LIGHT PROPAGATION IN A MEDIUM WITH RANDOM REFRACTIVE INDEX INHOMOGENEITIES IN THE MARKOV RANDOM PROCESS APPROXIMATION

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Propagation of light in a medium with random inhomogeneities of the refractive index is investigated in the parabolic-equation ("quasioptics") approximation. In the first part of the paper it is shown that the solutions of the problem obtained by the Rytov method or by perturbation theory including higher approximations essentially correspond to the model of inhomogeneities delta-correlated along the direction of wave propagation. It is found that this assumption itself is sufficient for obtaining an exact solution of the problem without using perturbation theory. In the second part of the paper, exact closed equations for the mean value of the field strength and for the mutual coherency function are derived for the delta-correlated inhomogeneity model, and their solution in some of the simplest cases is considered. In the third part it is shown that for this model the characteristic field functional satisfies an equation of the Fokker-Planck type, and that closed equations for moments of arbitrary order can be obtained from it. In the last part of the paper an equation for fluctuations of the field strength is considered which is valid for weak as well as strong fluctuations of the field strength.

1. STATEMENT OF THE PROBLEM

In recent years the problem of strong fluctuations in the intensity of light propagating in a turbulent medium has attracted much attention. A number of experimental and theoretical papers have been devoted to this question. Because of the high degree of mathematical complexity of this problem, the presently known experimental facts are still far from being quantitatively explained by theory. The present paper is an attempt to develop a theory of strong fluctuations, starting not from the series summation of perturbation theory, as was done in[6-8], but on the basis of an exact solution of the problem corresponding to a special model of the medium, which nonetheless is completely applicable to real conditions.

The propagation of light in a medium with random inhomogeneities in the dielectric constant (e.g., due to turbulence) is described rather well by the parabolic equation

$$2ik\frac{\partial u(x,\rho)}{\partial x}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+k^2\varepsilon_1(x,\rho)u=0, \tag{1}$$

where $\rho = (y, z)$, and the x axis is taken along the initial direction of wave propagation. In Eq. (1), $\epsilon_1 = (\epsilon - \langle\epsilon\rangle)/\langle\epsilon\rangle$, where ϵ is the dielectric constant, the angular brackets indicating averages, and $k = (\omega/c)\langle\epsilon\rangle^{1/2}$ is the wave number. We introduce the notation

$$\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=\Delta,\quad \left(\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)=\nabla_\perp.$$

The equation of Rytov's method of smooth perturbations (MSP) is obtained from Eq. (1) if we introduce a new unknown function $\psi = \ln u$:

$$2ik\frac{\partial\psi}{\partial x}+\Delta\psi+(\nabla_\perp\psi)^2+k^2\varepsilon_1=0. \tag{2}$$

The exact solutions of (1) and (2) are equivalent.

In the first approximation of MSP, the term $(\nabla_\perp\psi)^2$ is left out, i.e., we consider the equation

$$2ik\frac{\partial\psi}{\partial x}+\Delta\psi+k^2\varepsilon_1(x,\rho)=0. \tag{A}$$

It is known that Eq. (A) is not suitable for the description of the statistical properties of the solution in the region where field intensity fluctuations are no longer small. However, the transition to Eq. (A) is not the only simplification usually made in solving the problem in the first approximation of MSP. Besides (A), further simplification is made, which greatly facilitates the calculation and leads to very small errors in the final results. The essence of this simplification is the following. The two-dimensional spectral density of the fluctuations of the dielectric constant $F_\epsilon(x - x', \kappa)$, which is connected with the correlation function $B_\epsilon(x, \rho; x', \rho') = \langle\epsilon_1(x, \rho)\epsilon_1(x', \rho')\rangle$ by the relation

$$B_\varepsilon(x-x',\rho-\rho')=\int\int_{-\infty}^{\infty}F_\varepsilon(x-x',\varkappa)\exp\{i\varkappa(\rho-\rho')\}\,d^2\varkappa,$$

is a "narrow" function concentrated in the region $\kappa|x - x'| \lesssim 1$. Because of the "narrowness" of the function $F_\epsilon(x - x', \kappa)$, the integrals arising in the calculations are calculated approximately with the aid of the relation

$$\int_0^{L_1}dx_1\int_0^{L_2}dx_2F_\varepsilon(x_1-x_2,\varkappa)f(x_1,x_2,\varkappa)\approx\int_{-\infty}^{\infty}F_\varepsilon(\xi,\varkappa)\,d\xi\cdot\int_0^{\min(L_1,L_2)}f(\eta,\eta,\varkappa)\,d\eta.$$

