show the exact solution of Eqs. (40)—(42), in which the velocity is parallel to the channel axis ($v_r = v_\varphi = 0$, $v_z = v(r)$), while the magnetization has only the φ component $M_\varphi = M(r)$. The latter arises from the presence in the flow of a vortex **v** which creates a magnetic field **H′**, similar in its geometry to the field of linear flow. For the indicated **v** and **M**, Eqs. (40) are satisfied identically, and (41) and (42) give

$$\varepsilon=\text{const}, \quad (rv'+rM)'=-4\varepsilon r, \; M+\gamma v'-\delta\left(M''+\frac{M'}{r}-\frac{M}{r^2}\right)=0. \tag{45}$$

The velocity and magnetization of the fluid should remain finite over the entire cross section of the channel, including its center, and should satisfy the boundary conditions

$$v(1)=0, \quad M(1)=0 \tag{46}$$

on the surface of a solid non-magnetizable mass. The latter condition requires explanation. Inasmuch as M = 0 in the bulk material, the value of M(1) is equal to the jump in the φ component of the magnetization at the interface between the liquid and the solid. This jump determines[3] the surface current density $g_z = -cM(1)$. Thus the total surface current strength $-2\pi cM(1)$, which flows through the cross section of the channel, should differ from zero at $M(1) \neq 0$, which is impossible in a non-superconducting body.[3]

Solving Eq. (45) with the boundary conditions (46), we find

$$v(r)=\frac{\varepsilon}{1-\gamma}\left\{1-r^2-\frac{2\gamma}{kI_1(k)}[I_0(k)-I_0(kr)]\right\},$$
$$M(r)=\frac{2\gamma\varepsilon}{1-\gamma}\left[r-\frac{I_1(kr)}{I_1(k)}\right], \quad k^2=\frac{1-\gamma}{\delta} \tag{47}$$

(I_n is a Bessel function of imaginary argument). Returning to dimensionless units, we write out the formula which determines the discharge of fluid in the channel:

$$q=-\frac{\pi\rho R^4}{8\eta}\frac{\partial p}{\partial z}\left[1-\frac{4\gamma}{k}\left(\frac{I_0(k)}{I_1(k)}-\frac{2}{k}\right)\right]. \tag{48}$$

This expression can be interpreted as the Poiseuille formula

$$q=-\frac{\pi\rho R^4}{8\eta(R)}\frac{\partial p}{\partial z} \tag{48'}$$

in which, however, the viscosity coefficient now depends on the radius of the channel:

$$\eta(R)=\eta\left[1-\frac{4\gamma}{\xi R}\left(\frac{I_0(\xi R)}{I_1(\xi R)}-\frac{2}{\xi R}\right)\right]^{-1}, \quad \xi^2=\frac{\eta}{\eta_e\tau D_1}. \tag{49}$$

In the limiting cases $\xi R \to \infty$ (wide channel) and $\xi R \to 0$ (narrow capillary) we get

$$\eta(\infty)=\eta, \quad \eta(0)=\eta_e,$$

i.e., the effect of the rotational viscosity $\eta_e - \eta$ on the motion of the fluid is the greater the narrower the capillary. For intermediate values of $k = \xi R$, we have $\eta_e > \eta(R) > \eta$, so that the relative discharge of the fluid q/R^4 increases monotonically with increase in the radius of the channel.

In the presence of an external field ($\omega_0 \neq 0$), the nonlinear terms in Eqs. (41), (42) do not vanish identically, and it is not possible to find the exact solution of the problem. We shall carry out an approximate stationary solution of the set (40)–(42), which is valid for small pressure gradients ($\epsilon \ll 1$) in the absence of diffusion of the magnetization ($D_1 = D_2 = 0$). We seek a solution of the following form

	r	φ	z
$\mathbf{v}$	0	$u(r)$	$v(r)$
$\mathbf{M}$	$M_r(r)$	$M_\varphi(r)$	$M_0+m(r)$
$\mathbf{H}$	$H_r(r)$	0	H_0

From the equation for div **B**, we find

$$\omega_0 H_r + N\mu M_r = 0. \tag{50}$$

The projection of Eq. (42) on the direction r gives, with account of (50),

$$\omega_0 M_\varphi - \mu M_r = 0. \tag{51}$$

Eliminating H_r and M_r from the remaining equations of the system by means of (50) and (51), we obtain

$$\begin{gathered}(rv' + rM_\varphi)' + \frac{\omega_0 N}{2\pi} rm'M_\varphi = -4\varepsilon r \\ [m-(ru)'/r]' = 0, \quad M_0 = 4\pi\chi\omega_0/\mu N,\end{gathered} \tag{52}$$

For small ϵ the solution of these equations with the boundary conditions $v(1) = u(1) = 0$ can be sought in the form of a series in powers of ϵ:

$$\begin{Bmatrix} v \\ M_\varphi \end{Bmatrix} = \varepsilon \begin{Bmatrix} v_1 \\ M_1 \end{Bmatrix} + \varepsilon^3 \begin{Bmatrix} v_3 \\ M_3 \end{Bmatrix} + \ldots,$$

$$\begin{Bmatrix} u \\ m \end{Bmatrix} = \varepsilon^2 \begin{Bmatrix} u_2 \\ m_2 \end{Bmatrix} + \varepsilon^4 \begin{Bmatrix} u_4 \\ m_4 \end{Bmatrix} + \ldots .$$

After simple calculations, we get, with accuracy to ϵ^2,

$$\begin{gathered} v(r) = \varepsilon\frac{1+\omega_0^2}{1+\omega_0^2-\gamma}(1-r^2), \quad M_\varphi(r) = \frac{2\varepsilon\gamma r}{1+\omega_0^2-\gamma}, \\ u(r) = -\frac{\varepsilon^2\gamma^2 N}{1-\gamma}\frac{\omega_0}{(1+\omega_0^2-\gamma)^2} r(1-r^2), \\ m(r) = \frac{2\varepsilon^2\gamma^2 N}{1-\gamma}\frac{\omega_0}{(1+\omega_0^2-\gamma)^2}(2r^2-\gamma). \end{gathered} \tag{53}$$

As is then seen, the fluid, moving along the channel under the action of the pressure gradient, begins to rotate in the presence of an axial magnetic field. The trajectories of each element of the fluid are helices with a pitch

$$L = 2\pi r\frac{v(r)}{u(r)} = \frac{2\pi(1-\gamma)(1+\omega_0^2)(1+\omega_0^2-\gamma)}{\varepsilon\omega_0\gamma^2 N}$$

that does not depend on the radius ("solid helix").

The dependence of the φ components of the velocity on ω_0 has a resonance character: u passes through a maximum at $\omega_0^2 = (1 = \gamma)/3$. It is explained by the fact that in the absence of a field ($\omega_0 = 0$), there is no reason for generating rotation of the fluid, while in very strong fields ($\omega_0 \to \infty$) the coupling between **v** and **H** is disrupted. In this case, only one of the three magnetization components remains: $M_z = M_0$, i.e., the fluid is homogeneously magnetized along the axis of the channel. It must also be noted that the direction of rotation of the fluid does not depend on the sign of the pressure gradient and is determined exclusively by the direction of the applied field: the mean values of Ω_z and ω_0 over the crosssection of the channel have different signs.

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