SPECTRA OF LANGMUIR PLASMA TURBULENCE

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The nonlinear equations for the interaction between Langmuir waves in a plasma are used to determine the complete form of the quasistationary turbulence spectrum W_k. It is assumed that the stationary source produces turbulent energy in the region of very large wave numbers k and the energy is subsequently transmitted to the region of the fundamental turbulence scale $L = 2\pi/k_0$, where it is damped. The turbulence spectrum in the region of large values of k is calculated for various values of the plasma temperature and turbulence production power Q. An approximate analytic dependence of the fundamental turbulence scale on Q is found: $L = \text{const}\cdot Q^{1/2(\nu-1)}$, where ν is the exponent in the turbulence spectrum $W_k \sim k^{-\nu}$ in the range of small k. The effect of fast particles accelerated by turbulence on the spectrum of the latter is investigated. Self consistent distributions of fast particles and turbulent pulsations are found. For a sufficiently high level of the plasma turbulence the fast-particle spectrum should have the form $f(\epsilon) \sim \sqrt{\epsilon}$. The condition for vanishing of the maximum in the turbulence spectrum due to absorption by fast particles is found.

THE quasistationary spectrum of Langmuir turbulence of a plasma was calculated in[1] in the asymptotic region $k \gg k_0 \equiv 2\pi/L$, where L is the main scale of the turbulence. The calculation of[1] was limited by the assumption that $k \ll k_* = \frac{1}{3}\omega_0 v_i/v_e^2$, and the turbulence spectrum depends on the presence of fast particles accelerated by the turbulent pulsations[2] only if $k_* > \omega_0/c$. (For a hydrogen plasma and $T_e = T_i$, this condition yields $T_e < 12$ eV.) The purpose of the present paper is to construct the complete picture of the spectrum in the entire region of k, including $k > k_*$, and for both a sufficiently cold plasma ($k > \omega_0/c$) and a sufficiently hot plasma ($k_* < \omega_0/c$), and also to determine the main turbulence scale L as a function of the generation power Q, which equals the flux of the turbulent energy along the wave-number spectrum under stationary conditions. We also investigate the influence of fast accelerated particles on the turbulence spectra and determine the criterion for the vanishing of the maximum of the spectrum in the main turbulence scale obtained in[1]. The main premises of the present work are the same as in[1]. Namely, it is assumed that the generation occurs at large wave numbers near $1/\lambda_e$, where $\lambda_e = v_e/\omega_0$ is the Debye length, and that subsequently the energy of the turbulence is transformed, in the region of small k down to k_*, via scattering by ions and electrons, and when $k < k_*$ it is transformed as a result of four-plasmon processes. The fastest process is isotropization of the pulsations[3,4], and we shall therefore consider here the case of an isotropic three-dimensional turbulence.

1. INTERACTION OF SHORT-WAVE LANGMUIR PULSATIONS ($k \gg k_*$).

In the region $k \gg k_*$ it is necessary to take into account the following types of nonlinear interactions: 1) four-plasmon processes of nonlinear scattering of Langmuir waves into Langmuir waves, 2) induced scattering of Langmuir waves by plasmon electrons, 3) induced scattering by plasma ions, 4) decay of Langmuir waves into ion-acoustic waves ($l' \to l' + s$).

Unlike in the region $k \ll k_*$ (see[1]), in the region $k \gg k_*$ the four-plasmon interaction is strongly suppressed[4,5]. If the turbulence spectrum is sufficiently smooth (as is the case for stationary turbulence), then the effective increment of the energy transformation due to the four-plasmon interaction is estimated at[4]

$$\gamma^{(4)} \sim \frac{1}{4\cdot 3^5}\,\omega_0\Big(\frac{W^l}{nT_e}\Big)^2\Big(\frac{m_e}{m_i}\Big)^{3/2}(k\lambda_e)^{-5}, \qquad u = W^l/nT_e. \tag{1.1}$$

