INSTABILITY OF THE PINCH EFFECT IN AN ELECTRON-HOLE PLASMA

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Stability with respect to constrictions of pinching in the electron-hole plasma of an intrinsic semiconductor is investigated with no external field acting upon the carriers. It is shown that for oscillations having a longitudinal wavelength that exceeds the pinch radius a weak instability develops by a mechanism in which the self-magnetic field of the current plays an important role.

1. INTRODUCTION

THE pinch effect in a fully ionized gaseous plasma is known to be unstable with respect to constrictions and helical perturbations. A weakly conducting liquid cylinder is also unstable.[1] A similar instability of the pinch effect in solids might therefore be expected.[2]

Despite the qualitative analogy a solid-state pinch differs considerably from pinching in a gaseous plasma. The difference results not only from the greater carrier density in a semiconductor plasma, but mainly from the strong "frictional" interaction of electrons and holes with the lattice. The latter circumstance must also affect the stability of the pinch. The available experimental data indicate greater pinch stability than one would expect from an analogy with a gaseous plasma. In all the known experimental work[3] oscillations of the voltage amplitude were damped out during one or, at most, a few pulses. Indirect evidence also exists for recovery of a pinch that had been destroyed by an external magnetic field.[4]

We know of no special investigation concerning pinch stability in solid-state plasmas. In the present work an attempt is made to investigate pinch stability in the electron-hole plasma of an intrinsic semiconductor for the limiting case of zero applied magnetic field, when the effects of "friction" between carriers and the lattice are most pronounced. It is assumed that the electron mobility exceeds the hole mobility. Stability with respect to constrictions is investigated for a stationary state in which the longitudinal carrier-drift velocity is constant throughout the pinch cross section. A numerical solution of the equations for small oscillations showed that weak instability exists when $k_z a_0 < 1$, where a_0 is the pinch radius and k_z is the longitudinal wave number. The self-magnetic field of the current plays an important role in the development of the instability. When $k_z a_0 > 1$ the perturbations are completely suppressed by ambipolar diffusion.

2. BASIC EQUATIONS

We shall consider an infinite intrinsic semiconductor; we thus begin by excluding edge effects at the faces of the sample. We use cylindrical coordinates with an electric field applied along the z axis. We shall assume that in equilibrium, when carrier diffusion transverse to the self-magnetic field is balanced by their drift toward the center, all gradients are radially directed. For simplicity it will be assumed that the electron and hole temperatures are identical, constant across the entire cross section, and invariant during perturbations.[1)] Carrier inertia will be completely neglected, and a quasi-linear plasma will be assumed. Carrier recombination and generation will also be neglected; the total number of electron-hole pairs per unit length of the pinch will be taken as constant. Friction between the carriers and the lattice will be taken into account, but interactions between the carriers will be neglected.

We shall start with the equations of two-fluid hydrodynamics. Then

$$0 = \mp en\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}_{n,p}\mathbf{H}]\right) - \frac{m^*_{n,p}n}{\tau}\mathbf{v}_{n,p} - T\nabla n, \qquad (1)^*$$

where $m^*_{n,p}$ is the (scalar) effective mass of electrons or holes, τ is the pulse relaxation time ($\tau_{n,p} \equiv \tau$), H is the self-magnetic field of the current, T is the carrier temperature, and $n \equiv p$ for intrinsic conduction (n and p are the electron and hole densities). The last relation is conserved for perturbations because of the quasineutrality. Assuming

$$\frac{\mu_n H}{c} \equiv \frac{eH\tau}{m^* c} = \frac{\mu_n}{\mu_p}\left(\frac{\mu_p H}{c}\right) \ll 1, \qquad (1a)$$

$\mu_n \gg \mu_p$, and therefore $\mu_p H/c \ll 1$, we derive from (1) the following expressions for the electron and hole velocities, accurate up to terms with $(\mu_n H/c)^2$ and $(\mu_p/\mu_n)^2$:

$$\mathbf{v}_n = -\mu\mathbf{E} - D\frac{\nabla n}{n} - \mu\left[\frac{\mu\mathbf{H}}{c}\mathbf{E}\right] - D\left[\frac{\mu\mathbf{H}}{c}\frac{\nabla n}{n}\right],$$

$$\frac{\mu_n}{\mu_p}\mathbf{v}_p = \mu\mathbf{E} - D\frac{\nabla n}{n}. \qquad (2)$$

Here and henceforth the kinetic coefficients lacking subscripts will pertain to electrons; $\mu = e\tau/m^*_n c$ is the electron mobility, and $D = \mu T/e = T\tau/m^*_n$ is the electron diffusion coefficient.

