CONTRIBUTION TO THE THEORY OF CONDUCTIVITY OF PLASMA MEDIA IN CROSSED ELECTRIC AND MAGNETIC FIELDS

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Using an exact solution of the Boltzmann kinetic equation, we calculate the components of the conductivity tensor in crossed electric and magnetic fields. It is shown that if the momentum relaxation time depends on the particle energy, then the diagonal component of the conductivity tensor can be larger in a weak field than in the absence of a magnetic field.

In a conducting medium placed in a magnetic field, the connection between the current density and the electric field intensity in the region of small electric fields remains linear, but the components of the conductivity tensor depend on the magnitude of the magnetic field. In a homogeneous and isotropic medium, in crossed electric and magnetic fields, there are, besides the two independent diagonal components of the conductivity tensor, also nonzero two nondiagonal components, called the Hall components, which are equal in absolute magnitude.

For this to happen in a solid it is necessary to have, beside spatial isotropy, also the existence of spherical or ellipsoidal constant-energy surfaces of the carriers, but then the conductivity tensor is referred to a coordinate system which is connected with the principal axes of the energy ellipsoid[1]. It is therefore possible to confine oneself to a consideration of only a quadratic dependence of the energy on the momentum, and to a calculation of two components of the conductivity tensor, the diagonal transverse component σ, and the Hall component σ_H, since the diagonal longitudinal component does not depend on the magnetic field.

In a rarefied plasma media, even at moderate values of the magnetic field H, the particle has time to execute during scattering a sufficiently large number of revolution with cyclotron frequency $\Omega_c = eH/mc$, where e is the charge and m the effective mass of the particle, and c is the velocity of light in vacuum. In this case the particle trajectories are such, that the conductivity σ appears only as a result of the presence of a definite collision mechanism.

The Hall conductivity is connected with the existence of a constant drift with velocity $-cE/H$, where E is the intensity of the electric field and does not depend on the scattering mechanism. In a dense medium, when the particle does not have time to execute even a single revolution during the time between collisions, the motion trajectories can be close to straight lines (as in the absence of a magnetic field), but the presence of a constant drift causes the particles moving in one of the directions from the plane of the fields to have a larger kinetic energy than particles moving in the opposite direction from the plane or in the plane of the field.

The carrier free paths in different directions are different, and the mean free path can depend on cE/H. In the case of the motion of a charged particle in crossed fields, the integrals of the motion are the velocity along the magnetic field v_y and the energy of the rotational motion W, equal to

$$W = \frac{mv_z^2}{2} + \frac{m}{2}\left(v_x + \frac{cE}{H}\right)^2.$$

Thus, v_y and W remain unchanged between two scattering acts, whereas the kinetic energy ϵ of the particle changes. And if the particle trajectories are almost straight lines, then the current along the electric field **E** is the result of the compensation of the velocity within the effective collision time. At constant v_y and W we have

$$\Delta v_z = \frac{eE}{m}\left(1 + \frac{H}{cE} v_x\right)\Delta t,$$

and if the scattering of the particles is the same for all values, then Δt is constant and the average Δv_z does not depend on cE/H.

In the case when the particle free path time depends on its kinetic energy, Δt is the average time along the trajectory at constant values of v_y and W. In this case the conductivity depends on the dimensionless quantity $\alpha = v_T H/cE$, where $v_T = \sqrt{2kT/m}$, T is the absolute temperature, and k is Boltzmann's constant.

To obtain the functions $\sigma(H, \alpha)$ and $\sigma_H(H, \alpha)$ in crossed fields, we start from a solution of the Boltzmann kinetic equation for the distribution function $f(\mathbf{v})$:

$$\frac{e}{m}\left(\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right)\frac{\partial f}{\partial \mathbf{v}} + \frac{1}{\tau_0(\varepsilon)} f = \frac{1}{\tau_0(\varepsilon)} S\{f\}, \qquad (1)$$

where $\tau_0(\epsilon)$ is the total momentum relaxation time, ϵ is the energy of the particle in units of kT, and $S\{f\}$ is that part of the collision integral which represents the arrival of the particles into the considered velocity and energy region as a result of both elastic isotropic and anisotropic collisions and inelastic collisions.

