QUANTUM THEORY OF GAS LASERS WITH RING RESONATORS

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The two-level model of an atom near the generation threshold is used to derive the equation and to determine the photon density matrix of the radiation field of a gas laser with a ring resonator. Feedback between opposite waves that arises on reflection at the resonator mirrors is taken into account. The dispersion of the total number of photons, the dispersion of the number of photons in each of the directions, and the correlation function for amplitudes of the opposite waves, which defines the mean phase difference between them, are all determined as functions of the difference between the frequency of the resonator and the transition frequency.

1. INTRODUCTION

THE quantum theory of lasers with resonators of the Fabry-Perot type has been developed in sufficient detail (see, for example[1-3]). From the point of view of the statistical properties of the radiation, interest attaches to a gas laser with a ring resonator. Its distinguishing feature is the presence of a degree of freedom for opposing waves and weak feedback between them, giving rise to different generation regimes.

From experiment and from the quasiclassical theory[4-8] it follows that near the center of the Doppler amplification line there is realized, besides the standing-wave regime, also a traveling-wave regime with different amplitudes. The region of stability of the traveling-wave regime is determined by the feedback coefficient and becomes narrower with decrease of the latter, tending to the center of the amplification line[6].

When no account is taken of the feedback, the quasiclassical theory predicts the regime of unidirectional radiation in the frequency-deviation region $|\omega - \omega_0|/\gamma < \gamma/ks$ (γ—characteristic lifetime of the excited atom, ks—Doppler width, ω—resonator frequency, ω_0—transition frequency of the atom). In this region of the frequency deviation, the spatial modulation of the occupation number comes into play and leads to instability of the standing wave. A semiphenomenological approach to the theory of the ring laser, connected with introducing noise sources into the equation of motion, has been developed in[9], where the "anticorrelation" of the opposing waves, which has been observed experimentally[10], is explained.

The purpose of the present paper is to develop a rigorous quantum approach to the theory of the ring laser near the generation threshold. The photon density matrix is sought in the photon occupation number representation without account of feedback. For a laser with feedback, the investigation of the density matrix is carried out in the representation of the coherent stage (P-representation)[11,12]. Below the generation threshold, the radiation is ordinary quantum "noise" (black-body radiation). At the threshold, the relative fluctuations of the total radiation intensity, as well as in each of the directions, are of the order of unity. Above the threshold, the relative fluctuations of the intensity of the opposing waves are always larger than the fluctuations of the total energy, and on approaching the center of the amplification line it assumes the value on the order of unity. The relative dispersion of the total energy is small in the entire detuning region.

2. EQUATIONS OF MOTION FOR THE PHOTON DENSITY MATRIX

We represent a ring laser as a medium consisting of N two-level atoms (N spins), in which two electromagnetic waves with wave vectors $q = \pm k$ propagate ($k = \omega/c$). To describe the interaction of the opposing waves at the mirrors, we shall use the model of coupled oscillators and represent the Hamiltonian of the system in the form $\hbar = 1$)

$$H = \sum_{j=1}^{N} \omega_0 S_j^+ S_j^- + \omega \sum_{q=\pm k} a_q^+ (a_q + \varepsilon' a_{-q}) + g \sum_{j=1}^{N} A_j (S_j^+ - S_j^-), \quad (1)$$

$$A_j = i \sum_{q=\pm k} (a_q e^{iqr_j} - a_q^+ e^{-iqr_j}).$$

Here S_j^+ and S_j^- are the matrices of spin flip upward and downward, a_q^+ and a_q are the operators of photon creation and annihilation, $r_j = r_j^0 + v_j t$ is the coordinate of the j-th atom, v_j is its velocity, $g = d\sqrt{2\pi\omega_0/V_0}$, d is the dipole moment of the transition, V_0 is the volume of the system, and ϵ' is the feedback coefficient. For ordinary systems such as an He-Ne laser, $\omega\epsilon' \sim g \sim 10^3$–$10^4$ sec^{-1}.

Since the analysis of the problem is limited to the near-threshold regime (low radiation energies), the equation for the photon density matrix ρ is determined from perturbation theory accurate to fourth order in the interaction constant g. Omitting the intermediate calculations (see, for example,[1,3]), we present the final result:

$$\rho = -i\varepsilon'\omega \sum_q [a_q^+ a_{-q}, \rho] + \frac{\omega}{2Q} \sum_q [a_q, \rho a_q^+] - \frac{\sqrt{\pi}}{ks} g^2 \sum_q (N^+ [a_q, a_q^+ \rho]$$

$$- N^- [a_q, \rho a_q^+]) \left(1 - \frac{2g^2}{k^2 s^2} \Delta N\right) + \frac{\sqrt{\pi}}{\gamma^2 ks} g^4 \sum_q \Big\{ 2(N^+ [a_q, a_q^+ \rho a_q a_q^+]$$

$$- N^- [a_q, a_q^+ a_q \rho a_q^+]) + \frac{1}{(1 - i\delta)} (N^+ [a_q, a_{-q}^+ \rho a_{-q} a_q^+ + a_q^+ a_{-q} a_{-q}^+ \rho]$$

$$- N^-[a_q, a_q^+a_{-q}\rho a_{-q}^+ + \rho a_{-q}^+a_{-q}a_q^+])\Big\} + \text{h.c.} \quad (2)$$