In our approximation holes do not contribute to the current, and to the same order of accuracy [up to $(\mu H/c)^2$ terms] we have

$$\mathbf{j} = en\mu\mathbf{E} + eD\nabla n + e\mu n\left[\frac{\mu\mathbf{H}}{c}\mathbf{E}\right] + eD\left[\frac{\mu\mathbf{H}}{c}\nabla n\right]. \qquad (3)$$

For holes continuity is conserved:

1)It was shown in [5] that the carrier temperature is determined by the excitation energy $k\Theta_0$ of the optical phonons, because the heating of the carriers is arrested by their emission of optical phonons.

*$[\mathbf{vH}] \equiv \mathbf{v} \times \mathbf{H}$.

$$\partial n / \partial t = -\operatorname{div} n v_p. \tag{4}$$

Then Eqs. (2)—(4) together with Maxwell's equations

$$\operatorname{div} \mathbf{H} = 0; \quad \operatorname{rot} \mathbf{H} = \frac{4\pi}{c} \mathbf{j} \tag{5}$$

comprise a complete set of equations for our problem.

3. EQUILIBRIUM

In the equilibrium state the only non-zero current is parallel to the z axis:

$$j_z{}^0 = -\frac{2c}{H_\varphi{}^0} T \frac{dn_0}{dr}, \tag{6}$$

which is accompanied in the given geometry by only an azimuthal self-magnetic field:

$$H_\varphi{}^0 = -\frac{2}{n_0}\frac{dn_0}{dr}\frac{c}{\mu}\frac{D}{\mu E_z{}^0}, \tag{7}$$

here E_z^0 is the longitudinal electric field in the pinch.

In accordance with (2) only electrons are acted upon by the magnetic field of the current. An indirect influence of the magnetic field on the holes is manifested through the radial electric field with which the electrons confine the holes. In virtue of the quasi-neutrality, Eq. (2) shows that at equilibrium with $v_r^n = v_r^p = 0$ a radial electric field

$$E_r{}^0 = -\frac{\mu}{2c} E_z{}^0 H_\varphi{}^0 \left(1 - \frac{\mu_p}{\mu}\right), \tag{8}$$

is set up and prevents motion of the holes away from the axis.

On the basis of (6) and (7) together with the second equation in (5) we easily determine that the equilibrium distribution of carriers along the radius of the pinch has the familiar form of a Bennett distribution:

$$n_0 = N_0 f,$$
$$f = \frac{16}{(1+\rho^2)^2}; \quad N_0 = \frac{c^2 D}{2\pi e \mu v_{z0}{}^2 a_0{}^2};$$
$$\rho \equiv \frac{r}{a_0}; \quad a_0 = \frac{4c}{j_z{}^0}\left(\frac{N_0 T}{\pi}\right)^{1/2}. \tag{9}$$

This result is consistent with the fact that we are investigating the stability of the stationary state in which the axial drift velocity of carriers is uniform across the cross section of the pinch (v_z^0 = const) and the total number of carrier pairs per unit length of the pinch is also constant.

