ANISOTROPY OF THE CRITICAL CURRENT FOR SURFACE SUPERCONDUCTIVITY NEAR THE FIELD STRENGTH H_{c3}

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Submitted November 29, 1968

Zh. Eksp. Teor. Fiz. 56, 1441–1447 (April, 1969)

The problem of surface superconductivity in a state with nonvanishing total current is solved with the aid of the Ginzburg–Landau equations for $H - H_{c3} \ll H_{c3}$. The maximum value of the critical current in a direction perpendicular to the magnetic field (which lies in the plane of the sample) turns out to be 20% smaller than the value in the longitudinal direction, which is in agreement with the results of a numerical calculation.[3]

1. SAINT-JAMES and de Gennes[1] theoretically predicted the existence of a superconducting layer at the surface of a sample in a magnetic field H, exceeding the upper field H_{c2} of a bulk superconductor. For plane boundaries in a parallel field they found that superconductivity is preserved up to a field strength $H_{c3} = 1.69\,H_{c2}$. Later on the phenomenon of surface superconductivity gave rise to considerable interest from the point of view of both experimentalists and theoreticians. Abrikosov[2] calculated the value of the maximum superconducting current (the critical current) which may be maintained in a surface layer. One of Abrikosov's results was an assertion about the complete isotropy of the critical current in the plane of the sample. In the present article we show that the critical current depends on its direction relative to the applied field, i.e., the current is anisotropic, and we determine the nature of this dependence. Taking this fact into consideration allows us, in particular, to eliminate a certain discrepancy between the value for the critical current obtained by Abrikosov and the results of a numerical calculation by Park.[3]

2. We shall conduct the investigation on the basis of the Ginzburg–Landau equations[4] in the range of field strengths $H_{c3} - H \ll H_{c3}$, in which the quantity $\psi \ll 1$ serves as a physically small expansion parameter. In dimensionless form the system of Ginzburg–Landau equations for the order parameter ψ has the form

$$(i\nabla/\varkappa + \mathbf{A})^2\psi = \psi - |\psi|^2\psi, \quad (1)$$

$$(i\nabla/\varkappa + \mathbf{A})_n\psi = 0 \text{ (at the surface)}, \quad (1')$$

$$-\operatorname{rot}\operatorname{rot}\mathbf{A} = \mathbf{j}, \quad (2)$$

$$\mathbf{j} = |\psi|^2\mathbf{A} + \frac{i}{2\varkappa}(\psi^*\nabla\psi - \psi\nabla\psi^*), \quad (2')$$

where κ is a parameter of the theory, $\mathbf{j}$ is the current density, $\mathbf{A}$ is the vector potential of the magnetic field, $\mathbf{H} = \operatorname{curl}\mathbf{A}$. It is necessary to make the following substitutions in the final formulas in order to change to the usual units:

$$\psi \to \psi/\psi_0(T), \quad x \to x/\delta(T), \quad H \to H/\sqrt{2}H_{cm},$$

where $\psi_0(T)$ is the value of the order parameter in the absence of fields and currents, $\delta(T)$ is the penetration depth, and H_{cm} is the critical thermodynamical field.

Let the superconductor occupy the half-space $x \geq 0$, and let the magnetic field $\mathbf{H}$ be everywhere parallel to the plane of the sample, $x = 0$. It is convenient to carry out the calculations in a gauge in which the vector potential $\mathbf{A}$ only depends on x and lies in the same plane as $\mathbf{H}$, and the order parameter $\psi(x)$ is real. Then Eqs. (1) and (2) take an especially simple form:

$$-\varkappa^{-2}d^2\psi/dx^2 + A^2\psi = \psi - \psi^3, \quad (3)$$

$$(d\psi/dx)_{x=0} = 0, \quad (3')$$

$$d^2\mathbf{A}/dx^2 = \mathbf{j}(x) = \psi^2\mathbf{A}. \quad (4)$$

We shall be interested in a solution of the problem which corresponds to the bulk of the sample being in the normal state, that is, $\psi(x) \to 0$ as $x \to \infty$.