Here N^+ and N^- are the numbers of the excited and unexcited spins, $\Delta N = N^+ - N^-(\Delta n \gg \sqrt{N})$, $\delta = (\omega - \omega_0)/\gamma$, and Q is the quality factor of the resonator[1]. In the derivation of the equation it was assumed that $\omega/\gamma Q$, δ/ks, $\gamma/ks \ll 1$; the small factor $(ks)^{-3}$ was retained only for terms that are quadratic in the number of atoms. However, in the assumed approximation, the term with $(\Delta N)^2$, describing the collective effects in the spontaneous emission of the atom[3], is immaterial and introduces a small correction to the generation threshold.

3. PHOTON DISTRIBUTION FUNCTION IN A RING LASER WITHOUT FEEDBACK

In this section we shall seek the solution for the diagonal elements of the density matrix $\langle n_k, n_{-k} | \rho | n_k, n_{-k}\rangle \equiv \rho_{n_k, n_{-k}}$ without allowance for the feedback, $\epsilon' = 0$. In this case the equation (2) for the diagonal element takes the form

$$\frac{\omega}{Q}\dot{\rho}_{n_k, n_{-k}} = \sum_q (Q_{n_q, n_{-q}} - Q_{n_q - 1, n_{-q}}), \quad (3)$$

$$Q_{n_q, n_{-q}} = (n_q+1)\Big\{\rho_{n_q+1, n_{-q}} - \frac{(1+\xi)}{\Delta N}(N^+\rho_{n_q\, n_{-q}} - N^-\rho_{n_q+1, n_{-q}}) + \frac{(1+\xi)}{\Delta N}\beta\Big[2(n_q+1)(N^+\rho_{n_q, n_{-q}} - N^-\rho_{n_q+1, n_{-q}}) + \frac{(n_{-q}+1)}{1+\delta^2}\cdot \times (N^+\rho_{n_q, n_{-q}} - N^-\rho_{n_q, n_{-q}+1}) + \frac{n_{-q}}{1+\delta^2}(N^+\rho_{n_q+1, n_{-q}-1} - N^-\rho_{n_q+1, n_{-q}})\Big]\Big\}. \quad (4)$$

Here ξ is the relative excess of the pump over the threshold ($\xi \ll 1$) and β is the saturation parameter:

$$\xi = \frac{2\sqrt{\pi}g^2 Q\Delta N}{\omega ks} - 1, \quad \beta = \frac{g^2}{\gamma^2}, \quad (5)$$

n_k and n_{-k} are the numbers of the photons in the opposing waves.

The stationary distribution function $\rho_{n_k, n_{-k}}$ is found from Eqs. (3) and (4), using the condition for the absence of a probability flux $\Sigma_q Q_{n_q, n_{-q}} = 0$[1]. In the quasi-classical approximation ($\langle n\rangle \gg 1$)

$$\frac{N_+}{\Delta N}\ln\frac{\rho}{\rho_0} = \xi(n_k + n_{-k}) - \beta(n_k^2 + n_{-k}^2) - \beta\frac{2n_k n_{-k}}{1+\delta^2}, \quad (6)$$

where ρ_0 is a normalization constant. As seen directly from Eqs. (2)–(4) and the solution (6), the distribution function is symmetrical with respect to the variables n_k and n_{-k}.

Below the generation threshold ($\xi < 0$, $|\xi| \ll \sqrt{\beta}$), the radiation in the two directions is statistically independent

$$\rho(n_k, n_{-k}) = \rho(n_k)\rho(n_{-k})$$

and constitutes black-body radiation with effective temperature $T = \omega/|\xi|$[1].

At the generation threshold ($|\xi| \ll \sqrt{\beta}$) the complete distribution function cannot be represented in the form of a product of the distribution functions with respect to the directions. The relative fluctuations in this case, as in the preceding one, are of the order of unity. The dispersion of the total number of photons $n = n_k + n_{-k}$ is

$$\frac{\langle(\Delta n)^2\rangle}{\langle n\rangle^2} = \frac{2}{\pi}\frac{\sqrt{2+\delta^2}}{|\delta|}\operatorname{arc\,tg}\frac{|\delta|}{\sqrt{2+\delta^2}}\Big[\frac{2(1+\delta^2)}{|\delta|\sqrt{2+\delta^2}}\operatorname{arc\,tg}\frac{|\delta|}{\sqrt{2+\delta^2}} + 1\Big] - 1, \quad (7)$$

$$\langle n\rangle = \frac{\sqrt{\pi}|\delta|}{2\sqrt{\beta}\sqrt{2+\delta^2}} \Big/ \operatorname{arc\,tg}\frac{|\delta|}{\sqrt{2+\delta^2}}. \quad (8)$$

