

THE THEORY OF NONIDEAL BOSE GASES AND SIZE EFFECTS IN  
SUPERFLUID FILMS

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A theoretical investigation is carried out of the dependence of the Bose-condensation temperature of a nonideal Bose gas on the linear dimensions of the region occupied by the gas (dimensional effects). Two versions of the theory of the nonideal Bose gas are considered: a theory assuming the existence of nonvanishing quasi-expectations  $\langle a_0 \rangle$  and  $\langle a_0^+ \rangle$  where  $a_0$  and  $a_0^+$  are creation-annihilation operators of particles in the ground state, and a theory which makes use of a pair Hamiltonian, and in which the spectrum of one-particle excitations has a gap for small momenta  $p \rightarrow 0$ . The first theory leads to anomalous size effects, whereas the second one leads to normal size effects. For a normal size effect the dependence of the Bose-condensation temperature depends on the linear dimensions for arbitrarily large values of the smallest linear dimension.

THE investigation of the nonideal Bose gas carried out in [1-4] has shown that the particle density on the ground level depends on the relations among the linear dimensions of the region occupied by the gas, even in the thermodynamic limit. The thermodynamic limit means an indefinite increase in the particle number  $N$  and the volume  $V$  of the system, the density  $n = N/V$  remaining constant, such that the area of the boundary surface  $S$  increases slower than the volume, i.e.,  $SV \rightarrow 0$ .

The dependence of the particle number density in the ground state (particle density in the condensate) on the ratios of the linear dimensions means essentially that the number of particles in the condensate cannot be considered as an additive extensive thermodynamic function.

Since the existence of a macroscopic particle number in the ground state is usually related to the presence of superfluid properties, one should expect in this case a dependence of the temperature of superfluid transition on the ratios of the linear dimensions of the system. Experiments have repeatedly shown a dependence of the superfluid transition temperature on the thickness of thin helium films, [5-8] however, no correlations of this temperature and the ratio of the thickness and the other dimensions were observed. We shall call the dependence of the temperature of the superfluid transition on the absolute value of the smallest dimension involved, the normal size effect, and the dependence of this temperature on the ratio between the linear dimensions—the anomalous size effect. Thus, if the theory of the ideal Bose gas predicts the existence of anomalous effects, the experiment has so far exhibited only normal size effects.

In this connection it seems interesting to investigate the size effects in a nonideal Bose gas, which is the purpose of the present paper. Two presently existing versions of the theory of a weakly nonideal Bose gas have been considered: the theory with distinguished condensate, [9-11] in which it is assumed that anomalous expectation values like  $\langle a_0 \rangle$  and  $\langle a_0^+ \rangle$  exist, with  $a_0$  and

$a_0^+$  the annihilation and creation operators of particles in the ground state, and a theory which uses the so-called "pair Hamiltonian" [12-14] (as in [12] we shall abbreviate the name to "pair theory"). It turned out that only the first theory predicts anomalous dimensional effects. Thus, an investigation of dimensional effects might help in distinguishing the two theories. The experimental absence of anomalous effects favors the pair theory. We also note that the free energy in the pair theory turns out to be smaller than in a theory with distinguished condensate.

### 1. SIZE EFFECTS IN THE THEORY WITH DISTINGUISHED CONDENSATE

The existence of the nonvanishing anomalous expectation values, or so-called quasi-expectation values  $\langle a_0 \rangle$  and  $\langle a_0^+ \rangle$  in a theory with distinguished condensate is related to a violation of the symmetry of the system with respect to gauge transformations, symmetry which is exhibited by the initial Hamiltonian of the system. [15]

The fact that  $\langle a_0 \rangle$  and  $\langle a_0^+ \rangle$  do not vanish allows one to derive from the general properties of the grand canonical ensemble a fairly strong inequality for the average occupation numbers of one-particle levels with  $p \neq 0$ , witnessing the presence of a singularity  $1/p^2$  in the momentum distribution of the particles. This is the content of the so-called  $1/p^2$ -theorem proposed by Bogolyubov. [15] The indicated inequality can be written according to Hohenberg [16] in the form

$$N(p) \geq -\frac{1}{2} + \frac{Tm N_0}{p^2 N}, \quad (1)$$

where  $N(p)$  is the average particle number of the level with momentum  $p$ ,  $m$  is the particle mass,  $T$  is the temperature, and  $N_0 = N(0)$ .