It is convenient to change over to the dimensionless quantities

$$E_0 = \frac{e\tau_0(1)E}{\sqrt{2mkT}}, \qquad \omega_0 = \frac{e\tau_0(1)H}{mc}, \qquad \alpha = \frac{\omega_0}{E_0}. \qquad (2)$$

Then in a spherical coordinate system, with the polar axis along the field $\mathbf{E}_0$, Eq. (1) takes the form

$$\omega_0 \frac{\cos\theta}{\sin\theta}\sin\varphi\frac{\partial f}{\partial\varphi} - \omega_0\cos\varphi\frac{\partial f}{\partial\theta} + E_0\sqrt{\varepsilon}\left(2\cos\theta\frac{\partial f}{\partial\varepsilon} - \frac{\sin\theta}{\varepsilon}\frac{\partial f}{\partial\theta}\right) + \frac{f}{\tau_0(\varepsilon)} = \frac{S\{f\}}{\tau_0(\varepsilon)}, \qquad (3)$$

where θ and φ are the polar and azimuthal angles, and $\tau_0(\epsilon)$ is now normalized in such a way that $\tau_0(1) = 1$.

When $E_0 \ll 1$, we can confine ourselves to the relaxation part of the collision integral (see the appendix), and if we assume for simplicity that the particle scattering is isotropic, then $S\{f\} = f_0(\epsilon)$, where $f_0(\epsilon)$ is the equilibrium distribution function. In this case the solution of such an equation is usually obtained by putting the function $f(\epsilon, \theta, \varphi)$ in the parentheses of the left side equal to the equilibrium value $f_0(\epsilon)$ (the so-called linearization with respect to the electric field; see, for example,[1]. Mathematically this corresponds to retaining in the expansion of $f(\epsilon, \theta, \varphi)$ in spherical functions of the variables θ and φ only the first spherical function with coefficients $f_1(\epsilon)$ and $f_1^1(\epsilon)$, the first of which is proportional to E_0 and the second to $\omega_0 E_0$.

Such a linearization, generally speaking, is possible only in the absence of a magnetic field or in magnetic fields when $\alpha \gg 1$. In the presence of a magnetic field, retaining in the parentheses of the left side of (3) the equilibrium function $f_0(\epsilon)$ alone, we neglect terms of order E_0^2 compared with the remaining terms of order $\omega_0 E_0$, or even smaller ones, of order $\omega_0^2 E_0$, if ω_0 is small. In the general case it is easy to obtain the solution (3) with respect to the right side $S\{f\} = S_0 + S_1$, where S_0 and S_1 are the symmetrical and antisymmetrical parts of $S\{f\}$ with respect to the variable $\mu = \cos\theta$. To this end it is necessary to change over to the variables

$$\varepsilon, \quad \zeta = \sqrt{\varepsilon(1-\mu^2)}\sin\varphi, \quad w = \varepsilon + \frac{2\sqrt{\varepsilon}}{\alpha}\sqrt{1-\mu^2}\cos\varphi. \tag{4}$$

In terms of the new variables, Eq. (3) takes the form

$$\frac{\partial f^\pm}{\partial \varepsilon} \pm \frac{f^\pm}{\omega_0\tau_0(\varepsilon)\sqrt{(\varepsilon_2-\varepsilon)(\varepsilon-\varepsilon_1)}} = \frac{\pm S_0 + S_1}{\omega_0\tau_0(\varepsilon)\sqrt{(\varepsilon_2-\varepsilon)(\varepsilon-\varepsilon_1)}}, \tag{5}$$

Here $f^+ = f$ when $\mu \geq 0$, $f^- = f$ when $\mu \leq 0$, and ϵ_2, ϵ_1 are the roots of the equation $\mu = 0$, which do not depend explicitly on ϵ, such that $\epsilon_1 < \epsilon < \epsilon_2$. The explicit dependence of ϵ_2 and ϵ_1 on the variables ϵ, θ, and φ, is

$$\varepsilon_{2,1} = \varepsilon + 2\varepsilon\beta(v_0+\beta) \pm 2\varepsilon\beta\sqrt{(v_0+\beta)^2+\mu^2},$$

$$\beta = 1/\alpha\sqrt{\varepsilon}, \qquad v_0 = \sqrt{1-\mu^2}\cos\varphi. \tag{6}$$

S_0 and S_1 are expressed in terms of θ and φ in accordance with (4).

As a result we get an expression for $f(\epsilon, \theta, \varphi)$:

$$f^+ = S^+ + \frac{e^{-\psi(\varepsilon)}}{\operatorname{sh}\psi(\varepsilon_2)}\left\{\int_\varepsilon^{\varepsilon_2}\operatorname{sh}[\psi(\varepsilon_2)-\psi(x)]\frac{dS_0}{dx}dx - e^{\psi(\varepsilon_2)}\int_{\varepsilon_1}^{\varepsilon}\operatorname{sh}\psi(x)\frac{dS_0}{dx}dx\right.$$
$$\left. - \int_\varepsilon^{\varepsilon_2}\operatorname{ch}[\psi(\varepsilon_2)-\psi(x)]\frac{dS_1}{dx}dx - e^{\psi(\varepsilon_2)}\int_{\varepsilon_1}^{\varepsilon}\operatorname{ch}\psi(x)\frac{dS_1}{dx}dx\right\},$$

