ARE WHITE DWARFS SUPERCONDUCTORS?

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According to the Bardeen-Cooper-Schriffer (BCS) theory the temperature for the transition to the superconducting state depends on density according to $T_c \sim n^{2/3}$. In the present paper we investigate on the basis of this dependence the hypothesis that stars of the white dwarf type which are characterized by $T \sim 10^6$ °K and $n \sim 10^{30}$ cm^{-3} can under certain conditions go over into the superconducting state, and that such a transition, which as a result of the Meissner effect is accompanied by the expulsion of the magnetic field and the stripping of the outer sheath, may have the nature of an explosion. The ion-acoustic oscillations of a strongly degenerate plasma play in this case the same role as is played by phonons in the ordinary BCS theory.

1. White dwarfs do not occur in Babcock's catalog[1] (1958) containing stars possessing appreciable magnetic fields (the presence of a field is determined by means of the Zeeman effect). Although this circumstance cannot be regarded as proof of the absence of a magnetic field in the case of stars belonging to this class (additional investigations are needed), nevertheless an analysis of this problem is of definite interest. Indeed, for the range of densities ($n \sim 10^{25}$—10^{35} cm^{-3}) typical for white dwarfs at first glance one might expect high values of electrical conductivity, and, consequently, a very efficient generation of magnetic fields arising as a result of turbulent pulsations of a medium which is a good conductor. Therefore one might expect that white dwarfs possess superstrong magnetic fields.

Under these conditions the absence of a field should be almost unambiguously interpreted as an indication that white dwarfs are superconductors. We note that as a result of the Meissner effect a simply-connected superconductor cannot possess a magnetic field of its own.

2. Thus, for white dwarfs the following alternative possibilities exist: either they possess superstrong fields, or there should be no field at all. The possibility of realizing the superconducting state in the case of white dwarfs follows from the fact that the critical temperature T_c of the transition into the superconducting state depends on the electron density n_e in accordance with the expression

$$T_c \sim n_e^{2/3} \tag{1}$$

and if for a metal under ordinary conditions we have in order of magnitude $n_e \sim 10^{23}$ and $T_c \sim 10$°K then, on increasing the density by a factor of a million ($n_e \sim 10^{29}$), we obtain $T_c \sim 10^5$ °K. We note that white dwarfs are cold dying stars and temperatures of the order of a million degrees are a reasonable estimate for them.

Relation (1) can be derived in the following manner. According to modern ideas white dwarfs are stars in which the force of gravitational compression is in equilibrium with the pressure of a degenerate electron gas. At T = 0 the limiting Fermi momentum is equal to

$$p_F = (3\pi^2\hbar^3 n_e)^{1/3}. \tag{2}$$

In particular, $p_F = mc$ at $n_e \approx 0.6 \times 10^{30}$, so that $n_e \ll 10^{30}$ corresponds to the nonrelativistic case ($v_F \ll c$), while for $n_e \gtrsim 10^{30}$ relativistic considerations must be taken into account.

We first consider the nonrelativistic case when

$$\beta_F = \frac{v_F}{c} = \frac{p_F/mc}{\sqrt{1+(p_F/mc)^2}} \ll 1. \tag{3}$$

In the modern theory of superconductivity it is assumed that exchange of virtual phonons between two electrons leads to the formation of bound electron pairs and in the spectrum of elementary excitations an energy gap appears the width of which corresponds to the binding energy of a pair of electrons. For the critical transition temperature in this case one obtains the formula

$$T_c = \eta T_D, \tag{4}$$

where T_D is the so-called Debye temperature corresponding to the maximum phonon energy:

$$T_D = \frac{\hbar\omega_{max}}{\varkappa} = \frac{\hbar}{\varkappa} c_s k_{max}. \tag{5}$$