We show below that the inequality (1) implies the existence of anomalous size effects. [17]

We consider a parallelepiped with edges  $L_x \geq L_y \geq L_z$  and periodic boundary conditions, so that the values of the momentum components are determined by the

three quantum numbers  $k, l, s$ :

$$p_x = k \frac{h}{L_x}, \quad p_y = l \frac{h}{L_y}, \quad p_z = s \frac{h}{L_z}. \quad (2)$$

As in [41], we split the total particle number  $N$  into four components:

$$\begin{aligned} N &= N_0 + N_1 + N_2 + N_3, \\ N_0 &= N(0), \quad N_1 = \sum_{h=-\infty}^{\infty} N(\mathbf{p}), \\ N_2 &= \sum_{l=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} N(\mathbf{p}), \quad N_3 = \sum_{s=-\infty}^{\infty} \sum_{l, h=-\infty}^{\infty} N(\mathbf{p}). \end{aligned} \quad (3)$$

The primes on the summations in (3) indicate that the value zero of the summation index is to be omitted. The quantities  $N_0, N_1, N_2,$  and  $N_3$  determine the particle numbers in the condensate and in the one-dimensional, two-dimensional, and three-dimensional phases, respectively.

By means of the inequality (1) one can estimate from below the quantities  $N_1$  and  $N_2$ . For this estimation we replace  $N(\mathbf{p})$  in (3) by the right-hand side of (1), and compute the sum for those values of  $k$  and  $l$  for which the right-hand side of (1) is positive. It is easy to perform the one-dimensional summation for  $N_1$ , replacing the finite limits by infinite ones. For the computation of the two-dimensional sum  $N_2$  we replace the sum by a two-dimensional integral, eliminating the region of small momenta  $p < h/L_y$ , i.e., those which are smaller than the smallest momentum  $h/L_y$  occurring in the sum. It can be shown that this method of calculating the sums leads to a negligible relative error in the thermodynamic limit, i.e., for large  $L_x/\lambda, L_y/\lambda,$  and  $L_z/\lambda$ , where  $\lambda$  is the thermal wavelength

$$\lambda^2 = h^2 / 2\pi mT. \quad (4)$$

As a result we obtain the following estimates from below for the quantities  $N_1$  and  $N_2$ :

$$N_1 > \frac{\pi}{6} \frac{N_0}{N} \frac{L_x^2}{\lambda^2}, \quad (5)$$

$$N_2 > \frac{N_0}{N} \frac{L_x L_y}{\lambda^2} \ln \left( \sqrt{\frac{N_0}{\pi N}} \frac{L_x}{\lambda} \right). \quad (6)$$

If anomalous size effects are absent, then the quantity  $N_0/N$  should be a constant in the thermodynamic limit. However, this assumption leads to a contradiction for a limiting procedure in which  $\ln L_y$  increases faster than  $L_z$ . In that case it follows from (6) that for a sufficiently large ratio  $(\lambda/L_z) \ln(L_y/\lambda)$  the number of particles  $N_2$  in the two-dimensional phase becomes larger than the total number of particles  $N$ , which is proportional to the volume  $V = L_x L_y L_z$ . In the same manner the number  $N_1$  of particles in the one-dimensional phase  $N_1$  becomes larger than  $N$  if  $L_x$  increases faster than  $L_y L_z$ . If in the limit the quantities  $L_x \lambda / L_y L_z$  and  $(\lambda/L_z) \ln(L_y/\lambda)$  tends to zero, as happens for instance for a cube ( $L_x = L_y = L_z$ ), then the inequalities (5) and (6) allow for finite values of  $N_0/N$ , i.e., for the presence of Bose-condensation. Previously Hohenberg has made use of the  $1/p^2$ -theorem to prove the impossibility of Bose condensation in one- and two-dimensional systems.<sup>[16]</sup> From the above it follows that this theorem forbids Bose condensation also for three-dimensional systems with sufficiently large values of

the quantities  $L_x \lambda / L_y L_z$  and  $(\lambda/L_z) \ln(L_y/\lambda)$ . Thus, both in the ideal and nonideal Bose-gas the values of  $N_0/N$  and of the temperature of Bose condensation into the ground state depend on  $L_x \lambda / L_y L_z$  and  $(\lambda/L_z) \ln(L_y/\lambda)$ , i.e., anomalous size effects occur. This conclusion remains in force for arbitrarily strong interactions between bosons in a homogeneous system, as long as the quasi-expectation values  $\langle a_0 \rangle$  and  $\langle a_0^+ \rangle$  exist. However, in the interacting Bose gas not only  $N_0/N$  but also the free energy density depends on the quantities  $L_x \lambda / L_y L_z$  and  $(\lambda/L_z) \ln(L_y/\lambda)$ , since the free energy density depends on  $N_0/N$ . Therefore the violation of additivity occurs not only for  $N_0$ , but also for the free energy, in distinction from the ideal Bose gas, where the free energy does not depend on  $N_0$ .<sup>[41]</sup>