In the case of induced scattering by ions, only pulsations whose wave-number difference is much smaller than k_* interact effective in the region $k > k_*$. It can be stated that in this case the spectral redistribution occurs in physically infinitesimally small Δk ($\Delta k \ll k_*$), and is therefore differential for a broad spectrum $\Delta k \sim k$. From the general expression for scattering by ions[3]

$$\frac{\partial N(\mathbf{k})}{\partial t} = N(\mathbf{k})\,\frac{3}{8}\,\frac{\hbar\omega_0(T_e/T_i)}{m_e n_0 v_i(1+T_e/T_i)^2} \times\int\frac{N(\mathbf{k}')\,d\mathbf{k}'}{(2\pi)^{5/2}}\,\frac{(\mathbf{k}\mathbf{k}')^2}{k^2k'^2}\,\frac{k'^2-k^2}{|\mathbf{k}'-\mathbf{k}|}\exp\Big(-\frac{\omega_-^2}{2k_-^2v_i^2}\Big), \tag{1.2}$$

$$\mathbf{k}_- = \mathbf{k}-\mathbf{k}', \quad \omega_- = \omega_k-\omega_{k'} = {}^3/_2 v_e^2(k^2-k'^2)/\omega_0,$$

$$v_e = \sqrt{T_e/m_e}, \quad v_i = \sqrt{T_i/m_i},$$

assuming

$$-\frac{\omega}{\sqrt{2\pi}v_i^3k^3}\exp\Big(-\frac{\omega^2}{2v_i^2k^2}\Big) \to \delta'(\omega)$$

and integrating with respect to the angles for isotropic turbulence, we obtain

$$\frac{\partial W_k}{\partial t} = D_1 W_k\frac{\partial W_k}{\partial k}, \tag{1.3}$$

where

$$W_k = \frac{4\pi N_k k^2\omega_0\hbar}{(2\pi)^3}, \quad \int W_k\,dk = W^l, \qquad \omega_0 = \sqrt{4\pi e^2 n_0/m_e}, \quad D_1 = \frac{\pi\omega_0^3}{27 m_i n_0 v_e^4(1+T_e/T_i)^2} \tag{1.4}$$

Integrating with respect to the angles the well known expression for scattering by electrons[3]

$$\frac{\partial N(\mathbf{k})}{\partial t}=\frac{3v_e\hbar}{2m_en_0\omega_0}N(\mathbf{k})\int\frac{N(\mathbf{k}_1)d\mathbf{k}_1}{(2\pi)^{5/2}}\frac{(\mathbf{k}\mathbf{k}_1)^2}{k^2k_1^2}\frac{[\mathbf{k}\mathbf{k}_1]^2}{|\mathbf{k}-\mathbf{k}_1|^3}(k_1^2-k^2), \quad (1.5)$$

we obtain

$$\frac{\partial W_k}{\partial t}=\alpha' W_k\left\{\int_k^\infty W_{k_1}dk_1\frac{k^2}{k_1^3}(k_1^2-k^2)\left(\frac{1}{3}k_1^2+\frac{4}{7}k^2\right)\right. \quad (1.6)$$

$$\left.-\int_0^k W_{k_1}dk_1\frac{k_1^2}{k^3}(k^2-k_1^2)\left(\frac{1}{3}k^2+\frac{4}{7}k_1^2\right)\right\},$$

where

$$\alpha'={}^6/_5v_e\sqrt{2\pi}\,/\,m_en_0\omega_0^2.$$

Assuming that the spectrum is continuous and significant changes of the spectral density of the turbulence W_k occur in an interval Δk of the order of k, we find that four-plasmon interaction can be neglected practically always when $k > k_*$, since $\gamma^{(4)} < \gamma^{(i)}$. Scattering by electrons dominates over scattering by ions if

$$k > k_{**}=\left(\frac{m_e}{3m_i}\right)^{1/5}\left(1+\frac{T_e}{T_i}\right)^{-2/5}\frac{1}{\lambda_e}.$$

The decay processes of the Langmuir waves into ion-acoustic waves ($l \to l' + s$) are possible when $T_e \gg T_i$. We consider below only the case $T_i \gtrsim T_e$.

2. SPECTRUM OF PULSATIONS IN SCATTERING BY IONS

The general equation describing the change of the spectral density of the turbulence energy in the regions $k_* < k < k_{**}$, with allowance for scattering by ions, generation of turbulence, and scattering resulting from pair collisions, is of the form

$$\frac{\partial W_k}{\partial t}=D_1W_k\frac{\partial W_k}{\partial k}+\gamma_kW_k-\frac{\nu_e}{2}W_k-\gamma_sW_k, \quad (2.1)$$