Here κ is the Boltzmann constant, c_s is the speed of sound, while $k_{max} = 2\pi/\lambda_{min}$ is the maximum phonon propagation vector. In a one dimensional chain of atoms the minimum wavelength is evidently equal to $\lambda_{min} = 2a$ and then $k_{max} = \pi/a$ where a is the distance between neighboring atoms (in this case neighboring particles will have opposite phase). In Debye's theory of specific heats a somewhat different definition occurs:

$$k_{max} = (6\pi^2 n)^{1/3} = \frac{\pi}{a}\left(\frac{6}{\pi}\right)^{1/3}, \tag{6}$$

where $a = n^{-1/3}$. The speed of sound (the phase velocity of phonons) can be determined from the usual formula $c_s = (\partial p/\partial\rho)^{1/2}$. If the gas temperature T is much lower than the Fermi temperature (the temperature for degeneracy)

$$T_F = \varepsilon_F/\varkappa = p_F^2/2m\varkappa, \tag{7}$$

then in the first approximation (in the limit $T \to 0$) the electron gas pressure is determined by the adiabatic:

$$p_e = \text{const}\cdot n_e^{5/3}, \quad \text{const} = {}^1/_5 m^{-1}(3\pi^2\hbar^3)^{2/3}. \tag{8}$$

Considering the ions to be singly charged and taking into account the condition of quasineutrality ($n_i = n_e$), we obtain

$$c_s = \sqrt{\frac{\partial p}{\partial \rho}} \cong \sqrt{\frac{\partial p_e}{\partial \rho_i}} = \sqrt{\frac{\partial p_e}{M_i \, \partial n_e}} = v_F \sqrt{\frac{m}{3M}}. \tag{9}$$

In obtaining this in virtue of $p \sim m^{-1}$ we have neglected the ion pressure. Then from (6) and (9) we have for the Debye temperature

$$T_D = \frac{\hbar}{\varkappa} c_s \, k_{max} = \frac{2^{1/3}}{\sqrt{3}} \sqrt{\frac{m}{M}} T_F. \tag{10}$$

If, in particular, we assume that a white dwarf consists primarily of hydrogen ($M/m = 1830$) then $T_D \approx T_F/30$. It should be noted that under ordinary conditions relation (9) is satisfied by metals only in order of magnitude up to an accuracy of a factor of order 2—3. This occurs as a result of the fact that usually the Fermi energy ($\epsilon_F \sim 5$ eV) is comparable with the ionization energy I, i.e., with the binding energy of electrons in atoms. One might expect that at high densities when $\epsilon_F \gg I$ relation (9) will be satisfied with a high degree of accuracy. Then, as can be easily shown, the coefficient of proportionality η in formula (4) will not depend on the density and, consequently, all three characteristic temperatures T_F, T_D and T_c turn out to be proportional to $n_e^{2/3}$, and this completes the proof of formula (1).

Numerically the coefficient η in accordance with the theory of superconductivity[2] is equal to ($\ln \gamma = C = 0.577$)

$$\eta = 1.14 \exp\left(-1/V_{\text{int}}\rho_F\right) \quad (1.14 = 2\gamma/\pi), \tag{11}$$

where

$$\rho_F = 2\frac{4\pi p^2 V}{(2\pi\hbar)^3}\frac{dp}{d\varepsilon} = mVp_F/\pi^2\hbar^3$$

is the energy density of electron states near the Fermi surface, while V_{int} is the matrix element for the interaction between electrons in Bardeen's model Hamiltonian:

$$H = \sum_{\mathbf{p}} \xi_{\mathbf{p}} a_{\mathbf{p}}^{+} a_{\mathbf{p}} - V_{\text{int}} \sum_{\mathbf{p},\mathbf{p}'} a_{-\mathbf{p}'}^{+} a_{-\mathbf{p}} a_{\mathbf{p}'}^{+} a_{\mathbf{p}}. \tag{12}$$

$\xi_p = v_F(p - p_F)$ is the electron energy expressed with respect to the Fermi surface.