$$f^- = S^- + \frac{e^{\psi(\varepsilon)}}{\operatorname{sh}\psi(\varepsilon_2)}\left\{\int_\varepsilon^{\varepsilon_2}\operatorname{sh}[\psi(\varepsilon_2)-\psi(x)]\frac{dS_0}{dx}dx - e^{-\psi(\varepsilon_2)}\int_{\varepsilon_1}^{\varepsilon}\operatorname{sh}\psi(x)\frac{dS_0}{dx}dx\right.$$
$$\left. - \int_\varepsilon^{\varepsilon_2}\operatorname{ch}[\psi(\varepsilon_2)-\psi(x)]\frac{dS_1}{dx}dx - e^{-\psi(\varepsilon_2)}\int_{\varepsilon_1}^{\varepsilon}\operatorname{ch}\psi(x)\frac{dS_1}{dx}dx\right\}, \tag{7}$$

where

$$\psi(x) = \frac{1}{\omega_0}\int_{\varepsilon_1}\frac{dx}{\tau_0(x)\sqrt{(\varepsilon_2-x)(x-\varepsilon_1)}}, \tag{8}$$

and the derivatives under the integral signs are taken at constant values of the variables ζ and w. Multiplying (7) by μ and v_0 and integrating over the solid angle, we obtain

$$f_1(\varepsilon) = S_1 - \int\frac{\mu d\Omega}{\operatorname{sh}\psi(\varepsilon_2)}\left\{\operatorname{sh}\psi(\varepsilon)\int_\varepsilon^{\varepsilon_2}\left[\operatorname{sh}[\psi(\varepsilon_2)-\psi(x)]\frac{dS_0}{dx}\right.\right.$$
$$\left. - \operatorname{ch}[\psi(\varepsilon_2)-\psi(x)]\frac{dS_1}{dx}\right]dx + \operatorname{sh}[\psi(\varepsilon_2)-\psi(\varepsilon)]$$
$$\left.\times\int_{\varepsilon_1}^{\varepsilon}\left[\operatorname{sh}\psi(x)\frac{dS_0}{dx} + \operatorname{ch}\psi(x)\frac{dS_1}{dx}\right]dx\right\},$$

$$f_1^1(\varepsilon) = S_1^1 - \int\frac{v_0 d\Omega}{\operatorname{sh}\psi(\varepsilon_2)}\left\{\operatorname{ch}\psi(\varepsilon)\int_{\varepsilon_1}^{\varepsilon_2}\left[\operatorname{sh}[\psi(\varepsilon_2)-\psi(x)]\frac{dS_0}{dx}\right.\right.$$
$$\left. - \operatorname{ch}[\psi(\varepsilon_2)-\psi(x)]\frac{dS_1}{dx}\right]dx - \operatorname{ch}[\psi(\varepsilon_2)-\psi(\varepsilon)]$$
$$\left.\times\int_{\varepsilon_1}^{\varepsilon}\left[\operatorname{sh}\psi(x)\frac{dS_0}{dx} + \operatorname{ch}\psi(x)\frac{dS_1}{dx}\right]dx\right\}. \tag{9}$$

The densities of the conduction and of the Hall currents are expressed as follows:

$$j_z = \frac{2\pi e}{3}\sqrt{\frac{2kT}{m}}\int_0^\infty f_1(\varepsilon)\varepsilon d\varepsilon, \qquad j_x = \frac{2\pi e}{3}\sqrt{\frac{2kT}{m}}\int_0^\infty f_1^1(\varepsilon)\varepsilon d\varepsilon. \tag{10}$$

Just as in the absence of a magnetic field, when $E_0 \ll 1$ the deviation of the symmetrical part of the distribution function from its equilibrium value $f_0(\epsilon)$ is proportional to E_0^2. This can be verified by considering the equation

$$E_0\frac{\tau_0(\varepsilon)}{\sqrt{\varepsilon}}\frac{d(\varepsilon f_1)}{d\varepsilon} = -f_0 + S_0\{f_0\}\tau_0(\varepsilon),$$

which is obtained from (3) by integrating over the solid angle. Therefore, confining ourselves to weak electric fields, we retain in $S\{f\}$ only the relaxation part of the collision integral.