4. PERTURBATIONS

The stationary solution will be our zeroth approximation. In the next approximation we write $\mathbf{H} = \mathbf{H}_0 + \mathbf{H}'$, $n = n_0 + n'$, and $\varphi = \varphi_0 + \varphi'$. For the perturbations H', n', $\varphi' \sim f(\rho)\exp(-i\omega t + ik_z z + im\varphi)$ Eqs. (2)—(5) yield in a linear approximation

$$\left(\mu \mathbf{E}_0 + D\nabla + \left[\frac{\mu \mathbf{H}_0}{c}, \mu \mathbf{E}_0 + D\nabla\right]\right) n' - n_0\left(D\nabla + \left[\frac{\mu \mathbf{H}_0}{c}, D\nabla\right]\right)\varphi'$$
$$= \left[\frac{c^2}{4\pi\mu e}\nabla + D\nabla n_0 + \mu n_0 \mathbf{E}, \frac{\mu \mathbf{H}'}{c}\right];$$
$$\left(D\Delta - \mu\nabla \mathbf{E}_0 + i\omega\frac{\mu}{\mu_p}\right) n' + \mu(\nabla n_0 \nabla\varphi') = 0;$$
$$\nabla \mathbf{H}' = 0. \tag{10}$$

This derived system of five equations contains two unknown functions n' and φ', along with three components of the magnetic field perturbation. It is easily seen that the coefficients of the unknown perturbations in the given system of equations can be used to form three independent dimensionless parameters differing in their order of smallness with respect to $\mu H_0/c$. Equation (6) shows that our case $\mu H_0/c \ll 1$ corresponds to large longitudinal fields $E_z^0 \sim (c/\mu H_0)(D/\mu a_0)$ and large magnetic diffusion

$$D_m \sim \frac{c^2}{4\pi\mu e N_0} \sim D\Big/\left(\frac{\mu H_0}{c}\right)^2 \frac{H_0^2}{4\pi n_0 T} \sim D\Big/\left(\frac{\mu H_0}{c}\right)^2.$$

On the other hand, perturbations with $k_z a_0 \sim 1$ are of greatest interest. It follows from the third equation of (10) that in this region the magnetic field perturbations along all three coordinates are of an identical order of smallness. The relative perturbations of the potential and of the self-magnetic field of the current, which are induced by redistribution of the initial density, also have an identical order of smallness. Indeed, the current perturbation is proportional to the density perturbation, while $j'/j_0 \sim H'/H_0$. Therefore $n'/n_0 \sim H'/H_0$. On the other hand, the density perturbation obviously leads to a proportional redistribution of the basic (unperturbed) longitudinal electric field, so that $n'/n_0 \sim \varphi'/a_0 E_z^0 \sim (\mu H_0/c)(\mu\varphi'/D)$.

We shall here confine ourselves to simple constriction perturbations (m = 0), which appear to grow more rapidly than bending perturbations, just as in the case of a gaseous plasma. It is easily shown that in this case the redistribution of the density is associated only with a perturbation in the azimuthal magnetic field of the current.

The previous primed notation will now be replaced by more convenient dimensionless quantities for the perturbation of the potential, $\varphi = -if\mu\varphi'/D$, the density, $n = n'/N_0$, and the magnetic field, $y = \mu H'/c$. The equilibrium distribution of the self-magnetic field will be written as $y_0 = \mu H_\varphi^0/c$.

We now write (10) using the new notation and reduce all terms to an identical order of smallness. It must be remembered that in the new variables the perturbations have different orders of smallness. Thus the density perturbation is smaller by one order in $\mu H_0/c$, and the potential is smaller by two orders, than the field perturbation. After dropping $(\mu H_0/c)^2$ terms and higher-order terms, a simple calculation yields the following system of differential equations:

$$\left(2(\Delta_\rho - x^2) + \frac{1}{\rho}\frac{d}{d\rho}\rho y_0 - \gamma\right) n = -x\left(f - \frac{y_0{}^2}{2}\right)\varphi - \frac{1}{\rho}\frac{d}{d\rho}\rho f y;$$
$$f\frac{d}{d\rho}\left(\frac{\varphi}{f}\right) = xy;$$
$$n + x\varphi = \frac{1}{\rho}\frac{d}{d\rho}\rho y; \tag{11}$$

where $x \equiv k_z a_0$ and $y_0 = 8\rho/(1+\rho^2)$ in accordance with (7).

In our case of an intrinsic semiconductor, when the phase velocity of the perturbation is zero, we write $\gamma = -i(\omega a_0^2/D)(\mu/\mu_p)$, and consider henceforth that γ is a real quantity.

