QUADRATIC CORRECTIONS TO THE LAGRANGIAN DENSITY OF THE GRAVITATIONAL FIELD AND THE SINGULARITY

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The generalized Lagrangian density of the gravitational field proposed by A. D. Sakharov is investigated in connection with the problem of the nonsingular transition from constriction to expansion within the framework of a homogeneous and isotropic cosmological model. It is shown that the solutions which at later stages go over into the Friedmann solution, are singular at $t = 0$.

FOR the derivation of the field equations of the general theory of relativity (GTR) one postulates the action

$$S = -\frac{c^3}{16\pi G}\int (R + 2\Lambda)\sqrt{-g}\, d\Omega, \quad (1)$$

where Λ is the cosmological constant.

In[1], Sakharov proposed a possible explanation of (1), which leads to corrections to S which depend on the quadratic invariants of the curvature tensor; these corrections can, in principle, be calculated. The Lagrangian density is written in the form of a series

$$L(R) = L(0) + AR + BR^2 + CR^{ik}R_{ik} + DR^{iklm}R_{iklm} + ER^{iklm}R_{ilkm}, \quad (2)$$

where the first two terms lead to (1), and the remaining terms are quadratic corrections in R, which can become important only for large R (these values of R are, however, not of the order of the inverse square of the Planck length $l_0^{-2} = c^3/G\hbar \approx 10^{66}\ \mathrm{cm}^{-2}$, but somewhat smaller: $R \sim l_0^{-2}/137$).

The general form of the Lagrangian density of the gravitational field has been considered by Polievktov-Nikoladze[2] for the case of weak gravitational fields; a correction to the Lagrangian density of the form $R^2 \ln (R/R_0)$ due to quantum effects has been considered in unpublished work by T. Hill (cf.[3]), with the aim of obtaining a nonsingular solution for the collapse of a dust-like sphere. In the present paper we consider the possibility of applying (2) near the Friedman singularity in connection with the problem of the nonsingular transition from constriction to expansion in the simplest case of a homogeneous and isotropic space filled with matter with the equation state $p = \epsilon/3$.

1. Let us show first that not all quadratic invariants in (2) are independent. Indeed, multiplying the identity

$$R_{iklm} + R_{imkl} + R_{ilmk} = 0$$

by R^{iklm} and using $R_{iklm} = -R_{kilm}$, we find

$$R^{iklm}R_{ilkm} = {}^1/_2 R^{iklm}R_{iklm}.$$

Furthermore, we can verify by direct variation (cf. also[4]) that[1)]

$$\delta\int (R^{iklm}R_{iklm} - 4R^{ik}R_{ik} + R^2)\sqrt{-g}\, d\Omega = 0.$$

For the derivation of the field equations it is therefore sufficient to take the Lagrangian density of the form

$$L(R) = L(0) + AR + BR^2 + CR^{ik}R_{ik}. \quad (3)$$

The condition that a minimum of the action exists[5] leads to the requirement that the higher powers of the first time derivatives of the metric tensor enter with positive sign; this implies that the constants B and C (in contrast to A) are positive.

2. The equations of the gravitational field corresponding to (3) are derived (cf. the Appendix) with the help of the principle of minimal action:

$$\delta\int (L(R) + L_m)\sqrt{-g}\, d\Omega = 0,$$

where L_m is the Lagrangian density of matter. The field equations have the form

$$R_i^k - {}^1/_2\delta_i^k(R + 2\Lambda) - B[2RR_i^k - {}^1/_2\delta_i^k R^2 + 2(\delta_i^k g^{lm} - \delta_i^m g^{kl})R_{;l;m}]$$
$$- C\,[2R_{ih}R^{ik} - {}^1/_2\delta_i^k R^{lm}R_{lm} + g^{lm}R^k_{i;l;m} + \delta_i^k R^{lm}{}_{;l;m} - 2g^{km}R^l_{i;l;m}] = \frac{8\pi G}{c^4}T_i^k. \quad (4)$$

Contracting these equations over i and k and using a consequence of the Biancchi identity[5]

$$R^k_{i;k} = {}^1/_2 R_{;i}, \quad (5)$$

we obtain

$$R + 4\Lambda + \alpha g^{lm}R_{;lm} = -\frac{8\pi G}{c^4}T, \quad (6)$$

where $\alpha = 6B + 2C$.