where $\nu_e = \omega_0 \ln\Lambda/n_0\lambda_e^3$ is the frequency of the Coulomb collisions, $\ln\Lambda$ is the Coulomb logarithm, γ_k is the instability increment (the Landau damping is negligibly small in the region under consideration), and γ_s takes into account the absorption by the fast particles accelerated by the turbulence. Since it is our purpose to explain the qualitative picture of the turbulence spectrum, we confine ourselves first to the simplest case, when $\gamma_s = 0$ (or, more accurately, $\gamma_0 \gg \gamma_s$), i.e., there are no fast particles, and the increment γ_k is constant in a certain interval of wave numbers Δk_g 1):

$$\gamma_k=\gamma_0\begin{cases}1, & k_g<k<k_g+\Delta k_g,\\ 0, & k<k_g,\ k>k_g+\Delta k_g.\end{cases} \quad (2.2)$$

Let us examine W_k inside the interval Δk_g (γ_0 can be of the order of ν_e). In the quasistationary state we have

$$\frac{\partial W_k}{\partial k}=-\frac{1}{D_1}\left(\gamma_0-\frac{\nu_e}{2}\right). \quad (2.3)$$

1)The quantity γ_0 plays the role of the effective increment for isotropic pulsations. Since, by virtue of the fast isotropization of the pulsations, all the oscillations with close values of k turn out to be on par, although the excitation usually occurs at definite angles (for example, relative to the beam), to obtain γ_0 it is necessary to multiply the instability increment by the ratio of the phase volumes in k-space of the oscillations that are excited to the total number of oscillations having the same value of k.

Integrating (2.3) with allowance for the boundary condition $W_{k_g+\Delta k_g} = 0$, we obtain in the interval Δk_g a linear form of the spectrum:

$$W_k^{\mathrm{I}}=\frac{(\gamma_0-\nu_e/2)}{D_1}(k_g+\Delta k_g-k). \quad (2.4)$$

On the boundary of the interval under consideration, at $k = k_g$, we have

$$W_{k_g}=(\gamma_0-\nu_e/2)\Delta k_g/D_1$$

We note that the total power of the generation sources of the waves is connected with γ_0 and Δk_g by the formula

$$Q=\int_{k_g}^{k_g+\Delta k_g}\gamma_0W_kdk=\frac{\gamma_0(\gamma_0-\nu_e/2)}{D_1}\frac{(\Delta k_g)^2}{2}. \quad (2.5)$$

We now consider an interval in which there is no generation of waves ($k < k_g$). The solution (2.1) for $k < k_g$ in the quasistationary case, joined together with the solution in the neighboring region (2.4), will then take the form

$$W_k^{\mathrm{II}}=\frac{\gamma_0\Delta k_g}{D_1}-\frac{\nu_e}{2D_1}(k_g+\Delta k_g-k). \quad (2.6)$$

If we neglect collisions, then W_k^{II} = const. Extrapolating the solution (2.6) to the point k_*, we obtain

$$W_*\equiv W_{k_*}=\frac{\gamma_0\Delta k_g}{D_1}-\frac{\nu_e}{2D_1}(k_g+\Delta k_g-k_*).$$

This spectrum should be joined together with the spectrum obtained in[1] for $k \ll k_*$.

The solution in region III ($k_0 \ll k \ll k_*$) was obtained in[1] and has a power-law character, $W_k \sim k^{-\nu}$, where $\nu = 2.84$ for a sufficiently high level of turbulence. We note that this value of ν is obtained only if one can neglect the nonlinear scattering by the ions compared with the four-plasmon processes:

$$u > 16\frac{v_i}{v_e}(k\lambda_e)^3, \quad (2.7)$$

where $u = W^l/nT_e$. If (2.7) is not satisfied, the ν increases to $\nu = 4$. Extrapolating the power-law form of the solution up to the points k_0 and k_* and joining with the solution (2.6) at the point k_*, we obtain

$$W_k^{\mathrm{III}}=W_*\left(\frac{k_*}{k}\right)^\nu=\left[\frac{\gamma_0\Delta k_g}{D_1}-\frac{\nu_e}{2D_1}(k_g+\Delta k_g-k_*)\right]\left(\frac{k_*}{k}\right)^\nu. \quad (2.8)$$

Thus, when $k = k_0$ we have an estimate of the values W_k at the maximum of the spectrum:

$$W_{max}=W_*(k_*/k_0)^\nu$$

Going over to the region IV of very small $k < k_0$, and recognizing that W_k should vanish when $k \to 0$, we can propose the dependence of W_k on k in the form

$$W_k^{\mathrm{IV}}=(k/k_0)^sW_{max}. \quad (2.9)$$

We note that although actually the form of the spectrum when $k < k_0$ is obviously not so simple, (2.9) suffices for qualitative estimates. The obtained spectrum is shown qualitatively in Fig. 1 (curve a).