## 2. SIZE EFFECTS AND THE PAIR THEORY

In the pair theory the initial Hamiltonian of the interacting bosons is replaced by a pair Hamiltonian, in which some terms of the initial Hamiltonian are absent:

$$\begin{aligned} H_P &= \sum_{\mathbf{k}} \frac{\hbar^2}{2m} a_{\mathbf{k}}^+ a_{\mathbf{k}} + \frac{1}{2V} \left\{ \sum_{\mathbf{k}, \mathbf{p}} v(\mathbf{p}) a_{\mathbf{k}}^+ a_{\mathbf{p}}^+ a_{\mathbf{p}} a_{\mathbf{k}} \right. \\ &\quad \left. \times \sum_{\mathbf{k} \neq \mathbf{p}} v(\mathbf{p} - \mathbf{k}) (a_{\mathbf{k}}^+ a_{\mathbf{p}}^+ a_{\mathbf{p}} a_{\mathbf{k}} + a_{\mathbf{k}}^+ a_{-\mathbf{k}}^+ a_{\mathbf{p}} a_{-\mathbf{p}}) \right\}, \end{aligned} \quad (7)$$

where  $v(\mathbf{p})$  is the Fourier transform of the interaction potential. The pair Hamiltonian differs from the Hamiltonian of the Hartree-Fock approximation by the presence of the terms  $a_{\mathbf{k}}^+ a_{-\mathbf{k}}^+ a_{\mathbf{p}} a_{-\mathbf{p}}$ , corresponding to the creation of a pair of bosons with equal and oppositely directed momenta. Further, we carry out the transformation

$$\begin{aligned} a_{\mathbf{k}}^+ a_{-\mathbf{k}}^+ &= N_{\mathbf{k}} + B_{\mathbf{k}1}, \\ a_{\mathbf{k}} a_{-\mathbf{k}} &= \eta_{\mathbf{k}} + B_{\mathbf{k}2}, \end{aligned} \quad (8)$$

where  $B_{\mathbf{k}1}$  and  $B_{\mathbf{k}2}$  are operators and  $N_{\mathbf{k}}$  and  $\eta_{\mathbf{k}}$  are c-numbers. If in the pair Hamiltonian one discards terms quadratic in  $B_{\mathbf{k}1}$  and  $B_{\mathbf{k}2}$ , the remaining Hamiltonian can be diagonalized by means of a Bogolyubov transformation. Further, the c-numbers  $N_{\mathbf{k}}$  and  $\eta_{\mathbf{k}}$  are selected by minimizing the thermodynamic potential of the grand canonical ensemble for a system with the Hamiltonian  $H_0$ .

Wentzel<sup>[13]</sup> has shown that the thermodynamic functions obtained in this manner are exact for a system described by the pair Hamiltonian  $H_P$  in the thermodynamic limit. The extremum of the (Gibbs) thermodynamic potential occurs under the following conditions:<sup>[14]</sup>

$$N_{\mathbf{k}} = \langle a_{\mathbf{k}}^+ a_{\mathbf{k}} \rangle = \frac{1}{2} \frac{f_{\mathbf{k}}}{\varepsilon_{\mathbf{k}}} \operatorname{cth} \frac{\varepsilon_{\mathbf{k}}}{2T} - \frac{1}{2}, \quad (9)$$

$$\eta_{\mathbf{k}} = \langle a_{\mathbf{k}}^+ a_{-\mathbf{k}}^+ \rangle = \langle a_{\mathbf{k}} a_{-\mathbf{k}} \rangle = -\frac{1}{2} \frac{\hbar_{\mathbf{k}}}{\varepsilon_{\mathbf{k}}} \operatorname{cth} \frac{\varepsilon_{\mathbf{k}}}{2T}, \quad (10)$$

$$\varepsilon_{\mathbf{k}} = \sqrt{f_{\mathbf{k}}^2 - \hbar_{\mathbf{k}}^2}, \quad (11)$$

$$f_{\mathbf{k}} = \frac{\hbar^2}{2m} \mathbf{k}^2 - \mu + \frac{v(0)}{V} \left( \sum_{\mathbf{p}} N_{\mathbf{p}} - \frac{1}{2} \right) + \frac{1}{V} \sum_{\mathbf{p}(\neq \mathbf{k})} v(\mathbf{k} - \mathbf{p}) N_{\mathbf{p}}, \quad (12)$$

$$\hbar_{\mathbf{k}} = \frac{1}{V} \sum_{\mathbf{p}(\neq \mathbf{k})} v(\mathbf{p} - \mathbf{k}) \eta_{\mathbf{p}}, \quad (13)$$

where  $\mu$  is the chemical potential.