The error arising through the use of the approximation (B) can easily be estimated for each specific case. For example, in calculating the mean square fluctuation of the phase difference

$$D_S(L,\rho)=\langle[S(L,\mathbf{r})-S(L,\mathbf{r}+\rho)]^2\rangle,$$

the relative error associated with the use of (B) for the case when the fluctuations in ϵ are caused by turbulence is of the order $(\delta D_S/D_S) \sim (\rho/L)^{1/3}$. In calculating the mean square fluctuation of the logarithm of the amplitude $\sigma_\chi^2 = \langle[\ln A - \langle\ln A\rangle]^2\rangle$ in the same problem, the relative error associated with (B) is of the order

$$\frac{\delta\sigma_\chi^2}{\sigma_\chi^2} \sim \left(\frac{\sqrt{\lambda L}}{L}\right)^{1/3}, \quad \lambda = \frac{2\pi}{k}.$$

Thus, application of the simplified formula (B) introduces errors that diminish with increasing L and are insignificant in cases of practical interest.

We now note that the approximate formula (B) is equivalent to the following approximation to the function $F_\epsilon(x - x', \kappa)$:

$$F_\varepsilon(x - x', \varkappa) = \delta(x - x') A(\varkappa), \tag{3}$$

$$A(\varkappa) = \int_{-\infty}^{\infty} F_\varepsilon(\xi, \varkappa)\, d\xi = 2\pi\Phi_\varepsilon(\varkappa), \tag{4}$$

where $\Phi_\epsilon(\kappa)$ is the three-dimensional spectral density of ϵ on the two-dimensional argument κ.

Thus, in the first approximation of MSP, two simplifications are in fact used—linearization of Eq. (2) and the approximation (3) for the two dimensional spectral density of ϵ. It is known that in the region of weak fluctuations, where the simple form (A) is valid, the results of calculations using (3) agree well with the experimental data (see[9]). Since the error associated with the use of Eq. (3) decreases with increasing L, it is natural to expect that in the region of strong fluctuations, the approximation (3) will introduce still smaller errors.

It is also easy to estimate the error associated with (B) in the higher terms of the MSP perturbation series. It turns out to be equal to the relative error of the first approximation, raised to a power equal to the number of the approximation, i.e., it decreases with this number. It is found that for the model of inhomogeneities described by (3), it is possible in principle to obtain an exact solution to the problem, i.e., to find the moments of the field $u(x, \rho)$ that obeys Eq. (1). This is equivalent to giving up the linearization of Eq. (2).

The solution obtained below is analogous to the known solution of the problem of excitation of parametric oscillations under the influence of a delta-correlated pump (see[10]). In the application to the latter problem, the method developed below leads to the same results as were obtained in[10].

2. DERIVATION OF THE EQUATIONS FOR THE AVERAGE VALUE OF THE FIELD AND THE MUTUAL COHERENCE FUNCTION

We shall assume that $\epsilon_1(x, \rho)$ is a Gaussian random field that is delta-correlated in x:

$$\langle \varepsilon_1(x, \rho)\rangle = 0, \tag{5}$$

$$\langle \varepsilon_1(x, \rho)\varepsilon_1(x', \rho')\rangle = \delta(x - x') A(\rho - \rho'), \tag{6}$$

$$A(\rho) = 2\pi \iint \Phi_\varepsilon(\varkappa) \exp[i\varkappa\rho]\, d^2\varkappa. \tag{7}$$

Equations (6) and (7) are a consequence of Eqs. (3) and (4). We average Eq. (1) and symbolize $\langle u(x, \rho)\rangle$ by $\bar{u}$. Then we obtain the equation

$$2ik \frac{\partial \bar{u}(x, \rho)}{\partial x} + \Delta\bar{u} + k^2\langle\varepsilon_1(x, \rho) u(x, \rho)\rangle = 0. \tag{8}$$

To find an expression for $g_1(x, \rho) = \langle \epsilon_1(x, \rho) u(x, \rho)\rangle$ we use

$$\langle \varepsilon_1(x, \rho) Z[\varepsilon_1]\rangle = \int dx' \iint d^2\rho' \langle \varepsilon_1(x, \rho)\varepsilon_1(x', \rho')\rangle \left\langle \frac{\delta Z[\varepsilon_1]}{\delta\varepsilon_1(x', \rho')}\right\rangle, \tag{9}$$

which is valid for a Gaussian random field $\epsilon_1(x, \rho)$ with zero average value, and the functional Z on ϵ_1. Equation (9) was obtained by Furutsu[11] and Novikov[12] by expanding the functional $Z[\epsilon_1]$ in series form. In the case of a non-Gaussian ϵ_1, we have a more complicated formula instead of (9), which contains higher cumulants of the random field ϵ_1. Since $u(x, \rho)$ is a functional on ϵ_1, then, using (9) and (6), we obtain

$$g_1(x, \rho) = \iint A(\rho - \rho') \left\langle \frac{\delta u(x, \rho)}{\delta\varepsilon_1(x, \rho')}\right\rangle d^2\rho'. \tag{10}$$