On curve a of Fig. 1, region I corresponds to generation of turbulence, region II to its transfer by the ions, region III to the asymptotic spectrum obtained in[1], and region IV to the main scale of turbulence, where there is a maximum; this maximum was quali-

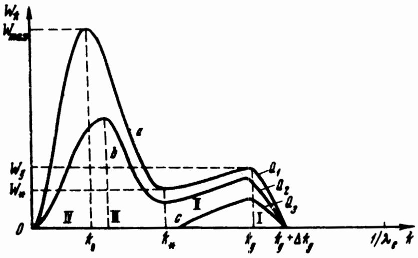

FIG. 1

tatively described in[1]. It should be noted that a spectrum of the type shown in Fig. 1 with a maximum k_0 can exist if $\nu_e k_g/2\gamma_0\Delta k_g \ll 1$ (we bear in mind that $\Delta k_g \le k_g$). This means that the power of the sources Q is so large that W_k, while decreasing with decreasing k, does not vanish in region II:

$$Q \gg \frac{\nu_e^2 k_g^2}{2D_1} = \frac{\nu_e^2 k_g^2}{2}\,\frac{27 m_i n_0 v_e^4 (1+T_e/T_i)^2}{\pi\omega_0^3}. \qquad (2.10)$$

A decrease of the source power corresponds to a deeper minimum of the spectrum at k_* (curve b in Fig. 1). In the case opposite to (2.10), we have curve c of Fig. 1. Figure 1 does not show the region of the largest $k > k_{**}$, since it is assumed that the source of turbulence is located at $k \ll k_{**}$. If the source is located at $k > k_{**}$, the spectrum of the stationary turbulence is determined by Eq. (1.6), which we transform into a differential equation by canceling out W_k, multiplying by k^α ($\alpha = -1, +1$), and differentiating with respect to k the required number of times:

$$2\frac{d^3W_k}{dk^3} + \frac{21}{k}\frac{d^2W_k}{dk^2} + \frac{46}{k^2}\frac{dW_k}{dk} + \frac{10W_k}{k^3}$$

$$= -\frac{7}{54\alpha'}\frac{d}{dk}\frac{1}{k}\frac{d}{dk}\frac{1}{k}\frac{d^2}{dk^2}\frac{1}{k}\frac{d}{dk}\frac{1}{k}\frac{d}{dk}k^3\gamma_k,$$

$$\alpha' = {}^6/_5 v_e\sqrt{2\pi}\,/\,m_e n_0\omega_0^2.$$

In the region where there is neither absorption nor generation of turbulence ($\gamma_k = 0$), the equation has three linearly-independent solutions $W_k^{(i)} = \text{const}\cdot k^{-\nu_i}$, $i = 1, 2, 3$, where $\nu_1 = {}^5/_2$ and $\nu_{2,3} = (5 \pm \sqrt{17})/2$ are the roots of the equation $2\nu^3 - 15\nu^2 + 29\nu - 10 = 0$.

The root ν_1 satisfies the condition for the locality of the energy transfer along the spectrum, namely when $2 < \nu_1 < 3$ the main contribution to both the first and the second integral terms of (1.6) is made by the region k_1 close to k. Only the solution ν_1 corresponds to a spectrum that does not depend on the source of turbulence, i.e., it justifies the assumption made above concerning the existence of a region with $\gamma_k = 0$.[2)]

[2)]For each of the $\nu_{2,3}$, one of the integrals in (1.6) diverges, i.e., strictly speaking, these solutions are not solutions of the integral equation (1.6). But the equation (1.6) itself is valid only in a limited region $k_{**} < k \ll 1/\lambda_e$. Depending on the boundary conditions in this interval, i.e., on the excitation and attenuation of the turbulence, one could also take into account the solutions $\nu_{2,3}$. If, however, the generation occurs at values of k that are close to $1/\lambda_e$, by virtue of the fact that the energy is transferred in the direction of smaller k, the divergence of the first integral (1.6) at large values of k would denote that the spectrum cannot be determined without allowance for γ_k. Therefore the solutions $\nu_{2,3}$ call for an allowance for γ_k. It is easy to obtain the solution of (1.6) with allowance for γ_k for different cases of plasma instability (in particular, two-stream instability, anisotropic instability, etc.), which, however, are not written out here.