For $H = H_{c3}$ one can linearize Eq. (3):

$$-\varkappa^{-2}d^2\psi_0/dx^2 + A_0^2\psi_0 = \psi_0, \quad (5)$$

where the vector potential $\mathbf{A}_0$ in the zero-order approximation describes a homogeneous external field H_{c3}. According to Eq. (4)

$$\mathbf{A}_0 = H_{c3}(x - x_0)\mathbf{a}, \quad (6)$$

where $\mathbf{a}$ is a unit vector lying in the plane of the sample ($\mathbf{a} \perp \mathbf{H}$). The choice of the integration constant x_0 determines the value of the field H_{c3}. In Appendix A it is shown that the largest value for H_{c3} is obtained upon fulfillment of the condition

$$\int_0^\infty A_0(x)\psi_0^2(x)\,dx = 0, \quad (7)$$

that is, for the absence of any total current in the ground state $\psi_0(x)$. An exact relation (see Eq. (A.4)) follows from Eqs. (5), (3′), and (7): $x_0 H_{c3} = 1$. By means of numerical integration Saint-James and de Gennes found that $H_{c3} = 1.69\,H_{c2}$.[1)]

3. Now let us determine the corrections to the solution under consideration which arise for $H < H_{c3}$ and also due to the fact that the total current does not vanish. The investigation will be carried out by the method of successive approximations. Starting from Eq. (4),

[1)]It is easy to see that the value $H_{c2} = \kappa$ corresponds to the choice $x_0 = 0$. In this case the problem admits an exact solution of the form ψ_0 $\psi_0(x) = \exp(-\kappa^2 x^2/2)$ (see [4]).

let us write the vector potential **A** correct to terms $\sim \psi_0^2$ in the following form:

$$\mathbf{A} = \mathbf{A}_0 + \boldsymbol{\zeta} + \mathbf{A}_2, \tag{8}$$

where

$$\mathbf{A}_2 = -\left(1 - \frac{H}{H_{c3}}\right)\mathbf{A}_0 + \int_0^x dx_1 \int_0^{x_1} \mathbf{A}_0(x_2)\psi_0^2(x_2)\,dx_2, \tag{9}$$

and $\boldsymbol{\zeta}$ is, for the present, an arbitrary constant vector lying in the plane of the sample. In the last expression the limits of integration are chosen so that the magnetic field at the surface of the sample will be equal to the applied field: $(dA/dx)_0 = H$. The vector $\boldsymbol{\zeta}$, as will be evident from what follows, specifies the current in the system and in magnitude it may not exceed the value $\psi_0 \sim \sqrt{1 - (H/H_{C3})}$. The second term in (9) describes the inhomogeneity of the magnetic field as a consequence of the superconducting properties of the surface layer ($\psi_0 \neq 0$).

We seek a solution of Eqs. (3) and (3′) in the form of the expansion

$$\psi = \psi_0 + \psi_2 + \psi_3, \tag{10}$$

where $\psi_2 \sim \psi_0^2$, $\psi_3 \sim \psi_0^3$, and so forth. From here, by substituting (8) into (3) and equating terms of the same order of smallness in ψ_0, we arrive at the following equations for the determination of ψ_2 and ψ_3:

$$-\varkappa^{-2}\psi_2'' + A_0^2\psi_2 - \psi_2 = -2(\boldsymbol{\zeta}\mathbf{A}_0)\psi_0, \tag{11}$$

$$-\varkappa^{-2}\psi_3'' + A_0^2\psi_3 - \psi_3 = -2(\boldsymbol{\zeta}\mathbf{A}_0)\psi_2 - 2(\mathbf{A}_0\mathbf{A}_2)\psi_0 - \zeta^2\psi_0 - \psi_0^3. \tag{12}$$

The solution of Eq. (11), satisfying the boundary condition (3′), has the form

$$\psi_2(x) = 2\varkappa^2\psi_0(x)\int_0^x \frac{dx_1}{\psi_0^2(x_1)}\int_0^{x_1}(\boldsymbol{\zeta}\mathbf{A}_0(x_2))\,\psi_0^2(x_2)\,dx_2. \tag{13}$$

One can easily verify this by performing the differentiation and using condition (7). It will not be necessary for us to have an expression for $\psi_3(x)$, and we shall not write it down in explicit form.

