# RADIATION BY A FAST CHARGED PARTICLE IN A NONUNIFORM PERIODIC MEDIUM

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The problem is considered of radiation by a charged particle in a medium with a density varying periodically in a plane perpendicular to the direction of the particle. It is shown that even a weak periodicity leads to an appreciable spatial redistribution of the radiation field near the resonance frequencies. Formulas are obtained for the electromagnetic field and spectral density of first order resonance radiation. The limits of applicability of the results obtained are discussed.

1. The radiation of sources moving in a medium depends not only on the nature of the radiator and its motion, but also on the electromagnetic properties of the medium.[1,2] In a uniform medium, a fast particle produces only Cerenkov radiation. The presence of periodically located nonuniformities leads, generally speaking, to interference of the radiation arising in (or traversing) different portions of the nonuniform medium. The problem of the radiation of photons by fast particles passing through a medium which changes its properties periodically only along the particle trajectory (the z axis), remaining uniform in the xy plane, was solved by Ter-Mikaelyan.[2]

The present paper is devoted to investigation of the radiation of a fast uniformly moving particle in a nonuniform medium which periodically changes its properties only in a plane perpendicular to the direction of the particle.

2. Assuming cylindrical symmetry of the problem, we can write Maxwell's equations for the Fourier component in time of the vector potential $\mathbf{A}(\omega, \rho)$ in the form

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A_z}{\partial\rho}\right)+q^2A_z=-\frac{2}{c}\delta^{(2)}(\rho),$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A_\rho}{\partial\rho}\right)-\frac{A_\rho}{\rho^2}+q^2A_\rho-\frac{1}{\varepsilon}\frac{\partial\varepsilon}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho)=i\frac{\omega}{v}\frac{1}{\varepsilon}\frac{\partial\varepsilon}{\partial\rho}A_z,$$

$$A_\varphi=0, \tag{1}$$

where

$$q=\left(\frac{\omega^2}{c^2}\varepsilon(\omega,\rho)-\frac{\omega^2}{v^2}\right)^{1/2} \tag{2}$$

(the charge of the incident particle is $e = 1$).

The systems of Eq. (1) and Eq. (2) describe the radiation field of the charge in a medium with a dielectric constant $\epsilon(\omega, \rho)$. At a large distance from the source ($qR \gg 1$) for the nonzero components of the vector potential we have the following expressions:

$$A_z(\omega,R)=\frac{i}{2c}\sqrt{\frac{2}{\pi R}}e^{-i\pi/4}\left\{\frac{1}{\sqrt{q}}\exp\left(i\int_0^R q\,d\rho\right)\right.$$

$$\left.+\int_0^\infty d\rho G_z(R,\rho)\frac{\partial^2}{\partial\rho^2}\left(\frac{1}{\sqrt{q}}\right)\exp\left(i\int_0^\rho q\,d\rho\right)\right\}, \tag{3}$$

$$A_\rho(\omega,R)=i\frac{\omega}{v}\int_0^\infty d\rho G_\rho(R,\rho)\frac{\partial}{\partial\rho}\left(\frac{1}{\varepsilon}\right)A_z(\omega,\rho), \tag{4}$$

where the Green's functions $G_z(R, \rho)$ and $G_\rho(R, \rho)$ satisfy the equations

$$\frac{\partial^2}{\partial\rho^2}G_z(\rho,\rho')+q^2G_z(\rho,\rho')=-\delta(\rho-\rho'), \tag{5}$$

$$\frac{\partial}{\partial\rho}\left(\frac{1}{\varepsilon}\frac{\partial}{\partial\rho}G_\rho(\rho,\rho')\right)+\frac{q^2}{\varepsilon}G_\rho(\rho,\rho')=-\delta(\rho-\rho'). \tag{5'}$$

After the Green's functions (5) and (5′) have been determined, formulas (3) and (4) solve the problem in general form. Note that the first term in the righthand side of Eq. (3) corresponds to Cerenkov radiation in a uniform medium.

3. It is well known that in a periodic medium when certain conditions are fulfilled (the Bragg equation, see for example ref. 3) diffracted waves can be propagated. We will consider radiation in a medium whose dielectric constant is

$$\varepsilon(\omega,\rho)=\varepsilon_0(\omega)+\varepsilon_1(\omega)\cos\frac{2\pi}{l}\rho;\quad |\varepsilon_1(\omega)|\ll|\varepsilon_0(\omega)| \tag{6}$$

($l$ is the period of the medium). Here we will be particularly interested in effects associated with reflected waves. Accordingly, the Bragg conditions and the condition for production of radiation, Eqs. (1) and (6), in our problem have the form

$$q_0(\omega)\equiv\frac{\omega}{c}\sqrt{\varepsilon_0}\sin\theta=\frac{\pi}{l}n\quad\left(n=1,2,\ldots\left[\frac{2l}{\lambda}\right]\right);$$

$$\frac{\omega}{c}\sqrt{\varepsilon_0}\cos\theta=\frac{\omega}{v}. \tag{7}$$

(The resonance conditions in particular, Eq. (7), determine the frequency and angle of the diffracted radiation, and the square brackets [ ] denote the integral part of a number.).