We recall the method of calculating the quantity V_{int}. Since we always have $T_D \ll T_F$, then the phonon oscillations affect only a small surface layer of electrons of width $\hbar\omega_D$ near the Fermi surface. If a density fluctuation occurs ($n \to n + \delta n$), then the energy of such a surface electron will be altered by the amount

$$\varepsilon_F(n + \delta n) = \varepsilon_F(n) + \frac{\partial \varepsilon_F}{\partial n}\delta n = \varepsilon_F(n) + \frac{2}{3}\varepsilon_F(n)\frac{\delta n}{n}. \tag{13}$$

Since $\delta n/n = -\operatorname{div} \xi$, where ξ is the displacement of the lattice, then summing the above increment over all the electrons and going over to second quantization we obtain the Hamiltonian for the interaction between electrons and phonons

$$H' = \sum \delta\varepsilon = -\frac{2}{3}\varepsilon_F \int dr \, n_e(\mathbf{r}) \operatorname{div} \xi(\mathbf{r}) = -i \sum_{\mathbf{p},\mathbf{q}} D_{\mathbf{q}} a_{\mathbf{p}+\mathbf{q}}^{+} a_{\mathbf{p}} (b_{\mathbf{q}} - b_{-\mathbf{q}}^{+}),$$

$$D_{\mathbf{q}} = \frac{2}{3}\varepsilon_F \sqrt{\frac{\hbar q}{2\rho_i V c_{\text{int}}}}. \tag{14}$$

Finally, eliminating first order terms with respect to D_q by means of a canonical transformation we obtain the Hamiltonian for the interaction in the form (12), and, moreover, we have

$$V_{\text{int}} = \frac{D_{\mathbf{q}}^2}{\hbar\omega_{\mathbf{q}}} = \frac{2\varepsilon_F^2}{9V\rho_i c_{\text{int}}^2} = \frac{\varepsilon_F}{3N}. \tag{15}$$

Then the parameter $V_{\text{int}}\rho_F$ appearing in (11) turns out to be simply equal to the number $V_{\text{int}}\rho_F = 1/2$ and we obtain

$$\eta = 1.14 \exp\left(-\frac{1}{V_{\text{int}}\rho_F}\right) = \frac{1.14}{e^2} = \frac{1}{6.5}. \tag{16}$$

In Table I we give values of the three temperatures $T_F = \epsilon_F/\kappa$, $T_D = T_F/30$ and $T_c = T_D/6.5$ calculated by means of nonrelativistic formulas for a hydrogen plasma.

Table I

n, cm^{-3}	ε_F, eV	T_F, °K	T_D, °K	T_c, °K
10^{22}	1,5	$18 \cdot 10^3$	600	92
10^{23}	7,2	$83 \cdot 10^3$	$2,8 \cdot 10^3$	430
10^{24}	33,3	$386 \cdot 10^3$	$13 \cdot 10^3$	$2 \cdot 10^3$
10^{25}	150	$1,8 \cdot 10^6$	$60 \cdot 10^3$	$9.2 \cdot 10^3$
10^{26}	720	$8,3 \cdot 10^6$	$280 \cdot 10^3$	$43 \cdot 10^3$
10^{27}	$3.3 \cdot 10^3$	$38.6 \cdot 10^6$	$1,3 \cdot 10^6$	$200 \cdot 10^3$
10^{28}	$15,5 \cdot 10^3$	$180 \cdot 10^6$	$6 \cdot 10^6$	$0,92 \cdot 10^6$
10^{29}	$72 \cdot 10^3$	$0,83 \cdot 10^9$	$28 \cdot 10^6$	$4,3 \cdot 10^6$
10^{30}	$333 \cdot 10^3$	$3,86 \cdot 10^9$	$130 \cdot 10^6$	$20 \cdot 10^6$

It should be noted that for n equal to 10^{22} and 10^{23} the Fermi energy respectively amounts to 1.5 and 7.2 eV which is less than the ionization energy for hydrogen. Therefore such hydrogen does not yet possess metallic properties and our formulas are not applicable to these densities. Only beginning with approximately $n = 10^{24}$ will the formulas hold for a hydrogen plasma. We note that the density of liquid hydrogen is equal to $n = 2 \times 10^{22}$ and at this stage it does not yet possess metallic properties.