3. MAIN TURBULENCE SCALE

The results make it possible to estimate k_0. Let us write down the energy balance equation for the waves, taking into account such processes as linear generation and absorption of waves in pair collisions. The nonlinear processes of four-plasmon decay and nonlinear scattering by ions redistribute the energy over the spectrum, and it can be readily shown that they drop out when the balance equation is set up. (Nonlinear scattering by ions leads to a small energy absorption, which is neglected here.)

The balance equation, from which we determine k_0, obviously is of the form

$$Q = \int_{k_g}^{k_g+\Delta k_g} \gamma_0 W_k\,dk = \frac{\nu_e}{2}\int_0^{k_g+\Delta k_g} W_k\,dk = \frac{\nu_e}{2}W^l. \qquad (3.1)$$

The integral on the right side of (3.1) corresponds to the rate of energy absorption in paired collisions over the extent of the entire spectrum W_k. Breaking this integral up into terms corresponding to regions I, II, III, and IV, and using the concrete formulas for W_k in the corresponding intervals, we can readily obtain an expression for the main scale of the turbulence k_0 (in the simplifications we neglect $\nu_e/2\gamma_0$, k_*/k_g, $(k_*/k_0)^{\nu-1}$, and $\nu_e k_g/2\gamma_0\Delta k_g$ compared with unity):

$$k_0 = k_*\left[\frac{\nu+s}{(s+1)(\nu-1)}\,\frac{k_*\nu_e}{\Delta k_g\gamma_0}\right]^{1/(\nu-1)}. \qquad (3.2)$$

We note that in practice a maximum of the spectrum occurs near k_0 only if (3.2) yields a ratio $(k_*/k_0) > 5$. If we specify not the generation increment γ_0 but the source power Q, then (3.2) can be rewritten in the form

$$k = k_*\left[\frac{3}{2\pi}\,\frac{(1+T_e/T_i)^2}{(s+1)^2 T_e/T_i}\left(\frac{\nu+s}{\nu-1}\right)^2\frac{\nu_e^2 m_e n_0 v_e^2}{\omega_0 Q}\right]^{1/2(\nu-1)} \qquad (3.3)$$

or in terms of the total energy of the Langmuir turbulence $W^l = 2Q/\nu_e$:

$$k = k_*\left[\frac{3}{2\pi}\,\frac{(1+T_e/T_i)^2}{(s+1)^2 T_e/T_i}\left(\frac{\nu+s}{\nu-1}\right)^2\frac{\nu_e}{\omega_0}\,\frac{n m_e v_e^2}{W^l}\right]^{1/2(\nu-1)} \sim \qquad (3.4)$$

$$\sim k_*\left(0{,}1\,\frac{\nu_e}{\omega_0}\,\frac{n m_e v_e^2}{W^l}\right)^{1/2(\nu-1)}$$

We see thus that at a sufficiently high source power, when condition (2.10) is satisfied (assuming $\nu = 2.84$), the main turbulence scale depends on Q like $k_0 \sim Q^{-0.27}$, or in the general case $k_0 \sim Q^{-1/2(\nu-1)}$, i.e., the maximum of the spectral density W_k at k_0 increases with increasing Q and shifts to the left.

The change of the form of the spectrum at several values $Q_1 > Q_2 > Q_3$ corresponding to the curves a, b, and c, can be seen from Fig. 1.

4. THE INFLUENCE OF FAST PARTICLES ON THE TURBULENCE SPECTRUM

According to[2,1], the acceleration of fast particles by Langmuir turbulence is quite effective, and therefore absorption of pulsations by the accelerated particles can become appreciable. This in turn changes the plasma turbulence spectrum. The influence of fast particles appears only when $k > k_c = \omega_0/c$. Therefore the value of the ratio k_c/k_* is important. The region $k_c < k < k_*$ exists only in a cold plasma, and the influence of the fast particles in this region was investigated in[1]. We consider here the influence of fast particles on the spectrum in the region $k > k_*$, both in a hot and in a cold plasma. If $k_c < k_*$, then the fast particles play an important role in the entire region $k > k_*$.