The Green's functions (5) and (5′) are constructed in the usual way of two linear independent solutions of the corresponding homogeneous equations, which for the condition $\lambda \ll l$ and also Eq. (6), reduce to the Mathieu equation:

$$\frac{d^2u}{d\rho^2}+(a+2\alpha_1\cos2\rho)u=0, \tag{8}$$

where we have introduced the designations

$$a=\left(\frac{q_0l}{\pi}\right)^2,\quad \alpha_1=\frac{2\varepsilon_1}{\varepsilon_0}\left(\frac{l}{\lambda}\right)^2;\quad |\alpha_1|\ll1. \tag{9}$$

4. Note that for $|\alpha_1| \ll 1$ the criterion of the quasiclassical approximation is everywhere satisfied. However, the results of the quasiclassical approxima-

tion are not applicable for determination of the reflected waves in a periodic medium. A similar problem arises in consideration of the scattering of electrons under the action of a strong electromagnetic standing wave (see ref. 4 for more detail). It is appropriate to search for "correct" solutions of Eq. (8) in the form[5]

$$u(\rho)=e^{\mu\rho}\sum_{n=-\infty}^{\infty}b_n e^{2in\rho}. \tag{10}$$

After substitution of (10) into (8) we have a system of homogeneous equations

$$[a+(\mu+2in)^2]b_n+a_1(b_{n-1}+b_{n+1})=0, \tag{11}$$

and to determine $\mu$ as a function of the parameters a and $\alpha_1$ we obtain Hill's dispersion equation. For small values of $|\alpha_1|$ this equation takes the form

$$\operatorname{ch}\mu\pi=\cos\pi\sqrt{a}-\frac{\pi\alpha_1^2}{4(1-a)\sqrt{a}}\sin\pi\sqrt{a}+O(\alpha_1^4). \tag{12}$$

In the region of stability of the Mathieu equation (8), when $|\sqrt{a}-n|\gg|\alpha_1|$, the system of Eqs. (11) is solved by the method of successive approximations, which is equivalent to finding the amplitudes of the reflected waves by means of perturbation theory.[6] The greatest interest is attached to study of the radiation of a charge near the characteristic curves which divide the $(a,\alpha_1)$ plane into regions of stability and instability.

5. We will consider for simplicity resonance radiation of the first order, $a=1+\alpha_0$, $|\alpha_0|\ll 1$. Taking into account Eqs. (11) and (12), we find accordingly for the linearly independent solutions (10)

$$u_1(\rho)\approx\exp\left\{\frac{i}{2}\sqrt{\alpha_0^2-\alpha_1^2}\,\rho\right\}(e^{i\rho}-\beta e^{-i\rho}),$$
$$u_2(\rho)\approx\frac{u_1(-\rho)+\beta u_1(\rho)}{1-\beta^2}, \tag{13}$$

where $\beta$ defines the degree of deviation from the resonance condition:

$$\beta=\frac{\alpha_1}{\alpha_0+\sqrt{\alpha_0^2-\alpha_1^2}} \tag{14}$$

(in obtaining the solutions (13) we neglected terms of order $|\alpha_1|\ll 1$). Using the definition of Green's functions and also Eq. (13), we obtain the following expressions for the nonzero components of the vector potential:

$$A_z(\omega,R)=\frac{i}{2c}\sqrt{\frac{2}{\pi q_0 R}}\,e^{-i\pi/4}u_1(R),$$
$$A_\rho(\omega,R)=\frac{i}{2c}\sqrt{\frac{2}{\pi q_0 R}}\,e^{-i\pi/4}\beta\sin 2\theta\exp\left\{i\left(\frac{1}{2}\sqrt{\alpha_0^2-\alpha_1^2}-1\right)q_0R\right\}. \tag{15}$$

It is evident from (14) and (15) that the degree of approximation to the condition of first order resonance is determined by the ratio of the quantities $\alpha_0$ and $\alpha_1$. Here the parameter $\beta$ is the "amplitude" of the wave reflected from the periodic medium. At frequencies for which $|\alpha_0|\gg|\alpha_1|$, the radiation field of the charge is Cerenkov radiation.