3. For $n = 10^{30}$ we have $\beta_F = v_F/c = 0.7$, so that in order to be able to continue Table I into the domain of greater densities a relativistic calculation is necessary. The Fermi energy ϵ_F can in this case be determined by the formula (and correspondingly $T_F = \epsilon_F/\kappa$)

$$\varepsilon_F = \sqrt{m^2c^4 + c^2p_F^2} - mc^2, \quad p_F = (3\pi^2\hbar^3 n)^{1/3} \tag{17}$$

(in particular, in the ultrarelativistic case we obtain $\epsilon_F \approx cp_F \sim n^{1/3}$, $T_F \approx cp_F/\kappa \sim n^{1/3}$). The total energy of the gas for $T = S = 0$ is equal to

$$E(N, V) = \int_0^{p_F} \sqrt{m^2c^4 + c^2p^2} \cdot \frac{2dp\, dV}{(2\pi\hbar)^3} = \frac{V}{\pi^2\hbar^3} \int_0^{p_F(V)} \sqrt{m^2c^4 + c^2p^2}\, p^2\, dp. \tag{18}$$

From here we obtain for the pressure

$$p_e = -\frac{\partial E(N, V)}{\partial V}$$

$$= \frac{1}{\pi^2\hbar^3}\left(-\int_0^{p_F(n)} \sqrt{m^2c^4 + c^2p^2}\, p^2\, dp + \frac{p_F^3}{3}\sqrt{m^2c^4 + c^2p_F^2}\right), \tag{19}$$

and then for the speed of sound in place of formula (9) we obtain (we assume that $c_s < c_{\text{light}}$):

$$c_s = \sqrt{\frac{\partial p_e}{M \partial n_e}} = \frac{p_F}{m}\sqrt{\frac{m}{3M}}\left[1 + \left(\frac{p_F}{mc}\right)^2\right]^{-1/4}. \tag{20}$$

Accordingly the Debye temperature corresponding to the maximum phonon energy turns out to be equal to

$$T_D = \frac{\hbar}{\varkappa} c_s\, k_{max} = \frac{2^{1/3}}{\sqrt{3}} \sqrt{\frac{m}{M}}\, \frac{p_F^2}{2\varkappa m}\left[1 + \left(\frac{p_F}{mc}\right)^2\right]^{-1/4} \qquad (21)$$

(under ultrarelativistic conditions $T_D \approx T_F(2^{2/3}p_F/3Mc)^{1/2} \ll T_F$). We finally obtain the temperature for the transition to the superconducting state. From the definition (17) we find similarly to formula (13)

$$\varepsilon_F(n+\delta n) = \varepsilon_F(n) + \frac{\partial \varepsilon_F(n)}{\partial n}\delta n = \varepsilon_F + \frac{p_F v_F}{3}\frac{\delta n}{n}, \qquad (22)$$

and the Hamiltonian (14) for the interaction with phonons has the same form as before, but the coefficient $2\epsilon_F/3$ should now be replaced by the expression $p_F v_F/3$.

Thus, in the relativistic case we have

$$D_q = \frac{p_F v_F}{3}\sqrt{\frac{\hbar q}{2\rho_i V c_s}}, \quad V_s = \frac{D_q^2}{\hbar\omega_q} = \frac{p_F v_F}{6N}, \qquad (23)$$

where $v_F = p_F/m\sqrt{1+(p_F/mc)^2}$.

The density of states ρ_F is now equal to

$$\rho_F = \frac{\partial}{\partial\varepsilon}\int\limits_{(\Omega)} 2\frac{dp\,dV}{(2\pi\hbar)^3} = \frac{Vp_F^2}{\pi^2\hbar^3 v_F} \qquad (24)$$

and the quantity $V_{int}\rho_F = 1/2$ remains unchanged as before. The Bardeen Hamiltonian (12) is not altered, so that the fundamental formula (4) for the transition temperature is retained and η has its former value $\eta = 1.14\, e^{-2} = 1/6.5$, so that

$$T_c = \frac{1}{6.5}T_D. \qquad (25)$$

Utilizing the relativistic formulas (17), (21) and (25), we obtain the data for Table II, which is a continuation of Table I. We note that the densities $n > 10^{36}$ correspond already to neutron stars.

