THE POINCARE RECURRENCE THEOREM AND THE PROBLEM OF GRAVITATIONAL COLLAPSE

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It is demonstrated that the Poincare recurrence theorem is valid for a collapsing gravitational system if the velocity spread of the particles is not zero. This signifies a time reversal of the collapse (in a synchronous reference system). For zero velocity spread (Tolman solution), the Poincare theorem is inapplicable. The obtained results are extended to the cosmological model of the Universe which, for the indicated initial conditions, is a pulsating one.

At the present time, the gravitational collapse is generally recognized as an irreversible process for the external observer (cf. [1], and references cited). The reversibility of the gravitational collapse for the laboratory observer remains an open question [1,2].

In this contribution, we are attempting to use one of the general theorems of mechanics, i.e., the Poincare recurrence theorem (cf., e.g., [3,4]) to investigate the reversibility of gravitational collapse.

Arnowitt, Dezer and Misner [5] demonstrated in the framework of the general relativity theory that it is feasible to construct a Hamiltonian in canonic variables for a spherically-symmetrical system of gravitating particles; such a Hamiltonian depends on generalized coordinates q_i and momenta p_i of the particles only, but does not depend upon the gravitational field g_{ik}. In principle, this provides a basis for formulating the Liouville theorem for particles executing spherically-symmetrical motion[1)].

On account of the conservation of energy in the field of a spherically-symmetrical potential, the system in phase space moves over the energy surface $H(p, q) = E = \text{const}$. (The numerical value of the energy E coincides with the generally-accepted energy value obtained from the energy-momentum pseudo-tensor of the gravitational field by integration over a sufficiently distant surface, where the metric goes over into a Euclidean metric [2].)

To prove the Poincare theorem, the following three conditions must be satisfied [3,4]: 1) the Liouville theorem; 2) the magnitude of the energy-surface area that bounds the possible phase volume must be finite; 3) the measure μ determined by the relationship

$$\mu = \int\limits_{S_E} \frac{dS}{|\operatorname{grad} H|}$$

must be non-zero; here S_E is the area of the energy surface.

The first condition, as we pointed out, is satisfied. The following should be noted regarding the two remaining conditions. When the system particles have a velocity spread different from zero, the phase-volume element is different from zero. It follows that the measure (1), which is proportional to the phase-volume element, is not zero and (according to the Liouville theorem) is conserved. When the velocity (or pressure) spread is not zero, the condition of finiteness (2) is satisfied in a synchronous system of reference (which we shall use henceforth), by virtue of the boundedness of the gravitational potential at the center $v \to 0$ when $\tau \to 0$ (integrable singularity); this is indicated by the general solution for a spherically-symmetrical collapse [6] when $\tau \to 0$. We note that in classical mechanics, when the particle falls upon the center, the finiteness is violated on account of the singularity of the potential; the particle momentum at the point $r = 0$ becomes infinite.

It follows that a spherically-symmetrical collapsing system with an initial particle velocity spread satisfies the applicability conditions of the Poincare theorem with respect to the finite time of recurrence.

The Tolman solution describes the collapse of dust-like matter in a comoving system of coordinates [2]; corresponding to this system in the space of phase trajectories is the measure $\mu = 0$. It is for this reason that the Poincare recurrence theorem is not applicable in this case.

As indicated above, the reversible nature of the collapse in the time of the laboratory observer will still not be the same for the external observer, for whom the time of approach of the system to the Schwarzschild sphere tends to infinity[2)].

It is interesting to extend these results to the investigation of cosmological models. When dealing with Friedmann models, one can mentally separate any spherically-symmetrical domain and it will remain energetically closed. This is sufficient for the application of all aforementioned findings to the considered domain. All models of the uniform universe are defined by the comoving system of reference (which corresponds to the measure $\mu = 0$) and therefore the Poincare

[1)] The question as to the applicability of the Hamiltonian [5] at the point zero for a system consisting of a finite number of particles remains open.

[2)] In this sense, the "escape" of the system from under the Schwarzchild sphere takes place into a space which is in the absolute future for the external observer (similarly to the problem of the collapse of a charged sphere [7]).

theorem is not applicable to those models. (All solutions cannot be analytically extended into the domain $t < 0$; the point $t = 0$ is essentially a singular point—the "beginning of the Universe"[2].)

If we consider a Universe with an initial particle velocity spread (violation of uniformity and isotropy), then the Poincare theorem will be valid for sufficiently large domains of the Universe; these domains may be considered quasi-closed. In this instance, the system will be quasi-periodic. The latter prohibits a transition of the Universe into a state with an infinite gravitational potential (which would mean non-conservation of the phase volume) and requires the existence of analytical continuation into the region $t < 0$.

It follows that a spherically-symmetrical collapsing system with an initial particle velocity spread in a synchronous system of reference must, according to the Poincare theorem, inevitably and in finite time return to a state that is close to the initial one.

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