We now find $\delta u(x, \rho)/\delta\epsilon_1(x, \rho')$. For this we integrate (1) over x:

$$2iku(x, \rho) - 2iku_0(\rho) + \Delta\int_0^x u(\xi, \rho)\, d\xi + k^2\int_0^x \varepsilon_1(\xi, \rho) u(\xi, \rho)\, d\xi = 0. \tag{11}$$

Here $u_0(\rho) = u(0, \rho)$ is the given boundary condition. We introduce the function

$$\Theta(\xi) = \int_{-\infty}^{\xi} \delta(\xi')\, d\xi' = \begin{cases} 0 & \text{for } \xi < 0 \\ 1/2 & \text{for } \xi = 0. \\ 1 & \text{for } \xi > 0 \end{cases}$$

Adding the factor $\Theta(x - \xi)\delta(\rho - \rho'')$ to the integrand of the last term in (11), we write (11) in the form

$$2iku(x, \rho) - 2iku_0(\rho) + \Delta\int_0^x u(\xi, \rho)\, d\xi + k^2\int_0^\infty d\xi \iint d^2\rho''\Theta(x - \xi)\delta(\rho - \rho'')\varepsilon_1(\xi, \rho'') u(\xi, \rho'') = 0. \tag{12}$$

Operating on (12) with the operator $\delta/\delta\epsilon_1(x', \rho')$ and taking into account that $\delta\epsilon_1(\xi, \rho'')/\delta\epsilon_1(x', \rho') = \delta(\xi - x')\delta(\rho'' - \rho')$:

$$2ik\frac{\delta u(x, \rho)}{\delta\varepsilon_1(x', \rho')} + \Delta\int_0^x \frac{\delta u(\xi, \rho)}{\delta\varepsilon_1(x', \rho')}\, d\xi + k^2\Theta(x - x')\delta(\rho - \rho') u(x', \rho') + k^2\int_0^\infty d\xi \iint d^2\rho''\Theta(x - \xi)\delta(\rho - \rho'')\varepsilon_1(\xi, \rho'')\frac{\delta u(\xi, \rho'')}{\delta\varepsilon_1(x', \rho')} = 0. \tag{13}$$

Now we note that by virtue of (11), $u(\xi, \rho)$ depends only on values of $\epsilon_1(x', \rho')$ for $x' < \xi$ (on prior values of ϵ_1). Hence $\delta u(\xi, \rho)/\delta\epsilon_1(x', \rho') = 0$ for $\xi < x'$. Consequently, the lower limit of integration in (13) can be replaced by x'. Transforming the last term in (13), we write

$$2ik\frac{\delta u(x, \rho)}{\delta\varepsilon_1(x', \rho')} + \Delta\int_{x'}^x \frac{\delta u(\xi, \rho)}{\delta\varepsilon_1(x', \rho')}\, d\xi + k^2\Theta(x - x')\delta(\rho - \rho') u(x', \rho') + k^2\int_{x'}^x \varepsilon_1(\xi, \rho)\frac{\delta u(\xi, \rho)}{\delta\varepsilon_1(x', \rho')}\, d\xi = 0. \tag{14}$$

But (10) contains the derivative $\delta u(x, \rho)/\delta\epsilon_1(x, \rho')$ for coinciding values of x. In (14) we set $x' = x$. In this case the integral terms disappear and, considering the relation $\Theta(0) = 1/2$, we obtain[1)]

$$\frac{\delta u(x, \rho)}{\delta\varepsilon_1(x, \rho')} = \frac{ik}{4}\delta(\rho - \rho') u(x, \rho);$$

$$\frac{\delta u^*(x, \rho)}{\delta\varepsilon_1(x, \rho')} = -\frac{ik}{4}\delta(\rho - \rho') u^*(x, \rho). \tag{15}$$

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1)Note that since the initial correlation function $B_\epsilon(x - x', \rho)$ is even with respect to $x - x'$, it is necessary in (6) to assume the δ-function to be even, which also leads to the relation $\Theta(0) = 1/2$.

Averaging (15) and substituting it into (10), we obtain

$$g_1(x, \rho) = ikA(0)\bar{u}(x, \rho)/4;$$

substituting the latter in (8), we arrive at the equation

$$2ik\frac{\partial \bar{u}(x, \rho)}{\partial x} + \Delta\bar{u}(x, \rho) + \frac{ik^3}{4}A(0)\bar{u}(x, \rho) = 0. \quad (16)$$

Along with the initial condition $\bar{u}(0, \rho) = u_0(\rho)$, Eq. (16) completely determines the average field $\bar{u}$, i.e., the coherent component of the field.