Table II

n, cm^{-3}	ε_F, MeV	T_F, 10^9 °K	T_D, 10^9 °K	T_c, °K
10^{30}	0,26	3	0.1 } $\approx \frac{T_F}{30}$	$15.4\cdot10^6$
10^{31}	0,83	9,6	0,37 }	$57\cdot10^6$
10^{32}	2.2	25,5	1,22	$188\cdot10^6$
10^{33}	5.2	60	3,9	$0,6\cdot10^9$
10^{34}	11,8	136	12.4	$1.91\cdot10^9$
10^{35}	26	300	39	$6\cdot10^9$
10^{36}	57	660	122	$20\cdot10^9$

4. We finally consider the transition of a white dwarf from the normal state into the superconducting state as the pressure is increased. In the absence of a magnetic field in the normal state one can obtain the following expression for the entropy of a strongly degenerate electron gas

$$S = \pi^2 m\varkappa^2 \frac{NT}{p_F^2}\left[1 + \left(\frac{p_F}{mc}\right)^2\right]^{1/2} = \pi^2\varkappa^2\frac{NT}{p_F v_F}. \qquad (26)$$

This expression is valid for any arbitrary degree of relativistic behavior. From this it follows that under adiabatic compression the temperature in the nonrelativistic case increases as $T \sim n^{2/3}$, while in the ultrarelativistic case it increases slower—as $T \sim n^{1/3}$. At the same time the parameter T_c—the critical temperature for the transition—increases in these two limiting cases respectively as $n^{2/3}$ and $n^{1/2}$, so that in the ultrarelativistic case the parameter T_c can exceed the temperature T, the result of which will be a transition into the superconducting state. Such a transition is not accompanied by liberation of heat, and is a transition of the second kind.

We now consider the case when a "frozen-in" magnetic field exists within a white dwarf. In this case a "supercooled" state is possible when the temperature is already lower than the critical temperature ($T < T_c$), but the transition of the dwarf into a superconductor is hindered by a magnetic field H. As is well known, for this to occur the value of H must exceed a certain critical value $H_c(T)$ which depends on the temperature T (for $T < T_c$). This dependence (both the theoretical one, and the experimental one for ordinary metals) is well approximated by the formula

$$H_c(T) = H_c(0)\,(1 - T^2/T_c^2), \qquad (27)$$

which is also valid in the relativistic case.[1)]

Further we shall in general utilize only the "relativistic" formulas, without stating this explicitly (cf., footnote [1)]). In our case the quantity $H_c(0)$ in (27) turns out to be equal to

$$H_c(0) = \frac{\pi^{3/2}\sqrt{6}}{\gamma}\varkappa T_c\sqrt{\frac{n}{p_F v_F}} \approx 2.62\frac{\hbar n^{5/6}}{\sqrt{M}} \qquad (28)$$

or numerically (H in gauss, n in cm^{-3}, taking into account $V_{int}\rho_F = 1/2$ and M = 1830 m) we have $H_c(0) = 2\times10^{-15}n^{5/6}$. From this, for example, we obtain that for $n = 10^{24}$ $H_c(0) = 2\times10^5$ gauss; for $n = 10^{30}$ $H_c(0) = 2\times10^{10}$ gauss, while for $n = 10^{36}$ $H_c(0) = 2\times10^{15}$ gauss.

In order to obtain an estimate as to how strong are these fields and to what extent can they influence the dynamic equilibrium of the star one should obtain an estimate of the ratio of the gas and the magnetic pressures

$$\beta_c = \frac{p_{gas}}{p_{mag}} = \frac{p_e}{H_c^2(0)/8\pi}. \qquad (29)$$

In the nonrelativistic case utilizing formula (8) for p_e one can establish that the quantity β_c generally does not depend on the density and is approximately equal to $\beta_c \approx 7\, M/m \approx 1.3\times10^4$. In the ultrarelativistic case from the general formula (19) we can obtain