The dispersion of the number of photons in each of the directions is

$$\frac{\langle(\Delta n_k)^2\rangle}{\langle n_k\rangle^2} = \frac{4}{\pi}\frac{(1+\delta^2)\sqrt{2+\delta^2}}{|\delta|^3}\operatorname{arc\,tg}\frac{|\delta|}{\sqrt{2+\delta^2}} \times\Big[\frac{2(1+\delta^2)}{|\delta|\sqrt{2+\delta^2}}\operatorname{arctg}\frac{|\delta|}{\sqrt{2+\delta^2}} - \frac{1}{1+\delta^2}\Big] - 1, \quad (9)$$

$$\langle n_k\rangle = \langle n\rangle / 2.$$

Greatest interest attaches to an investigation of the statistical properties of the radiation in the generation regime ($\xi \gg \sqrt{\beta}$). We note first that the points n_k^0 and n_{-k}^0, which determine the position of the absolute maximum $\rho(n_k, n_{-k})$. are connected by the classical stationary equations:

$$\xi - 2\beta\Big(n_k^0 + \frac{n_{-k}^0}{1+\delta^2}\Big) = 0, \quad \xi - 2\beta\Big(n_{-k}^0 + \frac{n_k^0}{1+\delta^2}\Big) = 0. \quad (10)$$

As is well known[7,8], the stationary solutions of (10), describing a standing wave, are stable only in the case of sufficiently large frequency deviations. Near the center of the line, this regime is unstable because of the influence of the spatial inhomogeneity of the occupation numbers. It is easy to see how the form of the distribution function changes when account is taken of spatial modulation. To this end it is necessary to replace the parameter $(1 + \delta^2)^{-1}$ in formulas (6) and (7) by $\mu + (1 + \delta^2)^{-1}$, where $\mu = (\gamma/ks)^2$ ($\mu \sim 10^{-3}$ for the He-Ne laser[7,8]).

Depending on the frequency, one can separate three characteristic regions with different statistical properties of the radiation:

$$1)\ \delta^2 - \mu \gg \sqrt{\beta}/\xi, \quad 2)\ \delta^2 = \mu, \quad 3)\ \mu - \delta^2 \gg \sqrt{\beta}/\xi.$$

In all three regions, the relative dispersion of the total number of photons is always small:

$$\langle(\Delta n)^2\rangle / \langle n\rangle^2 = 1/\langle n\rangle\xi, \quad (11)$$

$$\langle n\rangle = \xi/\beta\Big[1 + \mu + \frac{1}{1+\delta^2}\Big]. \quad (11')$$

Owing to the symmetry of the distribution function with respect to the variables n_k and n_{-k}, the average numbers of photons and the dispersions in the opposing waves are always equal:

$$\langle n_k\rangle = \langle n_{-k}\rangle = \langle n\rangle / 2, \quad \langle(\Delta n_k)^2\rangle = \langle(\Delta n_{-k})^2\rangle. \quad (12)$$

It should be noted, however, that $\langle n_k\rangle$ does not always correspond to the maximum of the distribution function. This is the cause of the strong dependence of the dispersion of $n_{\pm k}$ on the frequency near the center of the line.

1. At large detunings $\delta^2 > \mu$ we have

$$\langle(\Delta n_k)^2\rangle = \frac{\langle n\rangle}{2\xi}\frac{1+\delta^2}{\delta^2}. \quad (13)$$

2. On the boundary of the instability region of the standing wave $\delta^2 = \mu$ we have

$$\langle(\Delta n_k)^2\rangle = \langle n\rangle^2 / 12. \quad (14)$$

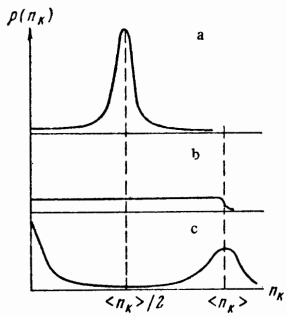

FIG. 1

3. At small detunings $\delta^2 < \mu$ we have

$$\langle(\Delta n_k)^2\rangle = \langle n\rangle^2/4. \tag{15}$$

The condition $\delta^2 < \mu$ determines the frequency region in which the standing wave (12) is unstable. We see that when $\delta^2 \leq \mu$ the relative fluctuations of the intensity in the opposing waves become of the order of unity.

The dependence of the dispersion of the number of photons in one of the directions on the parameter δ can be traced more clearly by determining $\rho(n_k)$:

$$\rho(n_k) = \rho_0 \int_0^\infty dn_{-k} \exp\left\{-\frac{\beta(1+\nu)}{2}\left[n - \frac{\xi}{\beta(1+\nu)}\right]^2 - \frac{\beta(1-\nu)}{2}(n_k - n_{-k})^2\right\},$$
$$\nu = \mu + 1/(1+\delta^2), \quad N^+ = \Delta N. \tag{16}$$

Figures 1a, b, and c show plots of $\rho(n_k)$ for $\delta^2 \gg \mu$, $\delta^2 = \mu$, and $\mu - \delta^2 \gg \sqrt{\beta}/\xi$ respectively. The curve in Fig 1a describes the distribution of the radiation in one direction in the standing-wave regime; ρ_b describes the generation in the position of indifferent equilibrium, when the amplitude of the radiation in one of the directions can be arbitrary while the total energy is conserved. The lower curve has two sharply pronounced extrema, the areas under which are equal, corresponding to unidirectional radiation (the radiation directions are equally probable). On the other hand, since the function $\rho_c(n_k)$ is continuous and $\rho_c > 0$, there is the finite probability of a transition from the state with $n_k = 0$ to the state with $n_k = \langle n\rangle$ (or vice versa), as a result of the finite repetition time of the large fluctuations. In other words, from the statistical point of view, the states with $n_k = 0$ and $n_k = \langle n\rangle$ are quasistationary.