We list the expressions of the total particle number

$N$ , of the energy  $E$ , and of the (Gibbs) thermodynamic potential  $\Omega$ :

$$N = \sum_{\mathbf{k}} N_{\mathbf{k}} = \sum_{\mathbf{k}} \left( \frac{1}{2} \frac{f_{\mathbf{k}}}{\epsilon_{\mathbf{k}}} \operatorname{cth} \frac{\epsilon_{\mathbf{k}}}{2T} - \frac{1}{2} \right) \quad (14)$$

$$E = U_0 + \sum_{\mathbf{k}} \frac{\epsilon_{\mathbf{k}}}{\exp(\epsilon_{\mathbf{k}}/T)}, \quad (15)$$

$$\Omega = -PV = U_0 + T \sum_{\mathbf{k}} \ln[1 - \exp(-\epsilon_{\mathbf{k}}/T)], \quad (16)$$

where

$$U_0 = -\frac{1}{2V} \sum_{\mathbf{k}, \mathbf{p}} \{ [v(0) + v(\mathbf{p}-\mathbf{k})] N_{\mathbf{k}} N_{\mathbf{p}} + v(\mathbf{p}-\mathbf{k}) \eta_{\mathbf{k}} \eta_{\mathbf{p}} \} \quad (17)$$

$$\times \frac{1}{2} \sum_{\mathbf{k}} (\epsilon_{\mathbf{k}} - f_{\mathbf{k}}),$$

$\epsilon_{\mathbf{p}}$  represents the quasiparticle energy, counted from the level of the chemical potential, and  $N_{\mathbf{k}}$  is the average number of particles on the level with momentum  $\mathbf{k}$ . In the pair theory there is also a breakdown of the gauge invariance, since the quasi-expectation values  $\eta_{\mathbf{k}} = \langle a_{\mathbf{k}} a_{-\mathbf{k}} \rangle = \langle a_{\mathbf{k}}^{\dagger} a_{-\mathbf{k}}^{\dagger} \rangle$  may be different from zero.

In order to consider the general case of a parallelepiped with unequal edges  $L_x \geq L_y \geq L_z$  it is necessary to decompose all three-dimensional sums over  $\mathbf{k}$  into four parts, as was done in Sec. 1 for the expression of  $N$  and only after that to replace in each part the summation by an integral. The four parts correspond: 1) to the condensate (the term of the original sum with  $\mathbf{k} = 0$ ), 2) to the one-dimensional phase (the line integral along the axis  $k_y = k_z = 0$ ), 3) to the two-dimensional phase (two-dimensional integral over the plane  $k_z = 0$ ), 4) to the three-dimensional phase (three-dimensional integral). In <sup>[12,14]</sup> a cubical region was considered, therefore the sums over  $\mathbf{k}$  were decomposed only into two parts: the condensate and the three-dimensional phase. Below we shall restrict our attention to the case of a film ( $L_x = L_y > L_z$ ), and therefore the sums over  $\mathbf{k}$  will be split into three parts: 1) the condensate, 2) the two-dimensional phase, 3) the three-dimensional phase.

Further, we assume that the interaction potential is a delta-function, i.e.,

$$v(\mathbf{p}) = v. \quad (18)$$

For this situation the solution of Eqs. (9)–(13) undergoes a considerable simplification. Owing to their nonlinearity, Eqs. (9)–(13) have several solutions. The first solution corresponds to the absence of any correlations ( $\eta_{\mathbf{k}} = \mathbf{h}_{\mathbf{k}} = 0$ ) and the absence of a condensed phase, (i.e., the condensate density  $n_0 = N_0/V$  decreases with the increase of the volume of the parallelepiped and for  $V \rightarrow \infty$ ,  $n_0 \rightarrow 0$ ). The vanishing of correlations makes the pair theory completely equivalent to the Hartree-Fock theory. The quasiparticle energy