Let the elastic scattering be isotopic and let (we omit the index zero throughout)

$$\tau(\varepsilon) = (1-\delta+\varepsilon\delta)^{-1}, \quad 0 \leqslant \delta \leqslant 1.$$

Making the change of variable

$$\varepsilon = {}^1/_2(\varepsilon_2+\varepsilon_1) - {}^1/_2(\varepsilon_2-\varepsilon_1)\cos\eta$$

we calculate from (8) the scattering factor $\psi(\eta)$:

$$\psi(\eta) = \frac{1}{\omega}\left\{\left[1-\delta+\frac{\delta}{2}(\varepsilon_2+\varepsilon_1)\right]\eta - \frac{\delta}{2}(\varepsilon_2-\varepsilon_1)\sin\eta\right\}. \tag{11}$$

Further, for the scattering mechanism under consideration we can write

$$2\psi(\varepsilon_2) - \psi(\eta) = \psi(2\pi-\eta),$$

as a result of which we obtain from (9) the following useful expressions

$$f_1(\varepsilon) = \int \mu e^{-\psi(\eta)}d\Omega\int_0^\infty e^{-\psi(\xi-\eta)}\frac{dS_0(\xi-\eta)}{d\xi}d\xi,$$

$$f_1^1(\varepsilon) = \int v_0 e^{-\psi(\eta)}d\Omega\int_0^\infty e^{-\psi(\xi-\eta)}\frac{dS_0(\xi-\eta)}{d\xi}d\xi. \tag{12}$$

From (12), (11), and (10) with $S_0 = f_0(\epsilon)$ we obtain for a nondegenerate carrier gas

$$\sigma = \frac{ne^2\tau(1)}{m}\int_0^\infty\frac{e^{-g(y)}dy}{(1+\delta y)^{5/2}}\left\{\cos\omega y\right.$$

$$-\frac{2\delta}{\alpha^2}\frac{1-\cos\omega y}{1+\delta y}\left(\frac{\sin\omega y}{\omega}+\delta\frac{1-\cos\omega y}{\omega^2}\right)\Big\}, \quad (13)$$

$$\sigma_H=-\frac{ne^2\tau(1)}{m}\int_0^\infty\frac{e^{-g(y)}\,dy}{(1+\delta y)^{1/2}}\Big\{\sin\omega y$$

$$-\frac{2\delta}{\alpha^2}\frac{1-\cos\omega y}{1+\delta y}\left[\frac{1-\cos\omega y}{\omega}+\frac{\delta}{\omega}\left(y-\frac{\sin\omega y}{\omega}\right)\right]\Big\},$$

where

$$g(y)=(1-\delta)y+\frac{2\delta}{\alpha^2}\Big[y-\frac{\sin\omega y}{\omega}$$

$$-\frac{\delta}{1+\delta y}\left(\frac{y^2}{2}-y\frac{\sin\omega y}{\omega}+\frac{1-\cos\omega y}{\omega^2}\right)\Big] \quad (14)$$

For a degenerate gas we have

$$\sigma=-\frac{n_0e^2\tau(1)}{m}\frac{3\sqrt{\pi}}{4\varepsilon_0^{3/2}}\int_0^\infty x^{3/4}\frac{df_0}{dx}\,dx\int_0^\infty e^{-g_0(y)}$$

$$\times\Big\{\frac{I_{1/2}(2\sqrt{ax})}{a^{1/4}}\Big[\cos\omega y-\frac{2\delta}{\alpha^2}\frac{\sin\omega y}{\omega}(1-\cos\omega y)\Big]$$

$$+\frac{2\delta^2}{\alpha^2}\sqrt{x}\frac{I_{3/2}(2\sqrt{ax})}{a^{3/4}}(1-\cos\omega y)\left(y\frac{\sin\omega y}{\omega}-\frac{1-\cos\omega y}{\omega^2}\right)\Big\}dy,$$

$$\sigma_H=\frac{n_0e^2\tau(1)}{m}\frac{3\sqrt{\pi}}{4\varepsilon_0^{3/2}}\int_0^\infty x^{3/4}\frac{df_0}{dx}\,dx\int_0^\infty e^{-g_0(y)}\Big\{\frac{I_{1/2}(2\sqrt{ax})}{a^{1/4}}\Big[\sin\omega y$$

$$-\frac{2\delta}{\alpha^2}\frac{(1-\cos\omega y)^2}{\omega}\Big]+\frac{2\delta}{\alpha^2}\sqrt{x}\frac{I_{3/2}(2\sqrt{ax})}{a^{3/4}}\Big[\frac{y}{\omega}(1-\cos\omega y)^2$$

$$-\frac{1}{\omega}\left(y-\frac{\sin\omega y}{\omega}\right)\Big]\Big\}\,dy, \quad (15)$$

where

$$g_0=(1-\delta)y+x\delta y+\frac{2\delta}{\alpha^2}\left(y-\frac{\sin\omega y}{\omega}\right),$$

$$a=\frac{\delta^2}{\alpha^2}\Big[y^2+\frac{2}{\omega^2}(1-\cos\omega y)-2y\frac{\sin\omega y}{\omega}\Big],$$

$$n_0=\frac{\varepsilon_0^{3/2}}{3\pi}\left(\frac{2mkT}{\hbar^2}\right)^{3/2}$$

ϵ_0 is the Fermi energy in units of kT, $\hbar$ is Planck's constant, and I_{nf_2} is a Bessel function of imaginary argument.