We assume that, the time of transition from the state $n_k = 0$ to the state $n_k = \langle n\rangle$ is of the same order as the repetition time of the state $n_k = \langle n\rangle/2$[13]:

$$T_0 = \tau_0/\langle n\rangle\rho_B(\langle n\rangle/2), \tag{17}$$

where τ_0 is the probability that the photon will leave the specified volume in a unit time. In our case, τ_0 coincides in order of magnitude with the lifetime of the photon in the resonator $\tau_0 \sim Q/\omega$. Then at resonance we have $(\delta = 0)$

$$T_0 \sim \frac{Q}{\omega\xi\mu\langle n\rangle}\exp(\xi^2\mu/8\beta). \tag{17'}$$

For the values $\xi = 10^{-1}$, $\beta = 10^{-7}$, $\mu = (1/8)\times 10^{-2}$, and $\omega/Q = 10^{-6}$ sec^{-1} we get $T_0 \sim 10^{-2}$ sec.

As seen from the calculations, there is a strong correlation between the intensities of the opposing waves. Following[9,10] and taking (12) into account, we determine the correlation coefficients in the following manner:

$$K_0 = \frac{\langle(\Delta n)^2\rangle - \langle(n_k - n_{-k})^2\rangle}{\langle(\Delta n)^2\rangle + \langle(n_k - n_{-k})^2\rangle}. \tag{18}$$

Since

$$\langle(n_k - n_{-k})^2\rangle = 4\langle(\Delta n_k)^2\rangle - \langle(\Delta n)^2\rangle, \tag{19}$$

we obtain, using formulas (11)–(15),

$$K_0 = -1/(1+\delta^2). \tag{20}$$

Qualitatively, the dependence of K_0 on δ coincides with the results of[9]. At the generation threshold (7)–(9) at the center of the line $(\delta^2 \ll 1)$ we have $K_0 = -(3\pi - 8)/(16 - 3\pi)$. With increasing detuning $(\delta^2 \gg 1)$ the correlation coefficient tends to zero:

$$K_0 = -(\pi^2 - 4\pi + 8)/\delta^2\pi(\pi - 2).$$

Below the generation threshold $(|\xi| \gg \sqrt{\beta})$ we have $K_0 = 0$. The fact that the correlation coefficients introduced above turns out to be negative has a simple physical meaning: at a well defined total energy, the total dispersion of the photon numbers of each of the directions is always larger than the dispersion of the total number of photons.

4. ALLOWANCE FOR FEEDBACK

The solutions of (2) with allowance for feedback without spatial modulation can be greatly simplified by changing over from the representation of the occupation numbers $n_{\pm k}$ to the representation of the coherent states[3] (the P-representation[11,12]):

$$\rho = \int P(z_k, z_{-k}) \prod_{\pm k} |z_k\rangle\langle z_k|\, d^2 z_k, \tag{21}$$

where $|z_k\rangle$ are the eigenfunctions of the operators a_k $(a_k|z_k\rangle = z_k|z_k\rangle)$.

After substituting the density matrix (21) in (2) we obtain for the distribution function P the Fokker-Planck equation

$$\frac{\partial P}{\partial t} + \frac{\omega}{2Q}\sum_{q=\pm k}(\nabla_q J_q + \text{c.c.}) = 0, \tag{22}$$

$$J_q = \left[\xi z_q - i\varepsilon z_{-q} - 2\beta z_q\left(|z_q|^2 + \frac{|z_{-q}|^2}{1 - i\delta}\right)\right]P - \frac{N^+}{\Delta N}\nabla_q P, \tag{23}$$
$$\nabla_q = \frac{1}{2}\left(\frac{\partial}{\partial x_q} - i\frac{\partial}{\partial y_q}\right), \quad \varepsilon = 2\varepsilon' Q.$$

In the derivation of (22) and (23) we took account of the fact that $|\xi| \ll 1$; in this approximation, the diffusion coefficient does not depend on the field.

As is well known from quasiclassical theory, the total energy of the opposing waves is well defined in the entire frequency region δ, regardless of whether feedback is taken into account or not. In the language of Eqs. (22) and (23), this means that the dispersion of the total energy is smaller than the dispersion of the energy in each direction. For example, when $\epsilon = 0$ and $|\delta| < \sqrt{\langle(\Delta n)^2\rangle}/\langle n\rangle$, the field energy in one direction is not defined completely (see formulas (14) and (15)). For this reason, it is advisable to separate in (22) and (23) the parts connected with the total energy. To this end, we go over to the coordinate system

$$x_k = \sqrt{n}\sin\frac{\theta}{2}\cos\left(\frac{\varphi}{2}+\frac{\psi}{2}\right), \quad y_k = \sqrt{n}\sin\frac{\theta}{2}\sin\left(\frac{\varphi}{2}+\frac{\psi}{2}\right),$$
$$x_{-k} = \sqrt{n}\cos\frac{\theta}{2}\cos\left(\frac{\varphi}{2}-\frac{\psi}{2}\right), \quad y_{-k} = \sqrt{n}\cos\frac{\theta}{2}\sin\left(\frac{\varphi}{2}-\frac{\psi}{2}\right), \tag{24}$$