At this point, we shall stop to consider the effects of just this interaction, between the fast particles and the Langmuir turbulence. When account is taken of the absorption of the waves by the fast particles, Eq. (2.1) for W_k assumes in this region the form

$$\frac{\partial W_k}{\partial t} = D_1 W_k \frac{\partial W_k}{\partial k} - \frac{\pi}{\sqrt{2}} \frac{\omega_0^3 m_e}{n_0 k^2} \sum_\alpha Z_\alpha^2 f^\alpha\left(\frac{m_\alpha \omega_0^2}{2k^2}\right) W_k - \frac{\nu_e}{2} W_k, \tag{4.1}$$

where α = e or i: m_α is the mass of the fast particle of type α, and Z_α is its charge. In the quasistationary case we have

$$\frac{\partial W_k}{\partial k} = \frac{\pi}{\sqrt{2}} \frac{\omega_0^3 m_e}{n_0 k^2} \frac{1}{D_1} \sum_\alpha Z_\alpha^2 f^\alpha\left(\frac{m_\alpha \omega_0^2}{2k^2}\right) + \frac{\nu_e}{2}. \tag{4.2}$$

We can see from (4.2) that the influence of fast particles reduces to the fact that W_k decreases with decreasing k. To find W_k it is necessary to solve in general the self-consistent problem, writing down the equation for the distribution function in the quasistationary case

$$\frac{\partial f^\alpha}{\partial t} = 0 = \frac{\partial}{\partial \varepsilon} D_2 \frac{\partial}{\partial \varepsilon} \frac{f^\alpha}{\sqrt{\varepsilon}} + \frac{\partial}{\partial \varepsilon} \nu_e m_e v_e^3 \sqrt{m_\alpha} \frac{f^\alpha}{\sqrt{\varepsilon}}, \tag{4.3}$$

$$D_2 = \frac{\pi}{2\sqrt{2}} \frac{Z_\alpha^2 \omega_0^4 m_\alpha \sqrt{m_\alpha}}{n_0} \int\limits_{\omega_0 \sqrt{m_\alpha / 2\varepsilon}} W_{k_1} k_1^{-3} dk_1,$$

and solving simultaneously (4.2) and (4.3). In the quasistationary state we obtain from (4.3)

$$D_2 \frac{\partial}{\partial \varepsilon} \frac{f^\alpha}{\sqrt{\varepsilon}} + \nu_e m_\alpha v_e^3 \sqrt{m_\alpha} \frac{f^\alpha}{\sqrt{\varepsilon}} = \text{const}. \tag{4.4}$$

Changing over from the variable k to the variable $\epsilon = m_\alpha \omega_0^2 / 2k^2$, and neglecting the last term of (4.2) (assuming absorption by the fast particles to be more appreciable than absorption in collisions), we obtain (in the case of one type of fast particles)

$$\frac{\partial W(\varepsilon)}{\partial \varepsilon} = -\mu_\alpha \frac{f^\alpha(\varepsilon)}{\sqrt{\varepsilon}},$$

$$\left(\int^{\varepsilon} W(\varepsilon)\, d\varepsilon\right) \frac{\partial}{\partial \varepsilon} \frac{f^\alpha(\varepsilon)}{\sqrt{\varepsilon}} + \eta \frac{f^\alpha(\varepsilon)}{\sqrt{\varepsilon}} = \text{const}, \tag{4.5}$$

where

$$\mu_\alpha = \frac{27 m_e m_i Z_\alpha^2 v_e^4 (1 + T_e/T_i)^2}{2\sqrt{m_\alpha}\, \omega_0}, \quad \eta = \frac{2\sqrt{2} n_0 m_\alpha \nu_e v_e^3}{Z_\alpha^2 \omega_0^2}, \quad \nu_e = \frac{\omega_0 \ln \Lambda}{n_0 \lambda_e^3}.$$

If it is recognized that the turbulence spectrum vanishes when $k > k_g$, it is easily seen that the constant in (4.5) must be set equal to zero. Thus, this system reduces to the equation

$$\left(\int^{\varepsilon} W(\varepsilon)\, d\varepsilon\right) \frac{\partial^2 W(\varepsilon)}{\partial \varepsilon^2} + \eta \frac{\partial W(\varepsilon)}{\partial \varepsilon} = 0. \tag{4.6}$$

Without solving (4.6) in the general case, let us analyze the effect of the interaction of the fast particles with a developed Langmuir turbulence in two limiting cases.