4. According to Eq. (4) the total current **J** flowing along the surface of the sample is given by

$$\mathbf{J} = \int_0^\infty \mathbf{A}(x)\psi^2(x)\,dx. \tag{14}$$

Expanding this expression in a series in powers of $\psi_0(x)$ and again taking condition (7) into account, to the first nonvanishing approximation we find that

$$\mathbf{J} = \boldsymbol{\zeta}\int_0^\infty \psi_0^2(x)\,dx + 2\int_0^\infty \mathbf{A}_0(x)\psi_0(x)\psi_2(x)\,dx. \tag{15}$$

With the aid of Eqs. (13), (6), and (7) one can represent the obtained result in the form

$$\mathbf{J} = \{\boldsymbol{\zeta} + p\mathbf{a}(\boldsymbol{\zeta}\mathbf{a})\}\int_0^\infty \psi_0^2(x)\,dx, \tag{16}$$

$$p = 4\varkappa^2\left(\int_0^\infty \psi_0^2(x)\,dx\right)^{-1}\int_0^\infty A_0(x)\psi_0^2(x)\,dx\int_0^x \frac{dx_1}{\psi_0^2(x_1)}\int_0^{x_1} A_0(x_2)\psi_0^2(x_2)\,dx_2. \tag{17}$$

For the case of a transverse current ($\boldsymbol{\zeta} \parallel \mathbf{a}$) expression (16) differs from the one used by Abrikosov by the presence of the second term. The origin of this term has a quite clear physical meaning and is connected with a change of the order parameter ψ due to the influence of ζ, i.e., due to the current. The coefficient ρ determines the magnitude of the anisotropy of the surface layer which, in turn, is caused by application of the field **H**.

In Appendix B is it shown that one can write an expression for p in the following simple form:

$$p = -1 + \psi_0^2(0)/H_{c3}\int_0^\infty \psi_0^2(x)\,dx, \tag{18}$$

where $\psi_0(0)$ is the amplitude of the order parameter at the surface of the sample. In the literature, the quantity

$$\Delta = \psi_0^{-2}(0)\int_0^\infty \psi_0^2(x)\,dx \tag{19}$$

is called the thickness of the surface layer; near H_{C3} it is $\sim 1/\kappa$ (in ordinary units $\Delta \sim \xi$ where $\xi = \delta/\kappa$ is the superconducting coherence length). Values of the function $\Delta(H)$ are tabulated in [5]; for $H = H_{C3}$ its exact value is given by $\Delta = 1.007/\kappa$. Thus, in our case the coefficient of anisotropy p is simply given by

$$p = \varkappa/H_{c3} - 1 \approx -0{,}41. \tag{20}$$

It remained for us to determine the amplitude of the order parameter $\psi_0(\boldsymbol{\zeta}, \mathbf{H})$ which enters into expression (16) for the current. For this purpose let us consider Eq. (12) for the correction ψ_3, and let us write down the condition under which it has a nontrivial solution satisfying the boundary condition (3′). This condition, as is well known, consists in the orthogonality of the right-hand side of Eq. (12) to the solution $\psi_0(x)$ of the homogeneous equation (5) combined with (3). Omitting simple transformations we at once write down the desired result

$$\int_0^\infty \psi_0^2(x)\,dx = Q^{-1}\left\{q\left(1 - \frac{H}{H_{c3}}\right) - \zeta^2 - p(\boldsymbol{\zeta}\mathbf{a})^2\right\}, \tag{21}$$

where

$$Q = \left[\int_0^\infty \psi_0^2(x)\,dx\right]^{-2}\left\{\int_0^\infty \psi_0^4(x)\,dx - 2\int_0^\infty dx\left[\int_0^x A_0(x_1)\psi_0^2(x_1)\,dx_1\right]^2\right\} \tag{22}$$

$$q = 2\int_0^\infty A_0^2(x)\psi_0^2(x)\,dx \Big| \left(\int_0^\infty \psi_0^2(x)\,dx\right)^{-1} > 0. \tag{23}$$

Hence

$$\mathbf{J} = Q^{-1}[\boldsymbol{\zeta} + p\mathbf{a}(\boldsymbol{\zeta}\mathbf{a})]\left\{q\left(1 - \frac{H}{H_{c3}}\right) - \zeta^2 - p(\boldsymbol{\zeta}\mathbf{a})^2\right\}. \tag{24}$$

The vector $\boldsymbol{\zeta}$ entering into this expression may have an arbitrary direction (in the plane of the sample), but in magnitude, as is evident from (21), it should not exceed a certain value ζ_{max}. The greatest value of the function $J(\zeta)$ in the interval $0 \leq \zeta \leq \zeta_{max}$ determines the value of the critical current J_c for the system. Starting from (24), by means of a direct calculation one can easily verify that

$$J_c(\theta) = \frac{2q}{3Q}\left[\frac{q(1 - |p|)}{3(1 - |p|\cos^2\theta)}\right]^{1/2}\left(1 - \frac{H}{H_{c3}}\right)^{3/2}, \tag{25}$$

where $\cos\theta = (\mathbf{J}\cdot\mathbf{H})/JH$. Here we have taken into account the fact proved earlier that $p < 0$ (see Eqs. (B.1) and (20)). The obtained angular dependence $J_c(\theta)$ describes a monotonic decrease of the critical current from the value $J_\parallel$ to the value $J_\perp$ during a change of the angle θ from 0 to 90°; the ratio $J_\perp/J_\parallel$ is given by

$$J_\perp / J_\parallel = \sqrt{1-|p|} \approx 0.77, \tag{26}$$

i.e., the anisotropy $\Delta J/J_c$ of the critical current amounts to $\approx 20\%$.