Putting $\delta = 0$, we obtain the mechanism of scattering with a constant relaxation time. In this case, as expected, the quantities σ and σ_H do not depend on α and coincide with those calculated by the method of linearizing the kinetic equation. However, when $\delta \neq 0$ or $\delta = 1$, the scattering depends on the energy and, as seen from (13)--(15), $d\sigma/d\alpha > 0$ and consequently, when $\omega \ll 1$ the quantity σ becomes larger than σ_0, i.e., the conductivity increases in the magnetic field until the trajectory twisting becomes appreciable ($\omega \sim 1$). At larger values of α and ω, i.e., when the thermal velocity greatly exceeds the magnetic-drift velocity cE/H, and there is a strong twisting of the trajectories between collisions, the results obtained by the method of linearizing the kinetic equation are valid.

The considered case at $\delta \neq 0$ is more likely an example of an exact calculation than an actually realized scattering mechanism. For all forms of scattering, the analytic calculation of σ and σ_H can be carried out only for limiting values of $\alpha(\omega \ll 1)$, thus obtaining small corrections to the components of the conductivity tensor. The correction for α, however, is always positive.

Let, for example, $\tau(\epsilon) = \epsilon^{-1/2}$, which in an isotropic semiconductor corresponds to scattering by the acoustic lattice vibrations. The influence of heating can be disregarded[2] when $E_0 \ll 2c_0\sqrt{m/kT}$, where c_0 is the speed of sound in the crystal. The scattering is obviously isotropic. When $\alpha \gg 1$, and consequently when

$$\gamma=(\varepsilon_2-\varepsilon_1)/(\varepsilon_2+\varepsilon_1)\ll 1,$$

we obtain from (8) accurate to terms in γ^2,

$$\psi(\eta)=\frac{1}{\omega}\sqrt{\frac{\varepsilon_2+\varepsilon_1}{2}}\left(1-\frac{\gamma^2}{16}\right)\eta$$

$$-\frac{\varepsilon_2-\varepsilon_1}{2\omega\sqrt{2(\varepsilon_2+\varepsilon_1)}}\left(1+\frac{\gamma}{8}\cos\eta\right)\sin\eta.$$

Just as in the preceding case, using (4), we obtain for a nondegenerate gas when $\omega \ll 1$

$$\sigma=\frac{4e^2n\tau(1)}{3\sqrt{\pi}m}\left(1+\frac{1}{\alpha^2}\right),\quad \sigma_H=-\frac{2}{3}\frac{ne^2\tau(1)}{m}\omega\left(1-\frac{16}{5\alpha^2}\right) \quad (16)$$

and σ in a magnetic field is larger than without a field.

In (16), the quadratic corrections in terms of the small quantity E_0 are disregarded, and therefore these formulas are valid in those regions where α^{-2} greatly exceeds E_0^2.

In anisotropic scattering by a screened Coulomb potential at large α and $\omega \ll 1$, it is necessary to retain in the relaxation part of the collision integral all the terms of the expansion in spherical functions. This can be verified by considering the obtained expression for the total function $f(\epsilon, \theta, \varphi)$ (7) and recalling that the derivatives of S are taken at constants ζ and w. Then the conductivity is determined not by a single transport cross section, but by all the moments of the scattering cross section, and calculations of σ and σ_H call for a separate investigation. However, for small values of α and $\omega \ll 1$ it is possible to confine oneself to the zeroth and first spherical functions.

Let us consider, for small values of $\alpha(\omega \ll 1)$, scattering by acoustic lattice vibrations and by a charged impurity, when $\tau_0(\epsilon) = \epsilon^{1/2}$. Since ϵ_2 is large when $\alpha \ll 1$, the calculation of the factors for scattering by acoustic vibrations and impurities, ψ_{ac} and ψ_q respectively, will be calculated by expanding in terms of the quantity ϵ_1/ϵ_2.