Using (5) and the formula $R_{;i;k;l} - R_{;i;l;k} = R_{;m}R^m_{ikl}$, it is easy to show that (4) contains the equations of motion

$$T^k_{i;k} = 0. \quad (7)$$

As already noted, for a homogeneous and isotropic space, the invariants $R^{ik}R_{ik}$ and R^2 in (3) are not independent in the derivation of the equations of motion (they differ by a total derivative!), and the equations (4) are accordingly simplified:[2)]

[1)]We mention also that in the case of a homogeneous and isotropic model the following relations hold:

$$R^{iklm}R_{iklm} - 2R^{ik}R_{ik} + {}^1/_3R^2 = 0, \quad \delta\int (R^2 - 3R^{ik}R_{ik})\sqrt{-g}\, d\Omega = 0,$$

i.e., two quadratic invariants are algebraically independent and only one suffices for the derivation of the field equations.

[2)]B. L. Altshuler has shown (private communication) that this result is also true for an anisotropic space with a metric of the first type:

$$ds^2 = c^2dt^2 - a^2(t)\,dx^2 - b^2(t)\,dy^2 - c^2(t)\,dz^2.$$

$$R_i^k - {}^1\!/_2\,\delta_i^k(R+2\Lambda) - {}^1\!/_6\,\alpha\,[2RR_i^k - {}^1\!/_2\,\delta_i^k R^2 + 2(\delta_i^k g^{lm} - \delta_i^m g^{kl})R_{;l;m}] = \frac{8\pi G}{c^4}T_i^k. \tag{8}$$

The constant α entering in (6) and (8) has the dimension of the square of a length, since the dimension of R is cm^{-2}. Evidently, the quantities $l^* = \sqrt{\alpha}$ and $t^* = \sqrt{\alpha}/c$ determine the characteristic space and time distances over which the quadratic corrections are important. Using the explicit expressions for the coefficients A, B, and C given in[1], one can show that $l^* \approx 10^{-31}$ cm, $t^* \approx 10^{-(41\text{ to }42)}$ sec. Since quantum effects appear at $l_0 = 10^{-33}$ cm, $t_0 = 10^{-44}$ sec, the classical approach to the problem is justified.

3. Let us consider the earlier stages of the cosmological expansion in a model with a plane co-moving space:

$$ds^2 = c^2dt^2 - b^2(t)(dx^2 + dy^2 + dz^2). \tag{9}$$

The cosmological constant plays no role in this case; we describe matter by the equation of state $p = \epsilon/3$. Here

$$T_0^0 = \varepsilon,\quad T_1^1 = T_2^2 = T_3^3 = -p,\quad T_\beta^\alpha = 0,\quad \alpha \neq \beta.$$

From the equation $T_0^k{}_{;k} = 0$ we find

$$\varepsilon b^4 = \text{const} \equiv 3c^4 b_1^2/8\pi G. \tag{10}$$

Substituting (9), and (10) in (8) with i = k = 0, we obtain an equation for the unknown function b(t), where the dot denotes differentiation with respect to ct:

$$\frac{\dot b^2}{b^2} + \alpha\left(2\frac{\dddot b\,\dot b}{b^2} - \frac{\ddot b^2}{b^2} + 2\frac{\ddot b\,\dot b^2}{b^3} - 3\frac{\dot b^4}{b^4}\right) = \frac{b_1^2}{b^4}. \tag{11}$$

For $\alpha = 0$ it goes over into the standard equation of relativistic cosmology[5] with the known Friedmann solution $b = \sqrt{2b_1 ct}$. Since b(t) is dimensionless, the arbitrary[3)] scale factor b_1 has the dimension cm^{-1}. It is also easy to see that the Friedmann solution satisfies (11) also for $\alpha \neq 0$, since the expression in parentheses vanishes for $b = \text{const}\sqrt{t}$.

The solution which has a regular minimum for t = 0 must, for small l, have the form

$$b(t) = b_0 + \frac{\ddot b_0}{2!}(ct)^2 + \frac{\dddot b_0}{3!}(ct)^3 + \ldots, \tag{12}$$

Substituting (12) in (11) and letting t tend to zero, we obtain a condition under which such an expansion is possible:

$$\alpha = -(b_1/b_0\ddot b_0)^2. \tag{13}$$

It follows from this that a solution with an extremum at zero exists only for $\alpha < 0$, which contradicts already the condition that the constants B and C be positive, which is required if the action corresponding to the Lagrangian density (3) is to have minimum.

4. Let us further investigate the asymptotic ($t \to \infty$) properties of the solution of (11); in particular, let us find an answer to the question whether solutions b(t) exist which have a regular minimum at t = 0 [formula (12)] and which go over into the Friedmann solution for large t.

[3)]It is seen from (9) that under the transformation b → const·b only the spatial coordinates x, y, z change, which do not enter explicitly in (10) and (11). It is clear therefore, that the physical quantities (for example, ϵ) do not depend on b_1.