$$\epsilon_{\mathbf{k}} = f_{\mathbf{k}} = k^2/2m - \mu + 2nv \quad (19)$$

differs from the energy of an ideal Bose gas by a shift by the amount of the interaction energy, which in the approximation used is independent of  $\mathbf{k}$ . Therefore the solution without condensate leads to the same distribution of particles among the two- and three-dimensional phases as the ideal Bose gas:<sup>[4]</sup>

$$n = n_2 + n_3 = \frac{1}{\lambda^2 L_z} g_1(z_0) + \frac{1}{\lambda^3} g^{3/2}(z_0), \quad (20)$$

where

$$g_s(x) = \frac{x}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} dt}{e^t - x},$$

$$g_1(x) = -\ln(1-x),$$

and  $\lambda = (h/2\pi mT)^{1/2}$  is the thermal wavelength. The parameter  $z_0$ , which in the canonical ensemble depends on the temperature according to (20), is related to the chemical potential by means of the relation

$$z_0 = \exp\{(\mu - 2nv)/T\}. \quad (21)$$

The magnitude of the free energy  $F_0$  differs from the free energy of the ideal gas by a temperature independent constant:

$$\frac{F_0}{V} = \mu n - \frac{\Omega}{V} = nv - T \frac{1}{L_z \lambda^2} [g_2(z_0) - g_1(z_0) \ln z_0] - \frac{T}{\lambda^3} [g^{3/2}(z_0) - g^{1/2}(z_0) \ln z_0]. \quad (22)$$

If one removes from (20) and (22) the terms which are inversely proportional to  $L_z$ , one obtains the equations for a cubical region  $L_x = L_y = L_z$ , in the absence of the condensate. In this case the equation (20) has a solution for  $z_0$  only at temperatures above the Bose-condensation temperature of an ideal three-dimensional Bose gas,  $T_3$ , temperature which is defined by the relation

$$n\lambda_3^3 = g^{3/2}(1) = 2.612, \quad (23)$$

where  $\lambda_3$  is the thermal wavelength for the temperature  $T_3$ .

In the case under consideration here  $L_x = L_y > L_z$  the condensate-free solution exists up to the temperature  $T_2$ , determined according to <sup>[4]</sup> from the relation

$$1.306 \frac{T_3}{T_2} \left( 1 - \left( \frac{T_2}{T_3} \right)^{3/2} \right) = \frac{\lambda_2}{L_z} \ln \frac{L_y}{\lambda_2}. \quad (24)$$

If for fixed  $L_z$  one lets  $L_x = L_y$  go to infinity, then  $T_2 \rightarrow 0$  and the solution without condensate exists for all nonzero temperatures.

We now consider the solutions of the equations (9)–(13) which yield a finite particle density in the condensate,  $n_0 = N_0/V$ . A necessary condition for this is that  $\epsilon_0 = 0$ .

In the presence of the condensate  $n_0 \neq 0$  there exists a solution without correlations, i.e.,  $\eta_{\mathbf{k}} = \mathbf{h}_{\mathbf{k}} = 0$  and  $f_0 = h_0 = 0$ , which is equivalent to the usual Hartree-Fock approximation. In this case in place of (19)–(22) there will be the following relations:

$$n = n_0 + n_2 + n_3 = n_0 + \frac{1}{\lambda^2 L_z} g_1(z_H) + \frac{1}{\lambda^3} g^{3/2}(z_H), \quad (25)$$

$$z_H = \exp\left(\frac{\mu - 2nv}{T}\right) = \exp\left(-\frac{n_0 v}{T}\right), \quad (26)$$

$$\frac{F_H}{V} = \left(n^2 - \frac{n_0^2}{2}\right) v - \frac{T}{\lambda^2 L_z} [g_2(z_H) - g_1(z_H) \ln z_H] - \frac{T}{\lambda^3} [g^{3/2}(z_H) - g^{1/2}(z_H) \ln z_H], \quad (27)$$

$$\epsilon_0 = f_0 = 0, \quad \epsilon_{\mathbf{k}} = f_{\mathbf{k}} = \frac{k^2}{2m} - \mu + 2nv = \frac{k^2}{2m} + n_0 v, \quad \mathbf{k} \neq 0. \quad (28)$$

In order to find the dependence of the free energy and chemical potential on the temperature, it is necessary to find from Eqs. (25) and (26) the dependence of  $z_H$  on  $T$ . From (28) it follows that the quasiparticle spectrum has a gap at  $\mathbf{k} \rightarrow 0$ . This gap is related to the fact that there is exchange interaction only for bosons

on different quantum levels. Therefore, for the particles in the condensate the gap is smaller by the amount of the exchange interaction with the particles in the condensate,  $n_0 v$ , than for particles on levels with small but nonvanishing  $\mathbf{k}$ .