The boundary conditions are obvious from considerations of symmetry and from the form of (11). For our constrictive type of perturbation the radial redistribution of density is symmetric about the axis, so that $(dn/d\rho)_0 = 0$. On the other hand, the third equation of (11) requires $y(0) = 0$ in the vicinity of the zero point. Then $(d\varphi/d\rho)_0 = 0$, indicating that the corresponding redistribution of potential is also symmetric about the origin. Our problem can be formulated with either of two kinds of boundary conditions at infinity. Thus if the sample is insulated by a medium with extremely small magnetic permeability, the perturbation of the current's magnetic field will vanish. On the other hand, if the sample is bounded by a shielding medium we have the boundary condition $\varphi(\rho_L) = 0$. In the latter case the perturbation of the magnetic field does not necessarily vanish. We shall consider only boundary conditions of the second type, for two different cases: when the shielding layer is located at a distance which is four times the pinch radius, and when the distance exceeds ten times the pinch radius (so that the shielding layer does not restrict the growth and development of perturbations).

The system (11) can be reduced to a single self-adjoint differential equation for the potential. Eliminating n and y, and introducing the notation

$$\lambda = x^2 + \gamma/4, \qquad \alpha = 2x^2 - f,$$
$$\beta = -x^4 + {}^1\!/_4 (f - {}^1\!/_2 y_0^2)(f + 2x)^2, \tag{12}$$

we obtain

$$\frac{1}{\rho}\left(\frac{\rho\varphi''}{f}\right)'' + \frac{1}{\rho}\left[\left(2(f-\lambda) - \frac{y_0}{2\rho} - \frac{1}{\rho^2}\right)\frac{\rho\varphi'}{f}\right]' + (\alpha\lambda + \beta)\frac{\varphi}{f} = 0, \tag{13}$$

where primes denote differentiation with respect to the radial dimension.

We must now solve a boundary problem for the eigenvalues of a fourth-order self-adjoint linear operator where the parameter λ appears in an unusually general manner.[6] With regard to the boundary conditions of (13) we can state that, in accordance with the foregoing discussion, for the investigated symmetric case all odd derivatives of the potential with respect to the radius must vanish, and we can assume $\varphi = \varphi' = 0$ at infinity.

5. DISPERSION EQUATION

We cannot obtain a general analytic solution for the spectrum of our boundary problem; numerical integration was required. Before presenting the solution we shall attempt to determine nonrigorously both the class of eigenfunctions of (13) that we can expect to find built up and the corresponding responsible terms. We multiply the self-adjoint form (13) by φ and integrate subject to the boundary conditions. Then the expression for the increment can be put into the perspicuous form

$$\frac{\gamma}{4} = -x^2 + \frac{x^4 + ax^2 + b}{2x^2 + c}, \tag{14}$$

where

$$c = -\frac{1}{\langle\varphi^2/f\rangle}\left\langle -2\frac{\varphi'^2}{f} + \varphi^2\right\rangle, \quad a = -\frac{1}{2\langle\varphi^2/f\rangle}\left\langle\left(f - \frac{y_0^2}{2}\right)\frac{\varphi^2}{f}\right\rangle,$$
$$b = -\frac{1}{\langle\varphi^2/f\rangle}\left\langle\frac{\varphi''^2}{f}\right\rangle + \frac{1}{\langle\varphi^2/f\rangle}\left\langle\left(2f - \frac{y_0}{2\rho} - \frac{1}{\rho^2}\right)\frac{\varphi^2}{f}\right\rangle$$
$$- \frac{1}{4\langle\varphi^2/f\rangle}\left\langle f\left(f - \frac{y_0^2}{2}\right)\frac{\varphi^2}{f}\right\rangle.$$

Here $\langle f\rangle = \int_0^\infty f\rho\, d\rho$. The signs and magnitudes of the coefficients a, b, and c will obviously depend entirely on the behavior of the potential φ/f.