By exactly the same means we can get the equation for the mutual coherence function

$$\Gamma(x, \rho_1, \rho_2) = \langle u(x, \rho_1)u^*(x, \rho_2)\rangle. \quad (17)$$

We multiply (1) by $u^*(x, \rho_2)$, and then we write the equation for $u^*(x, \rho_2)$ and multiply it by $u(x, \rho_1)$. Subtracting these equations and averaging, we obtain

$$2ik\frac{\partial\Gamma(x, \rho_1, \rho_2)}{\partial x} + \Delta_1\Gamma - \Delta_2\Gamma + k^2 g_2(x, \rho_1, \rho_2) = 0,$$

where Δ_1, Δ_2 are Laplacian operators in the variables ρ_1 and ρ_2, respectively and

$$g_2(x, \rho_1, \rho_2) = \langle[\varepsilon_1(x, \rho_1) - \varepsilon_1(x, \rho_2)]u(x, \rho_1)u^*(x, \rho_2)\rangle.$$

To find g_2 we again make use of Eq. (9), in which we set $Z = u(x, \rho_1)u^*(x, \rho_2)$, and Eq. (15). After simple calculations, we obtain the equation

$$2ik\frac{\partial\Gamma}{\partial x} + (\Delta_1 - \Delta_2)\Gamma + \frac{ik^3}{2}[A(0) - A(\rho_1 - \rho_2)]\Gamma = 0, \quad (18)$$

which we need to supplement with the boundary condition

$$\Gamma(0, \rho_1, \rho_2) = u_0(\rho_1)u_0^*(\rho_2).$$

Equations (16) and (18) permit us to find $\bar{u}$ and Γ for beams with arbitrary initial profiles $u_0(\rho)$.

In case the initial wave is plane, i.e., $u_0(\rho) = 1$, symmetry considerations tell us that $\bar{u}(x, \rho) = \bar{u}(x)$ and $\Gamma(x, \rho_1, \rho_2) = (x, \rho_1 - \rho_2)$. Hence $\Delta\bar{u} = 0$ and $(\Delta_1 - \Delta_2)\Gamma = 0$. The solutions of Eqs. (16) and (18) in this case have the form

$$\bar{u}(x, \rho) = \exp\left[-\frac{k^2}{8}A(0)x\right], \quad (19)$$

$$\Gamma(x, \rho_1, \rho_2) = \exp\left\{-\frac{k^2}{4}[A(0) - A(\rho_1 - \rho_2)]x\right\}. \quad (20)$$

We remark that Eq. (19) can be represented in the form $\bar{u} = \exp(-\sigma_0 x/2)$, where $\sigma_0 = k^2A(0)/4$ is the effective diameter of the scattering of unit volume in solid angle 4π found in the first Born approximation for Eq. (1).

In the case of arbitrary initial conditions the solution of Eq. (16) has the form

$$\bar{u}(x, \rho) = \frac{k}{2\pi ix}\int\int_{-\infty}^{\infty} u_0(\rho - \rho')\exp\left\{-\frac{\sigma_0 x}{2} + \frac{ik\rho'^2}{2x}\right\}d^2\rho'. \quad (19a)$$

Note that in the absence of fluctuations the corresponding expression would not contain the factor $\exp(-\sigma_0 x/2)$, so that (19a) represents the product of this factor and the solution of the diffraction problem in a medium without fluctuations.

Equation (2) for the coherence function may be written in the form

$$\Gamma(x, \rho) = \exp\{-{}^1/_2 D_1(x, \rho)\}, \quad (20a)$$

where

$$D_1(x, \rho) = \pi k^2 x\int\int[1 - \cos\varkappa\rho]\Phi_\varepsilon(\varkappa)d^2\varkappa \quad (21)$$

is the structure function for the complex phase found in the first approximation of MSP.[9] Equation (20a) was obtained by Chernov[13] as an approximation—in the first approximation of MSP with an additional assumption about the normality of the distribution law for ln u. Later, this same expression was also obtained in a number of other papers. In particular, de Wolf obtained it[5] by summing perturbation series. The derivation presented above justifies the use of (20) with less stringent limitations, in particular even for the region of strong intensity fluctuations. It should be noted that the agreement of Eq. (20) with the result of a calculation in the first approximation of MSP cannot be considered as a justification of this approximation in the region of strong fluctuations. As will be seen later, the expression for intensity fluctuations obtained from the exact equation does not agree with the corresponding expression from MSP.