$$p_e = {}^1/_4\, c p_F n = {}^1/_4\,(3\pi^2\hbar^3)^{1/3} c n^{4/3}$$

[1)]The generalization of the usual theory made by us for the relativistic case is not sufficiently consistent – a rigorous theory must be covariant, and must take into account production of electron-positron pairs and emission of photons. Since we are utilizing the same Bardeen Hamiltonian, then all the formulas of the ordinary theory remain valid, for example, the formulas from Sec. 36 of the book by Adriiosov, Gor'kov and Dzyaloshinskiĭ [3]. However, in these formulas one should carry out the replacement $mp_0 \to p_F^2/v_F$ (in the book cited above p_0 denotes the limiting Fermi momentum, i.e., p_F). After such a replacement the formulas are valid in the relativistic domain of p_F to the same extent as it is justified, for example, to treat an ultrarelativistic degenerate electron gas in the manner done in a number of textbooks (cf., for example, [14]). Therefore we do not reproduce a detailed derivation of the formulas used in the text, since using the prescription given above the reader can derive them himself from formulas of the ordinary theory. However, we note that in contrast to the usual theory in our formulas the matrix element for the interaction between electrons is taken to be equal to $V_{int} = p_F v_F/6N$ (cf., formula (23)).

and then we have

$$\beta_c \approx 8.6\, Mc / p_F \approx 1.6 \cdot 10^4\ (mc / p_F) \gg 1.$$

For example, for $n = 10^{35}$ we have $p_F/mc = 53$ and $\beta_c \approx 300$.

If the field exceeds the value $H_c(0)$ determined by formula (28), then a transition into the superconducting state is impossible. But as the star cools the magnetic fields in it diminish and approach the critical value $H_c(0)$, and after this the transition becomes possible. Since in the case of a transition into the superconducting state the magnetic field is forced out from the superconductor (the Meissner effect) such a transition must have the nature of an explosion of a nova or even a supernova.

As is well known, astronomical observations show that such explosions are accompanied by the formation around the star of gas nebulae the filamentary structure of which testifies to the presence of magnetic fields. Moreover, it is known that nebulae formed after supernova explosions are sources of intense cosmic radio emission, which testifies to the presence in them of a large number of relativistic electrons. The explosions of novae are not accompanied by appreciable radio emission and apparently they do not involve relativistic electrons.

All these phenomena find a qualitative explanation within the framework of the picture proposed by us. Indeed, the ejection of the external sheath of a star and the formation of a nebula can be explained by the fact that the transition into the superconducting state cannot occur all at once within the whole body of a white dwarf. The external sheath where the electron density is sufficiently small and where the effect of the temperature is essential cannot go over into a superconductor. The magnetic field pushed out from the interior portion of the star will be "frozen-in" in this sheath and the sheath will be pushed out by the field into the surrounding space.

In order to estimate the mass of the ejected sheath, we consider the optimum variant in which the energy of the expelled magnetic field is a maximum. Such a case occurs if we assume that the temperature is equal to zero, while the magnetic field at each point of the star before the transition has the maximum critical value $H_c(0)$. Then the total magnetic energy of the star will be equal to

$$W_{\mathrm{mag}}^{max} = \int \frac{H_c^2(0)}{8\pi}\, dV = \int \frac{p_e}{\beta_c}\, dV = \left\langle \frac{1}{\beta_c} \right\rangle \int p_e\, dV, \tag{30}$$

where $\langle 1/\beta_c \rangle$ is an average value, which for our estimates (cf., formula (29) and further) can be taken to be approximately equal to $\sim 10^{-3}$. Further, from the equation for equilibrium

$$\frac{dp}{dr} = m_i c_s{}^2 \frac{dn}{dr} = -\rho_i g = -\rho_i \frac{GM(r)}{r^2} \tag{31}$$

one can obtain the virial theorem

$$U_{\mathrm{gr}} = -3 \int p\, dV, \tag{32}$$

where U_{gr} is the gravitational energy of the star, and, thus, from (30) we have

$$W_{\mathrm{mag}}^{max} = \frac{1}{3} \left\langle \frac{1}{\beta_c} \right\rangle |U_{\mathrm{gr}}| \approx \frac{1}{3000} |U_{\mathrm{gr}}|. \tag{33}$$

The gravitational energy can be estimated in the following manner. If R is the radius of the star, and M is its total mass, then the energy required for the removal (lifting) of the mass ΔM from the surface of the star to infinity is evidently equal to

$$\Delta U = GM\Delta M / R. \tag{34}$$

From this it is seen that the total gravitational energy of the star can be calculated by means of the formula

$$U_{\mathrm{gr}} = -G \int_0^R \frac{M(r)\, dM(r)}{r} = -\frac{G}{2} \left[\frac{M^2}{R} + \int_0^R \frac{M^2(r)}{r^2} dr \right]. \tag{35}$$