Omitting terms of the order of

$$k_2=\left[\frac{1-(1-\varepsilon_1/\varepsilon_2)^{1/4}}{1+(1-\varepsilon_1/\varepsilon_2)^{1/4}}\right]^2\approx\frac{1}{64}\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2,$$

we obtain

$$\psi_{ac}=\frac{8\sqrt{\varepsilon_2k_2}}{\omega(1+\sqrt{k_2})}\left[\left(1+2\frac{\sqrt{k_2}}{\varepsilon_1}\varepsilon\right)\sqrt{\varepsilon(\varepsilon-\varepsilon_1)}+\varepsilon_1\ln\frac{\sqrt{\varepsilon}+\sqrt{\varepsilon-\varepsilon_1}}{\sqrt{\varepsilon_1}}\right],$$

$$\psi_q=\frac{2}{\omega_0\varepsilon_1\sqrt{\varepsilon_2}}\left(\sqrt{\frac{\varepsilon-\varepsilon_1}{\varepsilon}}+4\sqrt{k_2}\ln\frac{\sqrt{\varepsilon}+\sqrt{\varepsilon-\varepsilon_1}}{\sqrt{\varepsilon_1}}\right). \quad (17)$$

Substituting (17) in (9) we get

$$f_1(\varepsilon)=f_0(\varepsilon)\int\mu\left(\frac{d\psi}{dx}\right)^{-1}_{x=\varepsilon}d\Omega. \quad (18)$$

Further, the integration over the solid angle in (18) can be carried out by expanding the integrand in a series in powers of the small quantity α. As a result we obtain for acoustic scattering

$$\sigma=\frac{4}{3\sqrt{\pi}}\frac{ne^2\tau(1)}{m}\left(1+\frac{\alpha^2}{10}\right) \tag{19}$$

and for scattering by a charged impurity

$$\sigma=\frac{8ne^2\tau(1)}{\pi m}\left(1+\frac{\alpha^2}{5}\right). \tag{20}$$

In both cases, the conductivity σ increases in an external magnetic field.

In conclusion, I am grateful to L. P. Pitaevskiĭ and A. A. Rukhadze for a number of valuable remarks.

APPENDIX

TRANSFORMATIONS OF COLLISION INTEGRALS

The complete collision integral $I\{f\}$, which describes the change of the function $f(\epsilon, \theta, \varphi)$ upon collision of the carriers with other particles, is given by by[3]

$$I\{f\}=N\int\int dp_1 d\Omega q(u,\chi')u\,\{f(p')F(p_1')-f(p)F(p_1)\}, \tag{A.1}$$

where N is the number of particles per unit volume with which the carriers collide, and $F(p_1)$ in their distribution function normalized to unity; p' and p_1' are the momenta prior to collision, and p and p_1 are the momenta after the collision; $u = p/m - p_1/M$ is the relative velocity; $q(u, \chi')$ is the differential effective scattering cross section as a function of the relative velocity of the colliding particles and of the scattering angle between the velocities $\mathbf{u}$ and $\mathbf{u}' = p'/m - p_1'/M$. The internal integration over the scattering angles $d\Omega = \sin\chi' d\chi' d\kappa'$ is carried out at a fixed value of the momentum p_1.

Our problem is to reduce (A.1) to the form used in Eq. (1), and to ascertain the conditions under which it is possible to use the relaxation parts of the collision integral in the form $-(f - f_0)/\tau_0(\epsilon)$ in isotropic scattering.

In the case of the elastic collision with a heavy particle of mass $M \gg m$, the change of the energy $\Delta\epsilon$ is insignificant, and the change of the momentum is arbitrary, if there is no predominant small-angle scattering. It is therefore possible to expand the functions $f(p') = f(\epsilon + \Delta\epsilon, \theta', \varphi')$ and $F(p_1') = F(\epsilon - \Delta\epsilon, \theta_1', \varphi_1')$ in a series in $\Delta\epsilon$, and to consider such $q(u, \chi')$ for which small-angle scattering does not make a decisive contribution to the collision integral. The opposite case leads to the well known Landau collision integral, when the expansion of the functions f and F is better carried out in powers of $\Delta p = p' - \mathbf{p}$ and $\Delta p_1' = p_1' - p_1$[4].

From the energy and momentum conservation in elastic collisions we obtain

$$\Delta\varepsilon=\frac{2m'}{M+m}\left\{-\left(1-\frac{m}{M}\right)\sqrt{\frac{M}{m}\varepsilon\varepsilon_1}(1-\cos\chi')\cos\vartheta\right.$$
$$\left.+\frac{m}{m'}\sqrt{\frac{M}{m}\varepsilon\varepsilon_1}\sin\chi'\sin\vartheta\cos\varkappa'+(\varepsilon_1-\varepsilon)(1-\cos\chi')\right\},$$

where κ' is the angle between the planes $(\mathbf{u}, \mathbf{u}')$ and $(\mathbf{p}, \mathbf{p}_1)$; m' is the reduced mass. We shall assume that the heavy particles have a Maxwellian distribution: $F(p_1) = \pi^{-1/2}e^{-\epsilon S}$, and then the series expansion is possible if $2\sqrt{m\epsilon\epsilon_1/M} \ll 1$. This means that the collision integral obtained below is valid for carrier energy satisfying the inequality (in dimensionless units) $\epsilon \ll M/m$.