We assume first that the absorption of the Langmuir waves by fast particles changes the Langmuir wave spectrum little. We can then seek the solution of (4.6) in the form

$$W(\varepsilon) = W_g + \Delta(\varepsilon), \quad \Delta(\varepsilon) \ll W_g.$$

From (4.6) we readily obtain an equation for $\Delta(\epsilon)$:

$$\varepsilon \Delta''(\varepsilon) + \beta \Delta'(\varepsilon) = 0,$$
$$\beta = \eta / W_g = 2\sqrt{2} \ln \Lambda \cdot m_\alpha \omega_0^2 / \pi Z_\alpha^2 W_g. \tag{4.7}$$

One of the two integration constants of the last equation is obtained from the boundary condition $\Delta = 0$ at $k = k_g$, and the other can be readily obtained if it is recognized that

$$n_1 = \int\limits_{\varepsilon_g}^{\varepsilon_0} f(\varepsilon)\, d\varepsilon, \tag{4.8}$$

where n_1 is the total number of fast particles with energy ϵ, with

$$\frac{m_\alpha \omega_0^2}{2k_g^2} < \varepsilon < \varepsilon_0 = \frac{m_\alpha c^2}{2}$$

per unit volume, and also that

$$\Delta'(\varepsilon) = \frac{\partial W(\varepsilon)}{\partial \varepsilon} = -\mu_\alpha f(\varepsilon)/\sqrt{\varepsilon} = C_1 \varepsilon^{-\beta}(1 - \beta).$$

From this we get

$$W_k = W_g + \frac{n_1 \mu_\alpha}{\sqrt{\varepsilon_0}} \left(\frac{3}{2} - \beta\right)(1 - \beta)^{-1} \left[\left(\frac{k_c}{k_g}\right)^{2(1-\beta)} - \left(\frac{k_c}{k}\right)^{2(1-\beta)}\right] \tag{4.9}$$

$$f(\varepsilon) = n_1 \varepsilon_0^{-1} \left(\frac{3}{2} - \beta\right) \left(\frac{\varepsilon_0}{\varepsilon}\right)^{\beta - 1/2}. \tag{4.10}$$

It is easily seen from (4.9) and (4.10) that the assumption $\Delta \ll W_g$ is satisfied when $\beta = \eta / W_g < 1$ for $n_1 \mu_\alpha / \sqrt{\epsilon_0} \ll W_g$. In other words, for a sufficiently large spectral energy density of the Langmuir waves

$$\frac{k_c W_g}{n_0 T_e} > \frac{2\sqrt{2}}{\pi Z_\alpha^2} \frac{m_\alpha}{m_e} \frac{v_e}{c} \frac{\nu_e}{\omega_0}, \tag{4.11}$$

the solutions (4.9) and (4.10) are valid for n_1 such that

$$n_1 \ll W_g \frac{\sqrt{\varepsilon_0}}{\mu_\alpha} = n_0 \frac{W_g k_c}{n_0 T_e} \left(\frac{c}{v_e}\right)^2 \frac{m_\alpha}{m_i} \frac{\sqrt{2}}{27 (1 + T_e/T_i)^2} \tag{4.12}$$

We note that when $\beta \ll 1$, the fast-particle distribution function takes on the form

$$f(\varepsilon) \sim \sqrt{\varepsilon}. \tag{4.13}$$

Assume now that the absorption of the Langmuir waves greatly changes the spectral energy density W_k in the interval II (Fig. 2), and the spectral energy density of the Langmuir waves is sufficiently high

$$\frac{1}{\varepsilon} \int^{\varepsilon} W(\varepsilon)\, d\varepsilon \gg \eta. \tag{4.14}$$

The last condition is similar to $\beta \ll 1$, and for estimates it can be replaced by (4.12), bearing in mind in

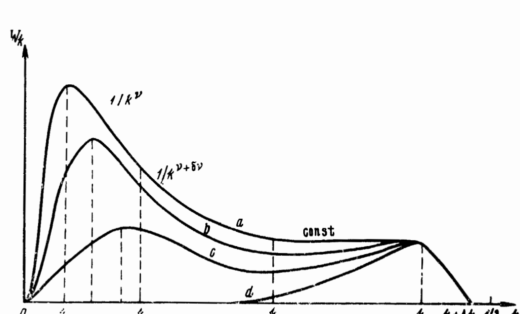

FIG. 2

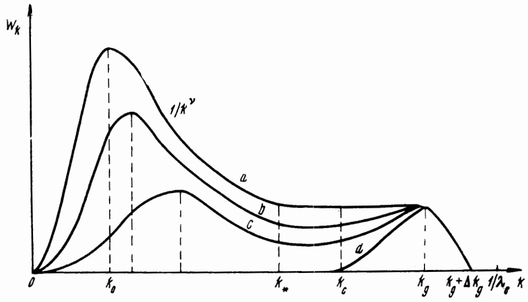

FIG. 3

place of W_g a certain average spectral energy density in the interval II.