3. DERIVATION OF AN EQUATION OF THE FOKKER-PLANCK TYPE FOR THE CHARACTERISTIC FUNCTIONAL

As is known, the probability distribution at time t of a random variable y(t) that satisfies a differential equation of the first order in time with a delta-correlated external force, satisfies the Fokker-Planck equation. Our case is analogous, with the role of time played by the coordinate x. However, for a fixed value of x, the function $u(x, \rho)$ is a random function of ρ, and it should be described by means of the characteristic functional

$$\Psi_x[v, v^*] = \langle\exp(iR_x)\rangle$$
$$= \left\langle\exp\left\{i\int\int[u(x, \rho)v(\rho) + u^*(x, \rho)v^*(\rho)]d^2\rho\right\}\right\rangle. \quad (22)$$

Here v and v* are treated as independent functions. We shall obtain below an equation satisfied by Ψ_x by following Novikov's method.[12]

Differentiating (22) with respect to x and using (1), we obtain

$$\frac{\partial\Psi_x}{\partial x} = \left\langle\exp(iR_x)\ i\int\int\left[v(\rho)\left(-\frac{1}{2ik}\right)(\Delta u + k^2\varepsilon_1(x, \rho)u)\right.\right.$$
$$\left.\left. + v^*(\rho)\left(\frac{1}{2ik}\right)(\Delta u^* + k^2\varepsilon_1(x, \rho)u^*(x, \rho))\right]d^2\rho\right\rangle. \quad (23)$$

The quantity $\langle\Delta u(x, \rho)\exp(iR_x)\rangle$ appearing in (23) may be represented in the form

$$\Delta\frac{1}{i}\frac{\delta\Psi_x[v, v^*]}{\delta v(\rho)},$$

similar to what was done in deriving the Hopf equation of the theory of turbulence.[14] We now find the quantity

$$g[v, v^*; x, \rho] = \langle\varepsilon_1(x, \rho)\exp(iR_x)\rangle, \quad (24)$$

in terms of which the terms containing ϵ_1 in (23) are expressed by differentiation with respect to v and v*. Applying Eqs. (9) and (6), we have

$$g[v, v^*; x, \rho] = \int\int A(\rho - \rho')\left\langle\frac{\delta[\exp(iR_x)]}{\delta\varepsilon_1(x, \rho')}\right\rangle d^2\rho'.$$

But

$$\frac{\delta[\exp(iR_x)]}{\delta\varepsilon_1(x, \rho')} = \exp(iR_x)i\int\int\left[v(\rho)\frac{\delta u(x, \rho)}{\delta\varepsilon_1(x, \rho')}\right.$$
$$\left. + v^*(\rho)\frac{\delta u^*(x, \rho)}{\delta\varepsilon_1(x, \rho')}\right]d^2\rho'.$$

Using now Eq. (15), we obtain

$$\left\langle\frac{\delta[\exp(iR_x)]}{\delta\varepsilon_1(x,\rho')}\right\rangle = -\frac{k}{4}\langle\exp(iR_x)[v(\rho')u(x,\rho')-v^*(\rho')u^*(x,\rho')]\rangle$$
$$= -\frac{k}{4}\left[v(\rho')\frac{1}{i}\frac{\delta\Psi_x}{\delta v(\rho')} - v^*(\rho')\frac{1}{i}\frac{\delta\Psi_x}{\delta v^*(\rho')}\right],$$

so that

$$g[v, v^*; x, \rho] = \frac{ik}{4}\iint A(\rho-\rho')\left[v(\rho')\frac{\delta\Psi_x}{\delta v(\rho')} - v^*(\rho')\frac{\delta\Psi_x}{\delta v^*(\rho')}\right]d^2\rho'. \quad (25)$$

Equation (25) permits calculating average values of the type

$$\langle\varepsilon_1(x,\rho)u(x,\rho_1)\ldots u(x,\rho_m)u^*(x,\rho_1')\ldots u^*(x,\rho_n')\rangle$$

as derivatives of g with respect to v, v*. Thus

$$\langle\exp(iR_x)\varepsilon_1(x,\rho)u(x,\rho)\rangle = \frac{1}{i}\frac{\delta g[v,v^*;x,\rho]}{\delta v(\rho)}.$$

Using this formula and substituting (25) into (23), we finally obtain the following equation for Ψ_x:

$$\frac{\partial\Psi_x[v,v^*]}{\partial x} = \frac{i}{2k}\iint\left[v(\rho)\Delta\frac{\delta\Psi_x}{\delta v(\rho)} - v^*(\rho)\Delta\frac{\delta\Psi_x}{\delta v^*(\rho)}\right]d^2\rho$$
$$-\frac{k^2}{8}\iint d^2\rho\iint d^2\rho' A(\rho-\rho')M(\rho)M^+(\rho')\Psi_x[v,v^*], \quad (26)$$

where for brevity we have introduced the Hermite operators

$$M(\rho) = M^+(\rho) = v(\rho)\frac{\delta}{\delta v(\rho)} - v^*(\rho)\frac{\delta}{\delta v^*(\rho)}. \quad (27)$$