where n determines the total number of photons, $\cos\theta$ the relative difference $(n_{-k} - n_k)/(n_k + n_{-k})$, and the angles φ and ψ correspond to the difference $(\varphi_k - \varphi_{-k})$ and the sum $(\varphi_k + \varphi_{-k})$ of the phases of the opposing waves. The choice of such a non-orthogonal system of coordinates is dictated by its lucid physical meaning, and also by the simplicity of the quasiclassical equations of motion in terms of these variables (see, for example,[14]).

In the new system of coordinates, Eqs. (22) takes the form

$$\frac{Q}{\omega}\frac{\partial P}{\partial t} + \frac{1}{n}\frac{\partial}{\partial n}n^2\left\{P\left[\xi - \frac{(2+\delta^2)}{(1+\delta^2)}\beta n\left(1+\frac{\delta^2\cos^2\theta}{2+\delta^2}\right)\right] - \frac{\partial P}{\partial n}\right\}$$
$$+\frac{1}{n\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\left\{nP\left(\frac{\beta n\delta^2}{1+\delta^2}\sin\theta\cos\theta - \varepsilon\sin\varphi\right) - \frac{\partial P}{\partial\theta}\right\}$$
$$+\frac{1}{n\sin^2\theta}\frac{\partial}{\partial\varphi}\left\{nP\left(\frac{\beta n\delta}{1+\delta^2}\sin\theta - \varepsilon\cos\varphi\right)\sin\theta\cos\theta - \right.$$
$$\left.-\left(\frac{\partial P}{\partial\varphi} + \cos\theta\frac{\partial P}{\partial\psi}\right)\right\} + \frac{1}{n\sin^2\theta}\frac{\partial}{\partial\psi}\left\{nP\left(\frac{\beta n\delta}{1+\delta^2}\sin\theta - \varepsilon\cos\varphi\right)\sin\theta\right.$$
$$\left.-\left(\frac{\partial P}{\partial\psi} + \cos\theta\frac{\partial P}{\partial\varphi}\right)\right\} = 0. \tag{25}$$

For simplicity it is assumed here that $N^+ = \Delta N$. The results obtained subsequently can be readily generalized to the case $N^+ \neq \Delta N$ by multiplying the diffusion coefficient by $N^+/\Delta N$.

It is easy to see that when $\epsilon = 0$ the stationary solution of Eq. (25), which depends on n and θ, can be determined accurately:

$$P(n,\theta)\,|_{\varepsilon=0} = P_0\exp n\left[\xi - \frac{(2+\delta^2)}{(1+\delta^2)}\frac{\beta n}{2}\left(1+\frac{\delta^2\cos^2\theta}{2+\delta^2}\right)\right]. \tag{26}$$

When this solution is substituted in (21), we obtain formula (6) for the photon distribution function $\rho(n_k, n_{-k})$ in the occupation-number representation.

In the case $\langle n\rangle\epsilon \ll 1$, which takes place below the generation threshold ($\langle n\rangle = 1/|\xi|$), the dependence of the stationary distribution function on the angles θ and φ can be obtained by expanding the solution in powers of ϵ.

Above the generation threshold, where $\langle n\rangle\epsilon \gg 1$, Eq. (25) can be simplified. We note for this purpose that the relative difference of the energies of the opposing waves is small for all frequencies up to $|\delta| \sim \sqrt{\beta}/\xi$ or $|\delta| \sim \epsilon$. This means that $|\cos\theta| \ll 1$ everywhere, with the exception of the frequency regions $|\delta| \sim \epsilon$ and $|\delta| \sim \sqrt{\beta}/\xi$, where $|\cos\theta| \sim 1$. However, $\delta^2\cos^2\theta \ll 1$ always. For this reason, it is possible to neglect the term $\delta^2\cos^2\theta$ in the radial part of (25), or else replace $\cos\theta$ by $\cos\theta_0$ in the case (33) when a stationary value $\cos\theta_0 \neq 0$ exists. Further, in the angular part of the equation, it is possible to replace n everywhere by $\langle n\rangle$, since the relative dispersion of the total energy is small. Then the angular and radial variables separate, and the solution takes the form

$$P = F(0,\ \varphi,\ \psi,\ t)\ \exp\left[-\xi(n-\langle n\rangle)^2/\langle n\rangle\right],$$
$$\langle n\rangle = \frac{\xi}{\beta}\frac{(1+\delta^2)}{(2+\delta^2)}\left(1-\frac{\delta^2\cos^2\theta_0}{2+\delta^2}\right). \tag{27}$$

A similar separation of the variables is possible also for the classical equations of motion.