During the derivation of expression (25) we did not make any specific assumptions except nearness of the field strength H to H_{c3}. Calculation of the coefficients q and Q must be carried out numerically. In order to determine the critical current Abrikosov used a variational method with a trial wave function of the zero-approximation form $\psi_0(x) = \exp(-\kappa^2 x^2/2)$, which corresponds to an eigenvalue which, according to Eq. (A.2), is equal to $\epsilon_0 \approx 0.6$ (the exact value is $\epsilon_0 = 0.59$). In this case the result has an especially simple form:

$$Q = (2/\pi)^{1/2}\varkappa[1 - 0.156\,\varkappa^{-2}], \qquad q = 1. \tag{27}$$

Thus,[2)]

$$J_c(\theta) = \frac{1}{3}\left[\frac{2\pi}{3}\frac{1-|p|}{1-|p|\cos^2\theta}\right]^{1/2}\frac{(1-H/H_{c3})^{3/2}}{\varkappa(1-0.156\varkappa^{-2})}. \tag{28}$$

For the longitudinal current ($\theta = 0$) this expression goes over into Abrikosov's formula. For the transverse current ($\theta = 90°$) an additional factor $\sqrt{1-|p|}$ arises in comparison with the previous case; this factor determines the anisotropy of the critical current (see Eq. (28)). This conclusion is in agreement with numerical calculations of the transverse current carried out by Park.[3] As follows from the derivation given, the presence of anisotropy is associated with the fact that a surface layer exists only in the presence of a strong field ($H_{c2} < H < H_{c3}$), which creates a preferred direction in the plane of the sample. In the absence of an external field the critical current will, of course, be isotropic if the anisotropic properties (usually small) of the material itself are neglected.

5. Expression (25) is quite general and is also applicable for a determination of the critical current of a plane film in a parallel field whose strength is near the critical value. In order to do this, in the formulas appearing here for the coefficients it is necessary to replace the limits of integration 0 and ∞ by the values $x = \pm d$ (the thickness of the film is equal to 2d). This problem was first solved by Ginzburg and Landau[4, 6] for films with $\kappa d \ll 1$. They showed that to the zero-order approximation in κd one can neglect the change of the ψ-function with respect to the film thickness. As a result the critical current is found to be isotropic and independent of the parameter κ. Near $H = H_c$ one finds

$$J_c^{GL} = \frac{8\sqrt{2}\,d}{3\sqrt{3}}\left(1-\frac{H}{H_c}\right)^{3/2}, \qquad \varkappa d \ll 1, \tag{29}$$

where $H_c = \sqrt{3}/d$ is the critical field of the film. It is easy to arrive at this result if one sets $\psi = \psi_0 = \text{const}$ and $A_0 = H_c x$ in Eqs. (22) and (23); to this approximation one should set the parameter p characterizing the current anisotropy equal to zero since, as already indicated, it is associated with spatial variations of the order parameter ψ. In the next approximation in κd, the correction which gives the dependence of the critical current on direction is of the greatest interest. According to Eq. (17),

$$p = \frac{2\varkappa^2}{d}\int_{-d}^{d} A_0(x)\,dx \int_{-d}^{x} dx_1 \int_{-d}^{x_1} A_0(x_2)\,dx_2 = -\frac{8}{5}(\varkappa d)^2 \ll 1. \tag{30}$$

Expanding the integrand in (25) in a series of powers of p, we find

$$J_c(\theta) = J_c^{GL}(1 + {}^4/_5\varkappa^2 d^2\cos^2\theta), \qquad \varkappa d \ll 1. \tag{31}$$

It is natural to expect that for thin films ($\kappa d \sim 1$) the anisotropy of the current will, generally speaking, be of the order of unity.