Finally, we consider the solution of Eqs. (9)–(13) in the presence of the condensate  $n_0 \neq 0$  and nonvanishing correlations  $\eta_{\mathbf{k}} \neq 0$ . As for the case of a cubical region, considered by Luban,<sup>[14]</sup> the relations obtained there hold:

$$f_0 = -\hbar_0, \quad \frac{\eta_0}{V} = n_0 \quad (29)$$

Let us consider the integral equations obtained from (9)–(13) for  $f_{\mathbf{k}}$  and  $h_{\mathbf{k}}$  which differ from Luban's equations<sup>[14]</sup> in that they contain double integrals over the plane  $k_z = 0$ , owing to the existence of the two-dimensional phase:

$$\begin{aligned} f_0 &= -\mu + nv(0) + F_2(0) + F_3(0), \\ h_0 &= -H_2(0) - H_3(0); \end{aligned} \quad (30)$$

for  $\mathbf{k} \neq 0$

$$\begin{aligned} f_{\mathbf{k}} &= \frac{k^2}{2m} - \mu + nv(0) + n_0 v(\mathbf{k}) + F_2(\mathbf{k}) + F_3(\mathbf{k}), \\ h_{\mathbf{k}} &= n_0 v(\mathbf{k}) - H_2(\mathbf{k}) - H_3(\mathbf{k}), \end{aligned} \quad (31)$$

where

$$\begin{aligned} F_2(\mathbf{k}) &= \frac{1}{2\hbar^2 L_z} \iint v(\mathbf{k}-\mathbf{p}) \left( \frac{f_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}} \operatorname{cth} \frac{\varepsilon_{\mathbf{p}}}{2T} - 1 \right) dp_x dp_y, \\ F_3(\mathbf{k}) &= \frac{1}{2\hbar^3} \iiint v(\mathbf{k}-\mathbf{p}) \left( \frac{f_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}} \operatorname{cth} \frac{\varepsilon_{\mathbf{p}}}{2T} - 1 \right) dp_x dp_y dp_z, \\ H_2(\mathbf{k}) &= \frac{1}{2\hbar^2 L_z} \iint v(\mathbf{k}-\mathbf{p}) \frac{h_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}} \operatorname{cth} \frac{\varepsilon_{\mathbf{p}}}{2T} dp_x dp_y, \\ H_3(\mathbf{k}) &= \frac{1}{2\hbar^3} \iiint v(\mathbf{k}-\mathbf{p}) \frac{h_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}} \operatorname{cth} \frac{\varepsilon_{\mathbf{p}}}{2T} dp_x dp_y dp_z. \end{aligned} \quad (32)$$

From (29) and (30) we obtain an expression for the chemical potential

$$\mu = nv(0) + F_2(0) + F_3(0) - H_2(0) - H_3(0). \quad (33)$$

However, substitution of the constant  $v$  in place of  $v(\mathbf{k})$  in  $H_2(\mathbf{k})$  and  $H_3(\mathbf{k})$  leads to divergent integrals for finite values of  $h_{\mathbf{k}}$ . We therefore assume that  $v(\mathbf{p}) = v$  for  $p < p_m$  and  $v(\mathbf{p}) = 0$  for  $p > p_m$ , and then go to the limit as  $p_m \rightarrow \infty$ . As a result we obtain

$$h_{\mathbf{k}} \rightarrow 0, \quad H_2(\mathbf{k}) + H_3(\mathbf{k}) \rightarrow n_0 v. \quad (34)$$

Taking (34) into account, the integral equations become algebraic equations. The remaining computations lead to relations which are structurally analogous to the expressions (25)–(28), obtained in the Hartree–Fock approximation:

$$n = n_0 + n_2 + n_3 = n_0 + \frac{1}{\lambda^2 L_z} g_1(z_P) + \frac{1}{\lambda^3} g'_{1/2}(z_P), \quad (35)$$

$$z_P = \exp\left(\frac{\mu - 2nv}{T}\right) = \exp\left(-\frac{2n_0 v}{T}\right), \quad (36)$$

$$\begin{aligned} \frac{F_P}{V} &= (n^2 - n_0^2)v - \frac{T}{\lambda^2 L_z} [g_2(z_P) - g_1(z_P) \ln z_P] \\ &\quad - \frac{T}{\lambda^3} [g'_{1/2}(z_P) - g'_{1/2}(z_P) \ln z_P], \\ f_0 &= -\hbar_0 = n_0 v; \end{aligned} \quad (37)$$

$$\varepsilon_{\mathbf{k}} = f_{\mathbf{k}} = \frac{k^2}{2m} - \mu + 2nv = \frac{k^2}{2m} + 2n_0 v; \quad \mathbf{k} \neq 0 \quad (38)$$