Let us consider the stationary solution of (25). In this case the angle part of P can depend only on θ and φ ($F = F_0(\theta, \varphi)$), since the stationary behavior means that $\langle a_k\rangle = \langle a_{-k}\rangle = 0$. The condition for the absence of a probability flux, which follows from (22), takes in the general case the form

$$\mathbf{J} = \operatorname{rot}\mathbf{M}, \tag{28}$$

where M is an arbitrary vector. Using the approximation (27), we obtain for $F_0(\theta, \varphi)$ the system of equations

$$\frac{\partial F_0}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial M_n}{\partial\varphi} = \langle n\rangle\,\xi\left(\frac{\delta^2}{2+\delta^2}\sin\theta\cos\theta - \frac{\varepsilon}{\xi}\sin\varphi\right)F_0,$$
$$\frac{\partial M_n}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial F_0}{\partial\varphi} = \langle n\rangle\,\xi\cos\theta\left(\frac{\delta}{2+\delta^2}\sin\theta - \frac{\varepsilon}{\xi}\cos\varphi\right)F_0, \tag{29}$$

where M_n is the radio component of the vector M. We note that when $\epsilon = 0$ we get $F_0 = F_0(\theta)$ and $M_n = F_0/\delta$.

We seek the solution of the system (29) near the stationary values of θ_0 and φ_0, determined from the condition $\partial F_0/\partial\theta = \partial F_0/\partial\varphi = 0$. Then $M_n = cF_0$, where c is a constant. From the condition of the compatibility of the system (29) we determine the value of c:

$$c = \frac{1}{\delta} - (2+\delta^2)\frac{2\varepsilon}{\xi\delta^2}\frac{\cos\varphi_0\sin\theta_0}{3\sin^2\theta_0 - 2} \tag{30}$$

Let us examine the behavior of the distribution function $F_0(\theta, \varphi)$ at small detunings $|\delta| \ll 1$. In this region there can exist both a standing-wave regime $\cos\theta_0 = 0$, and a traveling-wave regime $\cos\theta_0 \neq 0$[6].

For the standing wave $\sin\theta_0 = 1$ and $\cos\theta_0 = \pm 1$, we have

$$-\ln\left(\frac{F_{0s}}{P_0}\right) = \frac{\langle n\rangle\,\xi\delta^2}{4(1-(\xi\delta/4\varepsilon)\cos\varphi_0)}\left\{\left[\left(1-\frac{\xi\delta\cos\varphi_0}{2\varepsilon}\right)\right.\right.$$
$$\left.\times\left(1-\frac{\xi\delta\cos\varphi_0}{4\varepsilon}\right) + \frac{\delta^4\xi^2}{3\varepsilon^2}\right](\theta-\theta_0)^2$$
$$\left.+\frac{\xi\delta^2}{2\varepsilon}\cos\varphi_0(\varphi-\varphi_0)(\theta-\theta_0) + \left(1-\frac{\xi\delta\cos\varphi_0}{4\varepsilon}\right)(\varphi-\varphi_0)^2\right\}. \tag{31}$$

We see therefore that for frequencies $|\delta| > 2\epsilon/\xi$ there exists one stationary solution φ_0, determined by the condition $\delta\cos\varphi_0 < 0$. When $|\delta| < 2\epsilon/\xi$ there exist two stationary solutions $\cos\varphi_0 = \pm 1$, and F_{0s} describes the distribution over the phases near $\varphi \approx 0$ or π. In the frequency region $(1 - \xi|\delta|/2\epsilon) \gtrsim 2\beta/\epsilon^2$, the solution (31) is not valid, since the relative fluctuations of φ and θ near φ_0 and θ_0 become of the order of unity. In the classical theory, this region corresponds to an indifferent equilibrium position[6].

The solution (31) is likewise not valid at the center of the line $|\delta| \lesssim \sqrt{\beta}/\xi$. In this case it is possible to use perturbation theory, expanding the solutions in powers of δ. In the zeroth approximation we obtain

$$F_0 = \text{const}, \quad M_n = -\langle n\rangle\varepsilon F_0\cos\varphi\sin\theta. \tag{32}$$

Thus, at resonance the stationary distribution function depends only on the total energy. Indeed, as follows from the classical theory, an exact resonance over the total energy has a stationary value, and the relative difference of the energies of the opposing waves and the phase difference vary periodically in time. If the feedback coefficient is very small $\epsilon \sim \sqrt{\beta}$, but $\langle n \rangle \epsilon \gg 1$, i.e., the binding energy becomes comparable with the dispersion of the total energy $\langle n \rangle \epsilon \sim \xi\sqrt{\langle(\Delta n)^2\rangle}$, then the definition of the stationary phase difference becomes meaningless. In particular, the regions of strong fluctuations near $\delta = 0$ and $\delta = 2\epsilon/\xi$ overlap. In practice, however, the condition $\epsilon \gg \sqrt{\beta}$ is always satisfied ($\epsilon \sim 10^{-2}$, $\beta \sim 10^{-7}$), i.e., the stationary phase difference is well defined. We shall henceforth assume throughout that $\epsilon \gg \sqrt{\beta}$.