Equation (26) plays the role of the Fokker-Planck equation for our problem. It differs from the usual equation of this type in that it is written for the characteristic functional and not for the probability connected with this functional by the Fourier density. Hence Eq. (26) is itself the Fourier transform of the Fokker-Planck equation. A second difference is that (26) corresponds to a diffusion equation in a space of infinite dimensions with a variable (quadratic) coefficient of diffusion, in which connection it is an equation with variational derivatives. Since $A(\rho-\rho')$ is a correlation function in the plane x = const and hence is a positive definite kernel, the operator

$$K = -\frac{k^2}{8}\iint d^2\rho\iint d^2\rho' A(\rho-\rho')M(\rho)M^+(\rho'),$$

appearing in (26) is negative definite, which corresponds with the negative definiteness of the Laplace operator entering in the Fokker-Planck equation.

An important feature of Eq. (26) is its homogeneity: each operation of differentiation $\delta/\delta v(\rho)$ corresponds to multiplication by $v(\rho)$. This feature allows us to seek a solution of (26) in the form of the series

$$\Psi_x[v,v^*] = \sum_{m,n=0}^{\infty}\int\ldots\int K_{n,m}(x;\rho_1,\ldots,\rho_n;\ \rho_1',\ldots,\rho_m')$$
$$\cdot v(\rho_1)\ldots v(\rho_n)v^*(\rho_1')\ldots v^*(\rho_m')d^2\rho_1\ldots d^2\rho_n d^2\rho_1'\ldots d^2\rho_m', \quad (28)$$

whereby after substitution of (28) into (26) we obtain for each of the functions $K_{m,n}$ a closed differential equation in which functions K with other subscripts do not appear. The equation for $K_{1,0}$ is equivalent to the equation for $\bar{u}$; the equation for $K_{1,1}$ is equivalent to the equation for $\Gamma(x, \rho_1, \rho_2)$. The equation for $K_{2,2}$ describes intensity fluctuations and will be considered more thoroughly below. Thus, it follows from Eq. (26) that for moments of any order of the field u we obtain exact solutions of the equation in this approximation of delta-correlated inhomogeneities.

4. THE EQUATION FOR THE FOURTH MOMENT

The equation for the fourth moment

$$m_4(x,\rho_1,\rho_2,\rho_3,\rho_4) = \langle u(x,\rho_1)u(x,\rho_2)u^*(x,\rho_3)u^*(x,\rho_4)\rangle$$

can be obtained either from Eq. (26) or directly from (1) in a way similar to that by which Eq. (18) was derived. It has the form

$$\frac{\partial m_4(x,\rho_1,\rho_2,\rho_3,\rho_4)}{\partial x} = \frac{i}{2k}[\Delta_1+\Delta_2-\Delta_3-\Delta_4]m_4 \quad (29)$$
$$-\frac{k^2}{4}Q(\rho_1,\rho_2,\rho_3,\rho_4)m_4,$$

where

$$Q(\rho_1,\rho_2,\rho_3,\rho_4) = 2A(0) + A(\rho_1-\rho_2) + A(\rho_3-\rho_4) - A(\rho_1-\rho_3)$$
$$-A(\rho_1-\rho_4) - A(\rho_2-\rho_3) - A(\rho_2-\rho_4).$$

To Eq. (29) we need to add the initial condition

$$m_4(0,\rho_1,\rho_2,\rho_3,\rho_4) = u_0(\rho_1)u_0(\rho_2)u_0^*(\rho_3)u_0^*(\rho_4). \quad (30)$$

If the initial conditions are arbitrary, it is an extremely difficult problem to solve Eq. (29). Equation (29) is greatly simplified, however, if the incident wave is plane. In this case both the initial conditions and the function Q are invariant relative to shifts and rotations in the plane x = const. It turns out to be possible, by using these properties of invariance, to write Eq. (29) for the special case when $\rho_1-\rho_3=\rho_4-\rho_2$, i.e., for the case when the points $\rho_1, \rho_2, \rho_3, \rho_4$ are located at the vertices of a parallelogram. If R and R′ are the diagonals of this parallelogram and φ is the angle between them (so that when R = R′ and $\varphi = 0$ the point ρ_1 joins with ρ_3 and the point ρ_2 with ρ_4), then in the new variables Eq. (29) takes the form

$$\frac{\partial f(x,R,R',\varphi)}{\partial x} = \frac{i}{k}\left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial f}{\partial R}\right) - \frac{1}{R'}\frac{\partial}{\partial R'}\left(R'\frac{\partial f}{\partial R'}\right)\right.$$
$$\left.+\left(\frac{1}{R^2}-\frac{1}{R'^2}\right)\frac{\partial^2 f}{\partial\varphi^2}\right] - \frac{k^2}{4}Q_1(R,R',\varphi)f, \quad (31)$$