Accurate to terms of order m/M in the result of the series function are the functions $f(p')$ and $F(p_1)$, up to second order in $\Delta\epsilon$, we obtain for $I\{f\}$ the integral

$$I\{f\}=-\nu_0(\varepsilon)f(\mathbf{p})+\int\nu(\varepsilon,\chi')f'\,d\Omega'$$
$$+\frac{2m}{M}\frac{1}{\sqrt{\varepsilon}}\frac{\partial}{\partial\varepsilon}\left\{\varepsilon^{3/2}\int\nu_{tr}(\varepsilon,\chi')\left(\frac{df'}{d\varepsilon}+f'\right)d\Omega'\right\}$$

or, after expanding in a series in spherical functions:

$$f(\varepsilon,\theta',\varphi')=\sum f_n{}^m(\varepsilon)Y_{n,m}(\theta,\varphi'),$$
$$\nu(\varepsilon,\chi')=\frac{1}{4\pi}\sum(2n+1)\nu_n(\varepsilon)P_n(\cos\chi'),$$

$$I\{f\}=-\nu_0(\varepsilon)f(\mathbf{p})+\sum\nu_n(\varepsilon)f_n{}^m(\varepsilon)Y_{n,m}(\theta,\varphi)$$
$$+\frac{2m}{M}\sum Y_{n,m}(\theta,\varphi)\frac{1}{\sqrt{\varepsilon}}\frac{d}{d\varepsilon}\left\{\varepsilon^{3/2}\nu_{tr}{}^n(\varepsilon)\left(\frac{df_n{}^m}{d\varepsilon}+f_n{}^m\right)\right\}.$$

From this we obtain for isotropic scattering $\nu_0(\epsilon) = \nu(\epsilon)$

$$I\{f\}=-\frac{f-f_0}{\tau(\varepsilon)}+\frac{2m}{M}\frac{1}{\sqrt{\varepsilon}}\frac{d}{d\varepsilon}\left\{\varepsilon^{3/2}\nu(\varepsilon)\left(\frac{df_0}{d\varepsilon}+f_0\right)\right\}$$
$$-\frac{2m}{M}\sum_m Y_{1,m}(\theta,\varphi)\frac{1}{\sqrt{\varepsilon}}\frac{d}{d\varepsilon}\left\{\varepsilon^{3/2}\nu(\varepsilon)\left(\frac{df_1{}^m}{d\varepsilon}+f_1{}^m\right)\right\}.$$

By way of an example of an inelastic collision integral, let us consider the interaction between free electrons and lattice vibrations in a solid. For an isotropic crystal, without allowance for umklapp processes, the collision integral is given by

$$I\{f\}=\frac{V}{(2\pi\hbar)^3}\int w(q)d\mathbf{q}\{[(N(\mathbf{q})+1)f_{\mathbf{p+q}}^{\varepsilon+\omega}(1-f(\mathbf{p}))$$
$$-N(\mathbf{q})f(\mathbf{p})(1-f_{\mathbf{p+q}}^{\varepsilon+\omega})]\delta(\varepsilon_{\mathbf{p+q}}-\varepsilon_{\mathbf{p}}-\omega(\mathbf{q}))+[N(-q)f_{\mathbf{p+q}}^{\varepsilon+\omega}(1-f(\mathbf{p}))$$
$$-(N(-\mathbf{q})+1)f(\mathbf{p})(1-f_{\mathbf{p+q}}^{\varepsilon-\omega})]\delta(\varepsilon_{\mathbf{p+q}}-\varepsilon_{\mathbf{p}}+\omega(\mathbf{q}))\}, \tag{A.2}$$

where $N(\mathbf{q})$ is the number of phonons with momentum $\mathbf{q}$, $\omega(\mathbf{q})$ is the energy of the phonon in units of kT, $w(\mathbf{q})$ is the matrix elements of the transition as a function of the modulus of $\mathbf{q}$, and $d\mathbf{q} = q^2dq \sin\vartheta d\vartheta d\kappa$.

In scattering by acoustic phonons, the small parameter is the ratio

$$\omega(\mathbf{p})/\varepsilon_{\mathbf{p}}=\sqrt{2mc_0^2/\varepsilon_{\mathbf{p}}},$$

where $\omega(\mathbf{q}) = c_0 q$ and c_0 is the velocity of the longitudinal sound; it is therefore possible to expand the functions $f_{\mathbf{p+q}}^{\epsilon\pm\omega}$ and $\omega N(\pm q)$ in terms of the small quantity

$$\omega/kT=\sqrt{2mc_0^2\varepsilon}/kT\quad(q\leqslant 2p),$$

so that the integral obtained below for collisions with acoustic phonons will then be valid under the condition (in dimensionless units) $\epsilon \ll kT/mc_0^2$. Accurate to terms of order $\delta = 4mc_0^2/kT$, we obtain as a result by expanding in a series at $N(q) = 1/(e^{\omega} - 1)$:

$$I\{f\}=-\nu(\varepsilon)(f-f_0)+\frac{\delta}{\sqrt{\varepsilon}}\frac{d}{d\varepsilon}\left\{\varepsilon^{3/2}\nu(\varepsilon)\left[\frac{df_0}{d\varepsilon}+f_0(1-f_0)\right]\right\}$$
$$-\cos\theta\frac{\delta}{\sqrt{\varepsilon}}\frac{d}{d\varepsilon}\left\{\varepsilon^{3/2}\nu(\varepsilon)\left[\frac{df_1}{d\varepsilon}+f_1(1-f_0)\right]\right\}.$$

In the case of optical phonons $\omega_{opt} \gtrsim kT$ there is in general no small parameter, since the energy and momentum transfers from the electron to the lattice via the optical phonon are not small. Therefore, for valent crystals, when $w(q) = \text{const}$, $\omega_{\text{opt}}(q) = \omega_0$, and the phonons have an equilibrium distribution, we can readily obtain from (A.2), by going over to the variable $\cos\chi$, an exact expression for the collision integral:

$$I\{f\} = -\nu_0[N\sqrt{\varepsilon+\omega_0} + (N+1)\sqrt{\varepsilon-\omega_0}]f(\mathbf{p}) + \nu_0\{\sqrt{\varepsilon+\omega_0}(N+1-f(\mathbf{p}))f_0(\varepsilon+\omega_0) + \sqrt{\varepsilon-\omega_0}(N+f(\mathbf{p}))f_0(\varepsilon-\omega_0)\},$$

where $\sqrt{\epsilon - \omega_0}$ should be set equal to zero at $\epsilon \le \omega_0$.

Thus, the considered collision integrals, at an arbitrary $f(p)$ dependence, permit separation of a relaxation part in the form $(f - f_0)/\tau_0(\epsilon)$ (for isotropic scattering), provided the energy exchange between the carriers and the particles with which they collide is small and there is no predominant small-angle scattering. The remaining part of the collision integral describes under these conditions, in the main, the deviation of the symmetrical part of the distribution $f_0(\epsilon)$ from the equilibrium value, which causes the main part of the integral to vanish. Owing to the presence of the small parameter $\delta = 2m/M$ or $\delta = 4mc_0^2/kT$, in a sufficiently wide range of external constant electromagnetic fields, the deviation of $f_0(\epsilon)$ from equilibrium is negligible, so that we can confine ourselves to allowance for only the relaxation part of the collision integral.

In our case, integrating the kinetic equation (1) over the angles, we obtain

$$\frac{E_0}{\sqrt{\varepsilon}}\frac{d(\varepsilon f_1)}{d\varepsilon} = \frac{\delta}{\sqrt{\varepsilon}}\frac{d}{d\varepsilon}\left[\varepsilon^{3/2}\nu(\varepsilon)\left(\frac{df_0}{d\varepsilon} + f_0\right)\right]$$

or

$$E_0 f_1(\varepsilon) = \sqrt{\varepsilon}\,\nu(\varepsilon)\,\delta\left(\frac{df_0}{d\varepsilon} + f_0\right).$$

At sufficiently small E_0, the quantity $f_1(\epsilon)$ is determined by the rate of change of $f_0(\epsilon)$, or in other words,

$$f_1(\varepsilon) = -E_0 g(\varepsilon, \alpha)\frac{\sqrt{\varepsilon}}{\nu(\varepsilon)}\frac{df_0}{d\varepsilon}$$

where $g(\epsilon, \alpha)$ is a bounded function of ϵ and of the parameter $\alpha = v_T H/cE$. Therefore, under the condition $E_0^2 \ll \delta$, the contribution of $f_1(\epsilon)$ to the change of f_0 can be neglected, and we can confine ourselves in the collision integral to a relaxation part (in the general case of scattering) of the type

$$I\{f\} = \int \nu(\varepsilon, \chi) f(\varepsilon, \theta', \varphi')\, d\Omega' - \nu_0(\varepsilon) f(\mathbf{p}),$$

where now $f_0(\epsilon)$ is the fully defined distribution function of the carriers, namely the equilibrium function.

[1] F. J. Blatt, Theory of Mobility of Electrons in Solids, Academic Press, 1957.
[2] B. I. Davydov, Zh. Eksp. Teor. Fiz. 7, 1069 (1937).
[3] V. L. Ginzburg and A. V. Gurevich, Usp. Fiz. Nauk 20, 202 (1960).
[4] L. D. Landau, Zh. Eksp. Teor. Fiz. 7, 203 (1937).

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246