Equation (11) can be written in the following form:

$$u + \alpha\left(\frac{uu''}{b^2} - 2\frac{uu'}{b^3} - \frac{u'^2}{4b^2}\right) = b_1^2,$$

where $u(b) = (b\dot b)^2$, and the dash denotes differentiation with respect to b.

Since b and $\dot b$ are real, u is positive. Introducing $f = u^{3/4}$ and $\xi = (12)^{-3/4}b^3$, we obtain

$$f'' + \alpha^{-1}\xi^{-2/3}(f^{-1/3} - b_1^2 f^{-5/3}) = 0. \tag{14}$$

It is clear that it is sufficient to consider only the positive solutions of this equation. Let us also show that in the new variables, the Freidmann (particular) solution goes over into a constant $f = f_0 \equiv b_1^{3/2}$.

Equation (14) can be obtained by varying the action corresponding to the Lagrangian

$$L(f, f', \xi) = \frac{f'^2}{2} - \frac{3}{2\alpha\xi^{2/3}}(f^{2/3} + b_1^2 f^{-2/3}). \tag{15}$$

The corresponding Hamiltonian is

$$\mathscr{H} = f'\frac{\partial L}{\partial f'} - L = \frac{f'^2}{2} + \frac{3}{2\alpha\xi^{2/3}}(f^{2/3} + b_1^2 f^{-2/3}). \tag{16}$$

The "potential energy"

$$u = \frac{3}{2\alpha\xi^{2/3}}(f^{2/3} + b_1^2 f^{-2/3}),$$

has, as a function of f, an extremum for $f = f_0$ ($du/df = 0$), where $u(f_0) = 3b_1/4\alpha\xi^{2/3}$. The sign of the second derivative at the point $f = f_0$, $d^2u/df^2 = {}^4\!/_3\,\alpha b_1\xi^{2/3}$ depends on the sign of α. For $\alpha < 0$ the equilibrium position $f = f_0$ is unstable (maximum), for $\alpha > 0$ and fixed ξ, it is "stable" (minimum).

Let us show that the solutions with a regular minimum ($\alpha < 0$) do not go over into the Friedmann solution with increasing ξ. Multiplying (14) by f′ and integrating it from ξ_0, corresponding to t = 0, to an arbitrary ξ, we obtain

$$\mathscr{H} = \mathscr{H}_0 + \frac{1}{|\alpha|}\int_{\xi_0}^{\xi}\xi^{-5/3}(f^{2/3} + b_1^2 f^{-2/3})\,d\xi. \tag{17}$$

It is easy to see that for the solution b(t) with a regular minimum (12)

$$\mathscr{H}_0 \equiv \frac{f'^2(\xi_0)}{2} - \frac{3}{2|\alpha|\xi_0^{2/3}}(f^{2/3}(\xi_0) + b_1^2 f^{-2/3}(\xi_0)) = \frac{3}{8}\frac{\dddot b_0}{b_0}.$$

Since the integral in (17) is positive we have $\dddot b_0 \geq 0$, $\mathscr{H} > 0$, so that

$$f'^2(\xi) > \frac{3}{\alpha\xi^{2/3}}(f^{2/3} + b_1^2 f^{-2/3});$$

integrating this inequality we find that for large ξ

$$f(\xi) > (3/|\alpha|)^{3/4}\xi, \tag{18}$$

i.e., for large ξ the solution increases without limit, rather than going over into the Friedmann solution $f = f_0$. One can also obtain the explicit form of this asymptotic solution

$$f \approx |\alpha|^{-3/4}\xi(\ln\xi)^{3/4}, \tag{19}$$

which has the following form in the variables b and t:

$$b \approx \text{const}\cdot\exp\left(\frac{t^2}{12|\alpha|}\right),\quad t \to \infty, \tag{20}$$

Here

$$R = -6\left(\frac{\ddot b}{b} + \frac{\dot b^2}{b^2}\right) \approx -\frac{t^2}{6\alpha^2} \to -\infty.$$

We note that $\dddot{b}_0 = 0$ corresponds to a solution which is symmetric about $t = 0$. For $\dddot{b}_0 < 0$ the same divergence appears for large negative t.

For $\alpha > 0$ the solutions $f(\xi)$ will be oscillatory. For small deviations from the equilibrium position $f = f_0 + \varphi$ ($\varphi \ll f_0$) the Hamiltonian (16) becomes

$$\mathscr{H} \approx \frac{\varphi'^2}{2} + \frac{2}{3\alpha b_1^2 \xi^{2/3}} \varphi^2 \tag{21}$$

and describes an oscillator which oscillates with the frequency $\omega(\xi) = 2/\sqrt{3\alpha} b_1 \xi^{1/3}$, with $\mathscr{H} = I\omega$, where I is an adiabatic invariant. For large values of ξ [but not so large that the amplitude of the oscillations φ is of the order of, or larger than f_0] we find from an equation corresponding to (21)

$$\varphi \approx c_1 \xi^{1/6} \cos\left(\sqrt{\frac{3}{\alpha}} \frac{\xi^{2/3}}{b_1} + c_2\right) \tag{22}$$

(c_1 and c_2 are constants).