Thus, allowance for the correlation leads to a gap twice as large as the gap in the Hartree–Fock approximation. We note that although the anomalous expecta-

tions  $\eta_{\mathbf{k}}$ , which are related to correlations, are vanishingly small for  $\mathbf{k} \neq 0$  in the approximation under consideration, the sums  $\sum_{\mathbf{p}} \eta_{\mathbf{p}}$  are not small, as follows from (34).

As  $T \rightarrow 0$ , according to (35)–(38),  $F_P \rightarrow 0$ . Thus, the correlations are so effective in lowering the energy of the ground state that it becomes equal to zero, in the same manner as for the ideal Bose gas, i.e., the interaction does not lead to an increase in energy. The latter circumstance is related to the delta-like dependence of the potential on the distance. Negligibly small pair correlations, sufficient to prevent the presence of two bosons at the same place, make the interaction ineffective.

An analysis of Eqs. (25)–(28) and (35)–(38) shows that in both cases there exists a temperature  $T_m > T_3$  above which the solution ceases to exist, and below which the equations admit two solutions. If one considers  $n_0$  as a free parameter and searches for an extremum of the free energy with respect to this parameter, as is done in Huang's book<sup>[18]</sup> in the analysis of the non-ideal Bose gas according to the Hartree–Fock approximation,<sup>1)</sup> then one of the solutions corresponds to a maximum of the free energy, and the other to a minimum. It is natural that only the second solution is stable. If one discards the terms associated to the two-dimensional phase (i.e., one deals with a cubical region), the unstable branch of the solution exists only in the temperature interval  $T_m > T > T_3$ . For the stable branches of the Hartree–Fock solutions and the solutions with pair-correlations there is the inequality  $F_{P_2} < F_H$ . Thus, the Hartree–Fock solution is not stable.<sup>2)</sup>

We now consider the problem of the Bose-condensation temperature  $T_c$ . In Luban's paper<sup>[14]</sup> the temperature at which the solution without the condensate ceases to exist is considered as this temperature. This means that in the approximation considered in the present section this temperature equals the Bose-condensation temperature of an ideal Bose gas and exhibits anomalous dimensional effects. It is however more correct to define the Bose-condensation temperature as that temperature where the solution with condensate becomes more stable than the solution without condensate, i.e., we shall define the temperature  $T_c$  by means of the condition  $F_0(T_c) = F_P(T_c)$ .

For large  $L_z$ ,  $T_c$  is close to  $T_3$  and the inequalities  $1 - z_0 \ll 1 - z_P \ll 1$  hold. Therefore one may set  $z_0$  equal to one, and for functions of  $z_P$  one may use an expansion around the point  $z_P = 1$ .<sup>[19]</sup> The value of  $z_P$  for the temperature  $T_c$  is obtained from the  $F_0(T_c) = F_P(T_c)$ . Further, from (35) and (36) we find a rela-

<sup>1)</sup> The book [18] by Huang contains the incorrect assertion that free energy larger than  $T_3$  does not exhibit a minimum with respect to  $n_0$ , which leads to the impossibility of a stable solution with  $n_0 \neq 0$  for  $T > T_3$ .

<sup>2)</sup> One can show that the Hartree–Fock solution in the presence of the condensate is unstable with respect to the formation of pair correlations, in the same manner as the corresponding solution for the Fermi gas is unstable with respect to the formation of Cooper pairs. For this purpose it is sufficient to determine the energy of the state  $(a_0^\dagger a_0^\dagger - w a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger) (a_0^\dagger)^{N_0} |0\rangle$ , where  $|0\rangle$  is the vacuum state. This energy decreases with the growth of  $w$  from a value  $w = 0$  corresponding to the wave function of the ground state in the Hartree–Fock approximation. Thus the Hartree–Fock solution is not even metastable.