The solution describing the traveling-wave regime is given by[6]

$$\sin\theta_0 = \frac{2\varepsilon}{\xi|\delta|}, \quad \sin\varphi_0 = |\delta|\cos\theta_0, \quad \cos\varphi_0 = \left(1 - \frac{\delta^2}{2}\cos\theta_0\right)\operatorname{sign}\delta. \tag{33}$$

The total traveling-wave energy is smaller than the energy stored in the standing wave (27)

$$n_t = \frac{\xi}{2\beta}\left(1 + \frac{\delta^2\sin^2\theta_0}{2}\right). \tag{34}$$

From (29) we obtain the distribution function near the stationary state (33):

$$-\ln(F_{0t}/P_0) = \frac{\langle n\rangle \xi\delta^2}{4}\frac{(3\sin^2\theta_0 - 2)}{(2-\sin^2\theta_0)}[(\theta-\theta_0)^2\cos^2\theta_0 + (\varphi-\varphi_0)^2\sin^2\theta_0]. \tag{35}$$

In the last expression we have discarded, for simplicity, all corrections of order of δ and δ^2; these will be taken into account later in the calculation of the mean values.

From relation (35) follow the conditions for the stability of the solution (33)[6]

$$\sin^2\theta_0 > \frac{2}{3}, \quad \sqrt{6}\frac{\varepsilon}{\xi} > |\delta| > \frac{2\varepsilon}{\xi}. \tag{36}$$

When $\xi|\delta| = 2\epsilon$, the function F_{0t} coincides with F_{0s} for $\cos\varphi_0 = \operatorname{sign}\delta$. Further increase of the detuning $\xi|\delta| > 2\epsilon$ leads to a splitting of the maximum of the distribution with respect to θ near the value $\cos\varphi_0 = \operatorname{sign}\delta$, and to the appearance of two new maxima in θ ($\cos\theta_0 \gtrless 0$), corresponding to equally probable states with $n_k \gtrless n_{-k}$. The boundary between one stationary regime $\theta_0 = 0$ to the three regimes $\cos\theta_0 = 0$ and $\cos\theta_0 \gtrless 0$ becomes smeared out as a result of the fluctuations. The stationary state (33) will be well defined if

$$|\cos\theta_0| \gg \xi\sqrt{\beta}/\varepsilon^2.$$

The solution (35) is not valid at the stability boundary $|\delta| = \sqrt{6}\epsilon/\xi$. The condition

$$1 - 3\cos^2\theta_0 \sim \beta/\varepsilon^2$$

determines the width of the stability boundary. We see that the region of strong fluctuations near $\cos\theta_0 = 0$ is much broader than near $\cos\theta_0 = 1/\sqrt{3}$.

We proceed to calculate the average intensities of the fields and their fluctuations. The average value of a certain function of the operators a_k and a_k^+ is determined by integrals of the type (21), in which the stationary solutions (31) and (33) are saddle points. Noting that the introduction of ϵ does not change the general relations (12) and (19), and that the relative dispersion of the total energy is small in the entire frequency range and is determined by formula (11), we present the final results.

1. For the frequencies $2\epsilon/\xi > |\delta| \gg \sqrt{\beta}/\xi$ we get from (27) and (31)

$$\langle(\Delta n_k)^2\rangle = \frac{\langle n\rangle}{\xi\delta^2}\left[\left(1-\frac{\xi\delta}{4\varepsilon}\right)^2\left(1+\frac{\xi\delta}{2\varepsilon}\right)^{3/2} + \left(1+\frac{\xi\delta}{4\varepsilon}\right)^2\left(1-\frac{\xi\delta}{2\varepsilon}\right)^{3/2}\right]$$
$$\times\left(1-\frac{\xi^2\delta^2}{4\varepsilon^2}\right)^{-1}\left[\left(1-\frac{\xi\delta}{4\varepsilon}\right)\left(1+\frac{\xi\delta}{2\varepsilon}\right)^{1/2} + \left(1+\frac{\xi\delta}{4\varepsilon}\right)\left(1-\frac{\xi\delta}{2\varepsilon}\right)^{1/2}\right]^{-1},$$
$$\langle n\rangle = \frac{\xi}{2\beta}\left(1+\frac{\delta^2}{2}\right). \tag{37}$$

The correlation function of the amplitudes of the opposing waves is given by

$$\langle(a_k^+ a_{-k})^m\rangle/(\langle n_k\rangle\langle n_{-k}\rangle)^{m/2} = \langle\cos m\varphi\rangle =$$
$$= \begin{cases} 1, & m = 2l \\ \dfrac{(\xi\delta/2\varepsilon)^3}{2\left[\left(1-\frac{\xi\delta}{4\varepsilon}\right)\left(1+\frac{\xi\delta}{2\varepsilon}\right)^{1/2} + \left(1+\frac{\xi\delta}{4\varepsilon}\right)\left(1-\frac{\xi\delta}{2\varepsilon}\right)^{1/2}\right]}, & m = 2l+1 \end{cases} \tag{38}$$