Equation (14) shows that short-wave perturbations are suppressed by ambipolar diffusion, i.e., $\gamma \approx -2x^2$. When (14) is expanded in reciprocal powers of x^2 and we include one more term we have

$$\gamma \approx -2x^2 + 2a. \tag{15}$$

The second term can be positive or negative ($a > 0$ or $a < 0$).

For long-wave perturbations it is convenient to rewrite (14) as

$$\frac{\gamma}{4} = -\frac{x^4}{2x^2 + c} + \frac{(a - c)x^2 + b}{2x^2 + c}; \tag{16}$$

so that the sign and magnitude of the increment depend on the sign and magnitude of the coefficient b/c.

Let us assume that φ/f is a slowly varying function (which follows from the numerical solution) and let us neglect its derivatives. In first approximation, assuming a constant potential, we insert $\varphi/f = \text{const}$ into the integrals that appear in the expressions for a, b, and c. The integrals then vanish, so that $a = b = c = 0$. We see that the integrands in the coefficients a and c undergo a change of sign, exhibit a node, and at infinity approach zero from the negative-value side. This signifies that the coefficients a and c can be positive only for potentials that increase slowly from the z axis to the boundary.

The sign and magnitude of b depend on three terms which in the case of a constant potential are completely canceled out, so that $b = 0$. When, as previously, we analyze the behavior of the integrands, we easily determine that the second term makes a positive contribution for a potential that either increases or decreases toward the boundary; the third term can be positive only for a perturbation of the potential that increases towards the boundary, while the first term is always negative. Comparing orders of magnitude, we easily determine for our case (where the numerical solution shows that the potential is almost constant, increases only slightly between the axis and the boundary, and then drops to zero) the second term can be dominant, making b positive.

Summing up, we find that a buildup is possible, generally speaking, if the eigenfunctions of the boundary problem (11) include functions that increase slowly along the radius. It follows from the forms of a, b, and c that positive contributions come from terms which depend explicitly on the self-magnetic field of the current.

The foregoing qualitative conclusions were confirmed by the numerical calculation. The boundary problem was solved to obtain eigenvalues for two second-order linear differential equations in n and φ; the variable y had been eliminated from (11). The iteration method was used to derive eigenvalues and eigenfunctions. Each equation was written in terms of finite differences and was solved by a run-through method. For the perturbations, "physical infinity" was in one case located at a distance equal to four times the pinch radius; in a second case its distance was more than ten times the pinch radius. Figure 1 shows the dispersion curves obtained for these two cases; the dashed curve represents the second case.

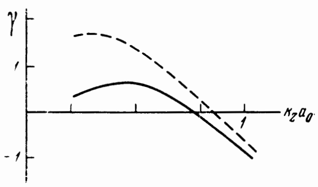

FIG. 1. Dispersion curves. The continuous curve represents the case of zero perturbations n and φ for $\rho = 4$; the dashed curve represents the case of $\rho = 10$, when there is no boundary effect; $\gamma = -i(\omega a_0^2/D) \times (\mu/\mu_p)$.

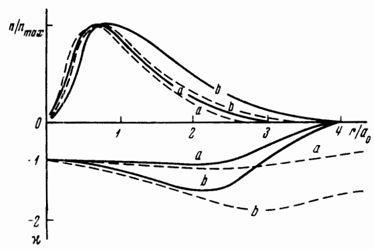

FIG. 2. Reduced density and potential perturbations: (a) for x = 0.5 and (b) for x = 1. The continuous curve represents the case of zero perturbations and φ for $\rho = 4$; the dashed curve represents $\rho = 10$, when there is no boundary effect; $\kappa = (\varphi/f)/|(\varphi/f)_0|$.

The iteration process converged rapidly for $x > 0.2$, but no convergence was obtained for $x \leq 0.2$. Figure 2 shows the behavior of the reduced potential $(\varphi/f)/|(\varphi/f)_0| = \kappa$ and the reduced density n/n_{max} for the two most interesting values, x = 1 and 0.5.