The occurrence of the small oscillations of $f(\xi)$ about the equilibrium position f_0 is explained by the circumstance that the bottom of the "potential well" $u(f_0)$ goes down with increasing ξ like $\xi^{-2/3}$, whereas the "energy level" $\mathscr{H}$ varies like $\xi^{-1/3}$. However, in the variables b and t, the solution has the following form in the approximation under consideration:

$$b(t) \approx \sqrt{2b_1 ct}\left(1 + \frac{\text{const}}{t^{3/4}} \sin\frac{t}{\sqrt{\alpha}}\right), \tag{23}$$

i.e., it approaches the Friedmann solution with increasing t.

Thus the theory of the gravitational field with the Lagrangian density (3) leads to the following results within a homogeneous isotropic model: a) even for $\alpha \neq 0$ the usual Friedmann solution is true; b) for $\alpha > 0$ all solutions are singular for $t = 0$ and go over into the Friedmann solution for large t; if $\alpha < 0$ there are solutions with a regular minimum at $t = 0$, however, these do not approach the Friedmann solution for large t.

We therefore arrive at the conclusion that the considered generalization of the GTR considered in this paper does not allow one to construct a satisfactory model for the continuous transition from constriction to expansion, but leads, as the usual GTR, to a singularity in the physically acceptable solutions.

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APPENDIX

Let us show how the equations (4) are derived. The necessary notation and formulas are taken from the book[5].

Let us write $\delta g_{ik} = h_{ik}$. Then

$$\delta g^{ik} = -h^{ik}, \quad \delta\sqrt{-g} = {}^1\!/_2 \sqrt{-g}\, g_{ik} h^{ik},$$

$$\delta\Gamma^i_{kl} = {}^1\!/_2 (h^i_{k;\,l} + h^i_{l;\,k} - h_{kl}{}^{;i});$$

$$\delta R_{ik} = (\delta\Gamma^l_{ik})_{;\,l} - (\delta\Gamma'_{il})_{;\,k} = {}^1\!/_2 (h^l_{i;\,k;\,l} + h^l_{k;\,i;\,l} - h_{ik;\,l}{}^{;\,l} - h_{;\,i;\,k}),$$

$$\delta R^2 = 2R\delta(g^{ik} R_{ik}) = 2R(g^{ik}\delta R_{ik} - R_{ik}h^{ik}),$$

$$\delta R^{ik} R_{ik} = 2R^{ik}\delta R_{ik} - R_{il}R^l_k h^{ik} - R_{mk}R^m_i h^{ik}.$$

The formula

$$\delta\int R\sqrt{-g}\, d\Omega = \int (R_{ik} - {}^1\!/_2 g_{ik} R)\sqrt{-g}\, h^{ik} d\Omega$$

is known.[5] We demonstrate the derivation of an analogous formula for R^2:

$$\delta S_1 \equiv \delta\int R^2\sqrt{-g}\, d\Omega = \int (\sqrt{-g}\,\delta R^2 + R^2\delta\sqrt{-g})\, d\Omega$$

$$= \int [-2RR_{ik}h^{ik} + {}^1\!/_2 R^2 g_{ik}h^{ik} + 2R(h^{ik}{}_{;i;k} - g_{ik}g^{lm}h^{ik}{}_{;l;m})]\sqrt{-g}\, d\Omega.$$

We integrate the term in parentheses twice by parts (Gauss theorem!). Since the variation of the field is zero on the hypersurface enclosing the entire four-volume, we obtain

$$\delta S_1 = \int [-2RR_{ik} + {}^1\!/_2 R^2 g_{ik} + 2(\delta^m_k \delta^l_i - g^{lm} g_{ik}) R_{;\,m;\,l}] h^{ik}\sqrt{-g}\, d\Omega.$$

By the same method we can obtain a formula for $R^{ik}R_{ik}$:

$$\delta S_2 = \int [-2R_{il}R^l_k + {}^1\!/_2 g_{ik}R^{lm}R_{lm} - g^{lm}R_{ik;\,l;\,m} - g_{ik}R^{lm}{}_{;\,l;\,m} + 2R^l_{i;\,k;\,l}]\sqrt{-g}\, h^{ik}\, d\Omega,$$

which leads to the left-hand side of (4). The right-hand part of these equations is obtained by the usual method from the Lagrangian density for matter.

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