tion between the temperatures  $T_C$  and  $T_3$ , expressing first  $n$  in terms of  $T_3$  by means of Eq. (23):

$$\frac{T_3^{3/2} - T_c^{3/2}}{T_c^{3/2}} g^{3/2}(1) = 2 \frac{\lambda_c}{L_z} \left( 1 + \ln \frac{3\sqrt{\pi}}{2R_1} \right) - \frac{4}{9} R_1 \xrightarrow{L_z \rightarrow \infty} -\frac{8}{9} \frac{a}{\lambda_c},$$

$$R_1 = \frac{a}{\lambda_c} + \sqrt{\frac{a^2}{\lambda_c^2} + \frac{9}{2} \frac{a}{L_z}}, \quad a = \frac{4\pi^2 m}{h^2} v. \quad (39)$$

For a temperature  $T_C$  the particle density  $n_{0c}$  in the condensate differs from zero and is equal to

$$n_{0c} = \frac{1}{\lambda_c^3} \left[ \frac{8}{9} R_1 + 2 \frac{\lambda_c}{L_z} \right] \xrightarrow{L_z \rightarrow \infty} \frac{16}{9} \frac{a}{\lambda_c^4}. \quad (40)$$

The temperature  $T_C$  defined in this manner for large  $L_z$  does not depend on the relations between the linear dimensions and is larger than the temperature  $T_3$ . Size effects occur only for sufficiently small  $L_z$ , when

$$L_z \sim \frac{\lambda_c^2}{a} \ln \frac{\lambda_c}{a}. \quad (41)$$

Thus the pair theory predicts normal size effects.

The same qualitative result could be obtained if one neglects the correlations and considers the usual Hartree-Fock approximation, defining  $T_C$  from the condition  $F_0(T_C) = F_H(T_C)$ . In this case we obtain in place of (39) and (40) the relations

$$\frac{T_3^{3/2} - T_c^{3/2}}{T_c^{3/2}} g^{3/2}(1) = 2 \frac{\lambda_c}{L_z} \left( 1 + \ln \frac{3\sqrt{\pi}}{R_2} \right) - \frac{21}{9} R_2 \xrightarrow{L_z \rightarrow \infty} -\frac{4}{9} \frac{a}{\lambda_c},$$

$$R_2 = \frac{a}{\lambda_c} + \sqrt{\frac{a^2}{\lambda_c^2} + 9 \frac{a}{L_z}}, \quad (42)$$

$$n_{0c} = \frac{1}{\lambda_c^3} \left[ \frac{4}{9} R_2 + 2 \frac{\lambda_c}{L_z} \right] \xrightarrow{L_z \rightarrow \infty} \frac{8}{9} \frac{a}{\lambda_c^4}. \quad (43)$$

## CONCLUSION

From what was said above it follows that the pair theory predicts normal size effects, whereas a theory with distinguished condensate predicts anomalous effects. In addition the pair theory leads to lower free energies whereas the theory with distinguished condensate has no advantage insofar as the free energy is concerned even when compared with the Hartree-Fock approximation.<sup>3)</sup>

We note the following two circumstances:

1. In the pair theory  $n_0$  is different from zero at the point  $T_C$ , where the state with condensate becomes stable (cf. (40)). Thus, there is a jump in  $n_0$ . This circumstance presents interest in relation with the fact that in experiments with helium films discontinuities of the density of the superfluid have been observed at the point of appearance of superfluidity.<sup>[8]</sup>

2. In the pair theory all states in which there are  $1/p^2$  type singularities in the momentum distribution are unstable for arbitrary relations between  $L_x$ ,  $L_y$ , and  $L_z$ . Such states could occur at a temperature  $T_3$  in the absence of condensate; however the state with condensate becomes stable for  $T_C > T_3$ .

Sometimes the fact that the existence of anomalous

<sup>3)</sup> The free energy in a theory with distinguished condensate has been calculated in [20-22].

expectation values  $\langle a_0 \rangle$  and  $\langle a_0^\dagger \rangle$  (or in the general form  $\langle \psi(\mathbf{r}, t) \rangle$  and  $\langle \psi^\dagger(\mathbf{r}, t) \rangle$ ) leads to a definite phase of the wave function is advanced as an argument in favor of the theory with distinguished condensate. The latter circumstance has been related in several papers<sup>[22]</sup> to the derivation of hydrodynamics of the superfluid and of the equations describing the Josephson effect, which was first investigated for superconducting systems. However these derivations require only the existence of a definite phase difference between two arbitrary points, and not the existence of a definite phase at each point.

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