6. In conclusion let us dwell briefly on the question of the experimental observation of the above-considered anisotropy of the critical currents. Experimentally the magnitude of the critical current turns out to be extremely sensitive to the presence of the normal component of the applied magnetic field. For example, in the experiments of Hart and Swartz[7] the critical current is decreased by an order of magnitude upon deviation of the field from the plane of the sample by an angle $\sim 1°$. An even greater decrease of the current (by four orders of magnitude) was observed by Akhmedov et al.[8] for very pure Nb with a ratio of resistances R(300°K)/R(4.2°K) = 14 000. A qualitative explanation of these results is that a superconducting surface layer goes over into a vertex state in the presence of an arbitrarily small normal component of the magnetic field.[8, 9] Taking into account the smallness of the critical current's anisotropy in a parallel field, the cited considerations indicate that for its observation there are rather stringent requirements on the degree of homogeneity of the magnetic field, the surface of the sample, and also on uniformity in the random distribution of impurities inside the sample.

The author thanks L. P. Gor'kov and G. F. Zharkov for a discussion of the results of this work and for their observations.

APPENDIX

A. In Eq. (5) we make the following change of variable: $\xi = \sqrt{\kappa H_{c3}}\,x$. As a result we arrive at the problem of determining the eigenvalues for the equation of motion of an oscillator

$$-d^2\psi_0/d\xi^2 + (\xi-\xi_0)^2\psi_0 = \varepsilon_0\psi_0 \tag{A.1}$$

with the boundary condition

$$(d\psi_0/d\xi)_{\xi=0} = 0, \tag{A.1'}$$

where $\xi_0 = \sqrt{\kappa H_{c3}}\,x_0$ and $\epsilon_0 = \kappa/H_{c3}$. We are interested in the largest value of the field $H = H_{c3}$ for which Eq. (6) has a nontrivial solution, i.e., the minimum eigenvalue ϵ_0. For the system of equations (A.1)–(A.1′) this is equivalent to the variational problem concerning the determination of the minimum of the expression[2]

$$\varepsilon_0 = \int_0^\infty \left[\left(\frac{d\psi}{d\xi}\right)^2 + (\xi-\xi_0)^2\psi^2\right]d\xi \Big/ \int_0^\infty \psi^2(\xi)\,d\xi, \tag{A.2}$$

in which ξ_0 plays the role of a free parameter.[2] Equating the derivative $\partial\epsilon_0/\partial\xi_0$ to zero, we arrive at the condition (in terms of the original notation)

2) We note that the nature of the dependence $J_c \sim Q^{-1} \sim \kappa^{-1}$ for $\kappa > 1$ follows at once from expression (22) if one makes the replacement $x \to \xi/\kappa$ (see Eq. (A.1)).

$$\int_0^\infty (x-x_0)\psi_0^2(x)\,dx=\int_0^\infty A_0(x)\psi_0^2(x)\,dx=0. \qquad \text{(A.3)}$$

The second equation explicitly shows that a total current is not present in the ground state. Now let us return to Eq. (5), and let us multiply it on both sides by $\psi_0'(x)$, and then integrate between the limits from 0 to ∞. Carrying out an integration by parts in the second term from the left and using Eqs. (3′) and (A.3), we finally find that

$$x_0 H_{c3}=1. \qquad \text{(A.4)}$$

B. One can easily convince oneself that the parameter p characterizing the anisotropy of the current is negative. In order to do this it is sufficient to integrate expression (16) by parts with respect to the variable x and to utilize property (A.3). We present the result:

$$p=-4\varkappa^2\int_0^\infty \frac{I^2(x)}{\psi_0^2(x)}\,dx\Big/\int_0^\infty \psi_0^2(x)\,dx<0, \qquad \text{(B.1)}$$

where

$$I(x)=\int_0^x A_0(x)\psi_0^2(x)\,dx. \qquad \text{(B.2)}$$

In order to prove Eq. (17) it is convenient to start directly from the equations themselves. Let us multiply Eq. (6) by $\psi_2'(x)$ and Eq. (11) by $\psi_0'(x)$, and then add the results. Integrating the resulting equation with respect to x between the limits $(0, \infty)$, after simple transformations we find that

$$\int_0^\infty A_0(x)\psi_0(x)\psi_2(x)\,dx=\frac{(\mathbf{a}\zeta)}{2H_{c3}}\Big[\psi_0^2(0)-H_{c3}\int_0^\infty \psi_0^2(x)\,dx\Big]. \qquad \text{(B.3)}$$

Comparison of (15) and (16) with (B.3) taken into account immediately leads to expression (17) for the quantity p.

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Translated by H. H. Nickle
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