6. DISCUSSION OF RESULTS

In the region $k_z a_0 < 1$ of the dispersion curves (Fig. 1) a buildup is observed, i.e., $\gamma > 0$; larger increments are reached for the second case (the dashed curve). Figure 2 shows that the positive eigenvalues are associated with eigenfunctions φ of the boundary problem that grow slowly in the radial direction and that are nodeless, like the density perturbations. These nodeless density perturbations can evidently only occur in conjunction with a strong inflow of carriers along the z axis, thus producing a nonuniform positive density increment in the entire cross section.

The following oscillation buildup mechanism is indicated reasonably by the foregoing discussion. If at some moment in some region of the plasma filament a density increment has appeared ($\delta n > 0$), it is easily seen that in this same region the maximum weakening of the longitudinal field E_z^0 will occur. Indeed, Fig. 3 shows that the field perturbation $E_z = -ik_z\varphi' \sim k_z\varphi$, with $\varphi < 0$, opposes the main field. However, the radial field perturbation $E_r' = -\partial\varphi'/\partial r \sim -i\partial\varphi/\partial r$ has been shifted one-half phase period from the density perturbation, and charges have arranged themselves in the regions a and b (shown in Fig. 3) with their maximum density in the boundary layer. The radial field perturbation at a is directed outward from the center and will reduce the main field; at b, however, the radial field will enhance the main field, being oriented in the same direction. The crossed fields (the radial field perturbation and the self-magnetic field) will induce a continual transfer of new plasma portions through the boundary layer into the region with $\delta n > 0$. Thus for the given potential distribution the buildup results from longitudinal Lorentz forces that contract the plasma through the surface layer and ultimately augment the initial perturbation.

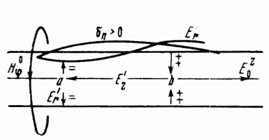

FIG. 3. Field and charge distributions in a perturbed pinch. a and b—regions of charge accumulation, with maximum charge density at the boundary. The crossed fields E_r' and H_φ^0 induce particle drift to the right at a, and to the left at b.

7. CONCLUSION

It has been our purpose in the foregoing investigation to determine the characteristics of instability in a solid-state pinch when a strong interaction exists between carriers and the lattice. For this purpose we required the given model (which is not too much different from reality), where no external magnetic field is applied and there is a constant total number of carriers per unit length of the pinch.

The numerical calculation has shown that in this model a pinch is unstable for constrictive perturbations with $k_z a_0 < 1$. For InSb below room temperature we find the characteristic growth time of this instability to lie in the interval $\tau \approx 10^{-6}-10^{-8}$ sec; here $a_0 \sim 10^{-2}$ cm and $D_p \approx 10^2-10^4$ cm^2/sec.[7] The instability itself is of absolute character, with a real increment. Under experimental conditions, however, since the growth increment of the perturbations is small, an appreciable stabilizing role can be played by carrier recombination and generation. We have not considered these effects.

At low temperatures, when impurities play a more important role, it becomes possible for perturbations to drift towards the sample faces in the direction of minority carrier motion.[8] In this case the character of the instability can change from absolute to drift type.

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[1] S. N. Breus, Zh. Tekh. Fiz. 30, 1030 (1960) [Sov. Phys.-Tech. Phys. 5, 960 (1960)]; A. Dattner, Arkiv Fysik 21, 71 (1962).

[2] A. P. Shotov, S. R. Grishechkina, and R. A. Muminov, Zh. Eksp. Teor. Fiz. 50, 1525 (1966) [Sov. Phys.-JETP 23, 1017 (1966)].

[3] M. Glicksman and R. A. Powlus, Phys. Rev. 121, 1659 (1961); B. Ancker-Johnson, Phys. Rev. Lett. 9, 485 (1962).

[4] K. Ando and M. Glicksman, Phys. Rev. 154, 316 (1967).

[5] H. Schmidt, Phys. Rev. 149, 564 (1966).

[6] E. Kamke, Differential gleichung, Lösungsmethoden und Lösungen [Russ. transl. of Vol. I on ordinary differential equations, IIL, 1965, p. 247].

[7] M. Glicksman, Phys. Rev. 124, 1655 (1968).

[8] R. A. Smith, Semiconductors, Cambridge Univ. Press, 1959 (Russ. transl., IIL, 1962, p. 256).

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