where $f(x, R, R', \varphi)$ and $Q_1(R, R_1', \varphi)$ are the values of the functions $m_4(x, \rho_1, \rho_2, \rho_3, \rho_4)$ and $Q(\rho_1, \rho_2, \rho_3, \rho_4)$ for $\rho_1-\rho_3=\rho_4-\rho_2$, expressed in terms of the variables R, R′, φ. To Eq. (31) we add the initial condition $f(0, R, R', \varphi) = 1$. Unlike Eq. (29), where m_4 depends on nine variables, Eq. (31) contains only four variables.

Let us consider in more detail the case when the fluctuations of the refractive index are due to turbulence. Then we may take[9]

$$\Phi_\varepsilon(\varkappa) = AC_\varepsilon^2\varkappa^{-11/3}\exp(-\varkappa^2/\varkappa_m^2), \quad (32)$$

where A is a numerical constant, κ_m is a wave number characterizing the size of the smallest inhomogeneity, and the quantity C_ϵ^2 characterizes the intensity of the fluctuations. The spectral density (32) goes to infinity for $\kappa = 0$; however, in this case this is not important, since the function Q corresponding to (32) exists (its existence for a spectral density of fluctuations with a singularity at zero implies that coarse inhomogeneities are not important for intensity fluctuations).

We introduce into Eq. (31) the longitudinal scale L and dimensionless coordinates $\xi = x/L$, $\eta = kR^2/4L$,

$\zeta = kR'^2/4L$. We introduce also the notation $D = \kappa_m^2 L/k$ for the wave parameter at distance L. Then in the new dimensionless variables Eq. (31), with the function Q corresponding to (32), can be written in the following form:

$$\frac{\partial f(\xi,\eta,\zeta,\varphi)}{\partial \xi} = i\left[\frac{\partial}{\partial \eta}\left(\eta\frac{\partial f}{\partial \eta}\right) - \frac{\partial}{\partial \zeta}\left(\zeta\frac{\partial f}{\partial \zeta}\right) + \frac{1}{4}\left(\frac{1}{\eta}-\frac{1}{\zeta}\right)\frac{\partial^2 f}{\partial \varphi^2}\right]$$
$$-\frac{11}{3}\cos\frac{\pi}{12}\frac{\beta_0^2(L)}{D^{5/6}}\left[2F\left(-\frac{5}{6},1,-\frac{D}{4}(\eta+\zeta+2\sqrt{\eta\zeta}\cos\varphi)\right)\right.$$
$$+2F\left(-\frac{5}{6},1,-\frac{D}{4}(\eta+\zeta-2\sqrt{\eta\zeta}\cos\varphi)\right)-F\left(-\frac{5}{6},1,-\eta D\right)$$
$$\left.-F\left(-\frac{5}{6},1,-\zeta D\right) -2\right] f(\xi,\eta,\zeta,\varphi). \qquad (33)$$

Here

$$\beta_0^2(L) = \alpha C_\varepsilon^2 k^{7/6} L^{11/6}, \qquad \alpha = 3\pi^3 A\left(11\Gamma\left(\frac{11}{6}\right)\cos\frac{\pi}{12}\right)^{-1},$$

which, as will be seen further on, is the mean square intensity fluctuation calculated by first-order perturbation theory for $D \to \infty$. If $D \to \infty$, then outside of a narrow region where $|\eta D| \lesssim 1$, $|\zeta D| \lesssim 1$, it is possible to use the asymptotic confluent hypergeometric function for large negative values of the argument $F(-5/6, 1, -z) \approx z^{5/6}/\Gamma(11/6)$ and obtain an equation not containing D:

$$\frac{\partial f}{\partial \xi} = i\left[\frac{\partial}{\partial \eta}\left(\eta\frac{\partial f}{\partial \eta}\right) - \frac{\partial}{\partial \zeta}\left(\zeta\frac{\partial f}{\partial \zeta}\right) + \frac{1}{4}\left(\frac{1}{\eta}-\frac{1}{\zeta}\right)\frac{\partial^2 f}{\partial \varphi^2}\right] - \frac{11\cos(\pi/12)}{3\Gamma(11/6)}$$
$$\times\beta_0^2(L)\left[2\left(\frac{\eta+\zeta+2\sqrt{\eta\zeta}\cos\varphi}{4}\right)^{5/6}\right.$$
$$\left.+2\left(\frac{\eta+\zeta-2\sqrt{\eta\zeta}\cos\varphi}{4}\right)^{5/6} - \eta^{5/6} - \zeta^{5/6}\right] f. \qquad (34)$$

Note, however, that the coefficient of f in (34) does not have derivatives at zero, so that when $\eta, \zeta \to 0$ one needs to use the "regularized" Eq. (33).