Here and in the preceding discussion G is the gravitational constant, while M(r) is the mass contained within a sphere of radius $r < R$. The value of U_{gr} depends on the radial distribution of the density which can be found only by means of a numerical solution of equation (31). However, we can always assume that $U_{gr} = -\xi GM^2/R$, where ξ is a numerical coefficient of the order of unity. If throughout the whole body of the star the nonrelativistic case $p_F \ll mc$, holds then $\xi = 6/7$, but if the ultrarelativistic case $p_F \gg mc$ holds, then $\xi = 3/2$. Thus, from (33) we obtain

$$W_{\mathrm{mag}}^{max} = \frac{\xi}{3} \left\langle \frac{1}{\beta_c} \right\rangle \frac{GM^2}{R} \approx G \frac{M}{R} \left(\frac{M}{3000} \right), \tag{36}$$

and comparison with (34) shows that the energy of the magnetic field would be sufficient to eject a shell of mass ΔM which amounts to approximately 1/3000 of the total mass of the star.

Utilizing the equation of equilibrium (31) we can find that the thickness δ of the shell ejected in this case can be determined from the formula

$$\varepsilon_e = 1 + 1 / k^2 d_e^2 \tag{37}$$

We also note that the ejected shell contains within itself both magnetic fields and relativistic electrons, so that in this picture there is no problem of describing the generation of magnetic fields and of subsequent acceleration in them of electrons up to relativistic energies.

As is well known, the two last problems usually lead to difficulties in interpreting radio emission from cosmic objects of the type of the Crab nebula (cf., for example, [5]). However, we note that the mass estimate usually accepted for the Crab nebula $\Delta M \sim 0.1\ \odot$, where $\odot = 2 \times 10^{30}$ kg is the solar mass, can hardly be explained by the mechanism proposed here, if we consider the solar mass to be a typical mass for white dwarfs (cf., however, below).

5. Apparently, the most difficult problem in the theory proposed above is the extension of phonon concepts to such a dense electron-ion plasma and, in particular, the problem of the maximum phonon energy or, and this is the same thing, of the minimum wavelength. Above we have everywhere assumed that in order of magnitude we have just as in ordinary metals $\lambda_{\min} \sim a = n^{-1/3}$ and correspondingly $\hbar\omega_D < \hbar c_s/a$.

However, the plasma considered above does not have any regular crystalline structure, and the ions in it do not oscillate around fixed equilibrium positions. Our phonons are essentially electrostatic oscillations of ions whose charge is compensated by electrons.

In the theory of an ordinary gas plasma such oscillations are usually called "ionic sound with an electron temperature." Since these oscillations are of low frequency for electrons and of high frequency for ions it is not difficult to obtain the dispersion equation for them if to the dielectric permittivity for electrons

$$\delta = R(\Delta M / 4.6\, M)^{2/3} \approx 0.02\, R. \qquad (38)$$

one adds an ion term

$$\Delta\varepsilon_i = -\frac{\omega_{0i}^2}{\omega^2}\left(1 - i\sqrt{\frac{\pi}{2}}\beta_i^3 e^{-\beta_i^2/2}\right). \qquad (39)$$

Here we have retained the imaginary part with $\beta_i = v_{phase}/v_{Ti}$, where $v_{Ti} = \sqrt{\Theta_i/m_i}$, which for high frequency electron oscillations leads to the well known "Landau damping" (cf., for example, [6]). Then from the equation $\epsilon_{\parallel} = \epsilon_e + \Delta\epsilon_i = 0$ we obtain the dispersion law

$$v_{ph}^2 = \frac{\omega^2}{k^2} = \frac{\omega_{0i}^2 d_e^2}{1 + k^2 d_e^2}\left(1 - i\sqrt{\frac{\pi}{2}}\beta_i^3 e^{-\beta_i^2/2}\right). \qquad (40)$$

Here $\omega_{0i}^2 = 4\pi ne^2/m_i$ and in this case Eq. (38)–(40) are valid both in the classical, and also in our quantum case, but the electron Debye radius d_e is expressed differently in the two cases:

$$d_e^{cl} = \sqrt{\Theta_e / 4\pi ne^2}, \quad \Theta_e = \varkappa T_e,$$
$$d_e^{qu} = \sqrt{{}^1/_3 p_F v_F / 4\pi ne^2} = n^{-1/3}\sqrt{11.2 v_F / c}. \qquad (41)$$

The last formula is also valid in the relativistic case, in particular, for the ultrarelativistic case we have $v_F = c$ and $d_e = 3.33\, n^{-1/3}$.