From the formulas (15) and also taking into account the Lorentz conditions, we find for the nonzero components of the electromagnetic field $\mathscr{E}_i(\omega,\mathbf{r})$ and $H_i(\omega,\mathbf{r})$

$$\sqrt{\varepsilon_0}\,\mathscr{E}_z(\omega,\mathbf{r})=-H_0\{\sin\theta e^{iRq_0}+\beta\cos 2\theta\sin\theta e^{-iRq_0}\}\times\exp\left\{i\frac{\omega}{v}z+\frac{i}{2}\sqrt{\alpha_0^2-\alpha_1^2}Rq_0\right\},$$
$$\sqrt{\varepsilon_0}\,\mathscr{E}_\rho(\omega,\mathbf{r})=-H_0\{-\cos\theta e^{iRq_0}+\beta\cos 2\theta\cos\theta e^{-iRq_0}\}\times\exp\left\{i\frac{\omega}{v}z+\frac{i}{2}\sqrt{\alpha_0^2-\alpha_1^2}Rq_0\right\},$$
$$H_\varphi(\omega,\mathbf{r})=H_0\{e^{iRq_0}-\beta\cos 2\theta e^{-iRq_0}\}\exp\left\{i\frac{\omega}{v}z+\frac{i}{2}\sqrt{\alpha_0^2-\alpha_1^2}Rq_0\right\}, \tag{16}$$

where we have introduced the designation

$$H_0=-\frac{\omega}{c^2}\sqrt{\frac{\varepsilon_0}{2\pi q_0 R}}\sin\theta e^{-i\pi/4}.$$

Let us compare the values of the spectral density of radiation per unit length near $q_0(\omega)\approx\pi/l$ in the radial and longitudinal directions, respectively:

$$\overline{\frac{d^2I}{dz\,d\omega}}=\left(\frac{d^2I}{dz\,d\omega}\right)_0(1-|\beta|^2\cos^2 2\theta)\exp\{-\operatorname{Im}\sqrt{\alpha_0^2-\alpha_1^2}Rq_0\},$$
$$\overline{\frac{d^2I}{d\rho\,d\omega}}=\left(\frac{d^2I}{d\rho\,d\omega}\right)_0(1+|\beta|^2\cos^2 2\theta)\exp\{-\operatorname{Im}\sqrt{\alpha_0^2-\alpha_1^2}Rq_0\} \tag{17}$$

(the bar above a quantity indicates averaging in the radial direction with the period of the medium $l$). Here

$$\left(\frac{d^2I}{d\rho\,d\omega}\right)_0 \text{ и } \left(\frac{d^2I}{dz\,d\omega}\right)_0=\operatorname{tg}\vartheta\left(\frac{d^2I}{d\rho\,d\omega}\right)_0$$

is the corresponding density of Cerenkov radiation in the medium $\epsilon_0(\omega)$.

The above discussion shows that the presence of even a weak periodicity of the medium, $|\epsilon_1(\omega)|\ll|\epsilon_0(\omega)|$, in a plane perpendicular to the direction of the particle leads to a sharp spatial redistribution of the radiation near the frequencies given by Eq. (7). From (16) it follows that if the resonance conditions are accurately satisfied with n = 1 the radiation penetrates into the medium a distance $\sim l/|\alpha_1|\pi$. Here the energy flux in a direction along the z axis is

$$\overline{\frac{d^2I}{d\rho\,d\omega}}\approx 2\left(\frac{d^2I}{d\rho\,d\omega}\right)_0$$

For the frequency interval of the first order resonance radiation we obtain the following estimate ($\alpha_0\approx\alpha_1$):

$$\delta\omega\sim\left(\frac{\varepsilon_1}{\varepsilon_0}\frac{1}{\lambda\!\!\!^{-}{}^2}\frac{d\omega}{dq_0^2}\right)_{\omega=\omega_*} \tag{18}$$

where $\omega_*$ is determined from the equation

$$\frac{\varepsilon_0(\omega_*)}{c^2}=\frac{1}{v^2}+\frac{\pi^2}{\omega_*^2 l^2}. \tag{19}$$

The applicability of the formulas obtained impose definite limitations on the collimation $\Delta\vartheta$ of the incident particle beam and the longitudinal dimension of the plate L:

$$\Delta\vartheta\ll\theta_0,\quad \theta_0=\theta_0(\omega_*), \tag{20}$$
$$l^2/|\alpha_1|\lambda\ll L\ll\theta_0^2 L_{\text{rad}}(E/E_s)^2. \tag{21}$$

Here $E_s$ = 21 MeV, and $L_{rad}$ is the radiation length. Relation (21) leads also to a condition imposed on the particle energy E.

In conclusion we note that for resonance radiation with $n > 1$ the degree of strictness of the limitations on the angular spread of the beam and the energy of the particle increases. Here the estimates (20) and (21) are preserved if we make the substitution $\alpha_1 \to \alpha_1^n$.

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221