2. In the frequency region $\sqrt{6}\epsilon/\xi > |\delta| > 2\epsilon/\xi$ we get from (27), (31), and (35)

$$\langle(\Delta n_k)^2\rangle = \frac{1}{2}\langle n\rangle^2\frac{\cos^2\theta_0}{1+\vartheta(\theta_0)}, \tag{39}$$

$$\langle n\rangle = \frac{\xi}{2\beta}\left(1+\frac{\delta^2}{2}-\frac{\delta^2}{2}\frac{\cos^2\theta_0}{1+\vartheta(\theta_0)}\right), \tag{40}$$

$$\langle\cos m\varphi\rangle = (-\operatorname{sign}\delta)^m\left[1-\frac{1-(-1)^m\sin^m\theta_0}{1+\vartheta(\theta_0)}\right] \tag{41}$$

$$\vartheta(\theta_0) = \frac{(1+2\sin\theta_0)(3\sin^2\theta_0-2)(1-\sin\theta_0)^{1/2}}{2(2-\sin^2\theta_0)(\sin\theta_0)^{1/2}}. \tag{42}$$

It is seen therefore that the fluctuations in each of the directions are large, and $\langle n \rangle$ is close in magnitude to the average number of photons in the traveling waves (34), since $0.3 > \vartheta(\theta_0) \geq 0$.

The meaning of relation (39) can be explained in the following manner. Unlike the preceding case (37) a traveling-wave regime exists in the region $\sqrt{6}\epsilon/\xi > |\delta| > 2\epsilon/\xi$ besides the standing wave. The relative dispersion of the distribution function near the states (31) and (33) is small, i.e., both the traveling and the standing waves are stable against small fluctuations. However, the large fluctuations occurring after a sufficiently long time, will change the system from one state to another. It is difficult to determine the average repetition time of such fluctuations (16) in this case, since for its determination it is necessary to know the magnitude and the position of the minima of the distribution function.

3. At large detunings $|\delta| \gg \sqrt{6}\epsilon/\xi$ the average energy and the dispersion coincide with expressions (11) and (13) at $\mu = 0$. The role of the feedback reduces to the appearance of a strong correlation between the amplitudes of the opposing waves

$$\langle\cos m\varphi\rangle = (-\operatorname{sign}\delta)^m. \tag{43}$$

The behavior of $\langle\cos\varphi\rangle$ as a function of δ is shown qualitatively in Fig. 2 ($\delta_1 = 2\epsilon/\xi$, $\delta_2 = \sqrt{6}\epsilon/\xi$).

5. ATTENUATION OF AVERAGE FIELDS

The attenuation of the average fields is due to phase fluctuations. The characteristic time of this process is much shorter than the relaxation time of the stationary distribution function. Usually, for a generator with a resonator of the Fabry-Perot type, this suffices to regard the phase fluctuations as independent of the amplitude fluctuations. In a generator with a ring resonator, the situation is radically changed. In the frequency region $|\delta| < \sqrt{6}\epsilon/\xi$ the distribution function

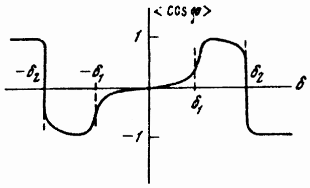

FIG. 2

has several extremal points. Therefore the states (31) and (33) are quasistationary and the time of transition between them can be comparable with the phase relaxation time. It is impossible in this case to separate the phase and amplitude fluctuations. At large detunings $|\delta| > \sqrt{6}\epsilon/\xi$ the stationary state (31) is uniquely determined, depending on the sign of δ, and the dispersion of the distribution function $F_0(\theta, \varphi)$ is small. Thus, the first fluctuations can be regarded independently of the amplitude fluctuation only for frequencies $|\delta| > \sqrt{6}\epsilon/\xi$. Since the phase difference is defined, the random quantity will be the sum of the phases ψ of the opposing waves.

Assuming that $|\delta| > \sqrt{6}\epsilon/\xi$, we represent the angular distribution functions in (27) in the form

$$F(\theta, \varphi, \psi, t) = F_0(\theta, \varphi) F_1(\psi, t). \quad (44)$$

Substituting this solution in (25), we get

$$\frac{Q}{\omega}\frac{\partial F_1}{\partial t} = \frac{1}{\langle n\rangle}\frac{\partial}{\partial\psi}\left[\frac{\partial F_1}{\partial\psi} - \xi\langle n\rangle\left(\frac{|\delta|}{2+\delta^2} + \frac{\varepsilon}{\xi}\right)F_1 \operatorname{sign}\delta\right]. \quad (45)$$

Expanding F_1 in a Fourier series in $\psi/2$ ($\langle a^m\rangle \sim \langle e^{im\psi/2}\rangle$), we obtain

$$F_1(\psi, t) = \sum_m F_{1m}(t) e^{im\psi/2}, \quad F_{1m}(t) \sim e^{i\nu_m t}, \quad (46)$$

where Re ν_m gives the frequency shift corresponding to the pulling effect

$$\operatorname{Re}\nu_m = -m\frac{\xi\omega}{2Q}\left(\frac{\delta}{2+\delta^2} + \frac{\varepsilon}{\xi}\operatorname{sign}\delta\right).$$

The imaginary part of ν_m determines the damping decrement of the average fields

$$\operatorname{Im}\nu_m = \frac{\omega\beta}{4Q\xi}\frac{2+\delta^2}{1+\delta^2}m^2. \quad (47)$$

In particular, Im ν_m at m = 1 represents the width of the emission line in each of the directions, and in the limit of small detunings $1 \gg |\delta| > \sqrt{6}\epsilon/\xi$ coincides with the corresponding expression obtained in[1,3].

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