Under the condition (4.14), we get

$$f(\varepsilon) \sim \sqrt{\varepsilon}, \quad W(\varepsilon) = C_2 - C_1\mu_\alpha\varepsilon. \tag{4.15}$$

On the boundary of region II we have $W_g = C_2 - \mu_\alpha C_1 \epsilon_g$, from which we get

$$W(\varepsilon) = W_g + C_1\mu_\alpha(\varepsilon_g - \varepsilon),$$

and the constant C_1 is determined from the normalization condition

$$n_1 = C_1 \int_{\varepsilon_g}^{\varepsilon_1} \sqrt{\varepsilon} d\varepsilon \approx \frac{2}{3} C_1 \varepsilon_0$$

(if we neglect ϵ_g compared with ϵ_0). As a result we obtain

$$W(k) = W_g + \frac{{}^3/_2 n_1\mu_\alpha}{\varepsilon_0^{3/2}} \frac{m_\alpha\omega_0^2}{2}\left(\frac{1}{k_g^2} - \frac{1}{k^2}\right)$$

It is easily seen that when $W_g \leq {}^3/_2 n_1\mu_\alpha/\sqrt{\epsilon_0}$, i.e., when

$$Q < \frac{3^5\pi}{16}\left(\frac{m_e}{m_\alpha}\right)^2\left(\frac{n_1}{n_0}\right)^2\left(\frac{v_e}{c}\right)^2 m_i n_0 v_e^2 \omega_0 Z_\alpha^4 (1 + T_e/T_i)^2 \tag{4.17}$$

the spectral energy density W_k has no maximum at $k < k_*$, owing to the absorption of the Langmuir waves by the fast particles.

Figure 2 shows the spectra of the Langmuir turbulence W_k of a cold plasma ($k_c < k_*$), in accord with the foregoing. Curve a corresponds to the absence of fast particles, and curves b and c correspond to an increasing number of particles at the same turbulence generation power Q. The fast particles, according to[1], have a power-law turbulence spectrum in the region $k_c < k < k_*$ (ν ranges from $\nu = 2.84$ to $\nu = 4$).

For a hot plasma, $k_c > k_*$, effects of the influence of the fast particles on the turbulence, analogous to those described above, take place only when $k > k_c$. When $k_* < k < k_c$, we get the results of Sec. 2, and in the absence of collisions W_k is constant. When $k < k_*$, the spectrum coincides with that obtained in[1], where in the case of intense turbulence $\nu = 2.84$ and the fast particles do not influence the spectrum (ν can be a function of Q only under conditions of not very strong turbulence, when the induced scattering is comparable with the four-plasmon scattering). Qualitatively, the spectra of the hot plasma are shown in Fig. 3. Curve a corresponds to the absence of fast particles, and curves b and c correspond to an increasing number of such particles.

If the number of fast particles is large, the spectrum has no maximum (curves d of Figs. 2 and 3). An increase in the intensity of the fast particles at such a spectrum has little effect on its intensity, since a small increase of the slope (an increase of the fall-off rate) of the spectrum strongly decreases the effects of acceleration and the gathering of energy by the fast particles. This is connected with the decrease of the acceleration like v_{ph}^3 ($v_{ph} = \omega_0/k$); such a process of self-regulation of the acceleration can establish an equipartition of energy between the fast particles and the turbulence.

Under astrophysical conditions, frequently even a small turbulence level ($W^l/nT_e \sim 10^{-6}$) effectively accelerates the electrons and ions of low energies (ions up to 30 MeV). The distribution of the fast particles (subcosmic rays) in this region should correspond to $f(\epsilon) \sim \sqrt{\epsilon}$, in accordance with formula (4.15). The absence of low-energy subcosmic rays agrees with the results of an analysis[6] based on ultraviolet radiation (curve b of Fig. 3).

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