From (40) it may be seen that for $kd_e \ll 1$ (i.e., $\lambda \gg d_e$) we have

$$v_{ph} = \omega_{0i} d_e = \begin{cases} \sqrt{\Theta_e/m_i} & \text{(classical case)} \\ \sqrt{p_F v_F/3m_i} = c_s & \text{(quantum case, cf., (20))} \end{cases} \qquad (42)$$

The imaginary part of ω will be small ($\mathrm{Im}\,\omega \ll \mathrm{Re}\,\omega$) and this means that the damping of the waves will be weak if $\beta_i = v_{ph}/v_{Ti} \gg 1$, which is possible in the classical case only for $T_e \gg T_i$. In the quantum case the condition $\beta_i \gg 1$ corresponds to strong degeneracy $T_F \gg T$ (for $T_i = T_e = T$). In order that the damping of the waves should be strong ($\mathrm{Im}\,\omega \approx \mathrm{Re}\,\omega$) it is necessary to have $\beta_i \approx 1$, i.e., $v_{ph} \approx v_{Ti}$, which, as can be easily seen from formula (40), in both cases corresponds to small wavelengths of the order of the Debye radius for ions (at the same time $kd_e \gg 1$)

$$k_{max}^{pl} \cong 1/d_i, \quad \lambda_{min}^{pl} \cong 2\pi d_i = 2\pi\sqrt{\Theta_i/4\pi ne^2}. \qquad (43)$$

The "plasma" values of k_{max}^{pl} and λ_{min}^{pl} obtained in this manner, which as a result of damping cut off the spectrum of ion-acoustic oscillations, should be compared to the "solid state" values for phonons utilized by us previously (cf., formula (6)):

$$k_{max}^{sol.\,st.} = k_D = (6\pi^2 n)^{1/3}, \quad \lambda_{min}^{pl} = \lambda_D = 1.6\, n^{-1/3}. \qquad (44)$$

If into formula (43) we substitute for T_i the temperature T_c of the transition into the superconducting state found earlier in Tables I and II, then we obtain approximately

$$\frac{k_{max}^{pl}}{k_D} \cong \begin{cases} 6 & \text{for } n = 10^{25} \\ 1 & \text{for } n = 10^{30}. \\ {}^1/_3 & \text{for } n = 10^{35} \end{cases} \qquad (45)$$

From this it may be seen that, for example, at a density of $n = 10^{25}$ the values of the temperatures T_D and T_c be increased by approximately a factor of six compared to their previous values in Table I. In accordance with this the critical magnetic field will also increase by a factor of six (cf., formula (28)) and this means that the mass of the stellar shell ejected as a result of the explosion can increase by a factor of 36.

We note in conclusion that the applicability of the formulas utilized in the present paper taken from the usual BCS theory in order to describe the superconductivity of a dense plasma requires, of course, a more detailed justification. Earlier Abrikosov[7] investigating the properties of a highly compressed substance under somewhat different conditions, and in particular for $T = 0$ and in the presence of a crystalline lattice, came to the conclusion that in such a "solid state" system superconductivity can be only an exponentially small effect (in particular, he had $T_c \sim \Delta \sim \exp(-\hbar v_F/e^2) \to e^{-137}$ for $v_F \ll c$). However, in this case he assumed that the strong inequality $\kappa = 1/d_e \ll k_F = p_F/\hbar$ was valid which actually does not hold (in [7] it is assumed that $v_F \ll c$), and he completely neglected the phonon attraction between electrons in the region $k < k_F$ although the latter is essential for the BCS theory. But the phonon spectrum of the lattice in the region $k \sim k_F$ is not known exactly, and therefore Abrikosov's conclusions also need to be made more precise.

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