In the region where $\beta_0^2(L)^{11/6} = \beta_0^2(x) \ll 1$, the desired function f in the last term on the right-hand side of (34) can be replaced by the initial condition f = 1. After this, we can solve Eq. (34) analytically and obtain $f_1(\xi, \eta, \zeta, \varphi)$ (the subscript 1 signifies the first iteration or first approximation of perturbation theory):

$$f_1(\xi,\eta,\zeta,\varphi) = 1 + \beta_0^2(L)\left\{ i\frac{12\cos(\pi/12)}{11\Gamma(11/6)}(\eta^{11/6}-\zeta^{11/6})\right.$$
$$-\frac{2^{1/3}\cdot 11\cos(\pi/12)}{6\Gamma(11/6)}\xi\left[(\eta+\zeta+2\sqrt{\eta\zeta}\cos\varphi)^{5/6}+(\eta+\zeta-2\sqrt{\eta\zeta}\cos\varphi)^{5/6}\right]$$
$$\left.-2\cos\frac{\pi}{12}\xi^{11/6}\left[i^{17/6}F\left(-\frac{11}{6},1,\frac{i\eta}{\xi}\right)+i^{-17/6}F\left(-\frac{11}{6},1,-\frac{i\zeta}{\xi}\right)\right]\right\}. \qquad (35)$$

When $\eta = \zeta = 0$, the formula $f_1(\xi, 0, 0, \varphi) - 1 = \beta_0^2(L)^{11/6} = \beta_0^2(x)$ follows from (35). Thus, the relative intensity fluctuations f − 1 in the first approximation are given by a formula coinciding with the solution obtained by MSP. If in (35) we set $\varphi = 0$ and $\eta = \zeta$, we obtain the first approximation for the correlation function of the intensity fluctuations, which coincides exactly with the expression found by MSP.[9]

It was mentioned above that the function $\Gamma(x, \rho)$ found from Eq. (18) coincides for the case of a plane incident wave with the solution obtained from MSP. This is explained by the fact that for a plane wave Eq. (18) does not contain the transverse derivatives and hence the exponent from the first approximation agrees with the exact solution. However, as is seen from Eq. (34) this does not occur for intensity fluctuations, since Eq. (34), even in the case of a plane wave, contains the transverse derivatives. Hence the solution obtained for intensity fluctuations in MSP is not the solution of Eq. (34).

Equation (33) is rather complicated, and its solution, obviously, can be obtained only by numerical methods. However, from the very form of this equation it follows at once that the only parameters on which the solution can depend are the wave parameter and the mean square intensity fluctuation calculated in the first approximation.

The region of strong intensity fluctuations was investigated in[6-8] by means of a partial summation of the perturbation series. However, in this summation certain classes of diagrams are discarded that differ from the diagrams taken into account only by numerical coefficients. Since in all these papers the assumption that the fluctuations in ϵ are delta-correlated in the direction of wave propagation is implicit, it is clear that the solutions obtained in this way have no advantages (other than simplicity, perhaps) over the solution contained in Eq. (33).

5. CONCLUSION

The method developed here is based on the following two assumptions about the fluctuations in dielectric constant ϵ: a) $\epsilon_1(x, \rho)$ is a Gaussian random field; b) $\epsilon_1(x, \rho)$ is delta-correlated along the direction of propagation of the incident wave.

The solution obtained may, however, serve as a basis for constructing such a solution of Eq. (1) in which one or both of these assumptions are not used. This construction can be accomplished as follows. Equations can be set up for the moments of the fields u and ϵ_1 which are not based on assumptions a) and b). These equations will form an infinite coupled system, the solution of which can be sought in the form of a new perturbation series, on the basis of which the solution obtained above is undertaken. In doing this, the consequent terms of the series will take into account either the deviation of the field ϵ from a delta-correlated field, or its deviation from a Gaussian field. It can be expected that the corresponding expansion must converge much faster than the usual perturbation series.

It should also be mentioned that the method developed can be easily generalized to the case when the "intensity" of the fluctuations in the dielectric constant is a function of x.

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Note added in proof. Equation (18) has been obtained by Dolin (Izv. vyssh. uch. zav., Radiofizika, 11, 840 (1968)) and Chernov (abstract of a report to the VI All-Union Conference on Acoustics, 1968); however, they essentially used the smallness of the fluctuations in ϵ. The special case of Eq. (29) was obtained by Pishov (Izv. vyssh. uch. zav., Radiofizika, 11, 866 (1968)), who, unlike de Wolf [6,8] summed all the essential diagrams for m_4.

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