GENERAL SOLUTION OF THE GRAVITATIONAL EQUATIONS WITH A PHYSICAL OSCILLATORY SINGULARITY

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This is a continuation of an earlier paper[1] in which we investigated a general solution of the gravitation equations with the same type of time behavior as that encountered in the particular case of a homogeneous metric of Bianchi type IX. The general solution is assumed in this case to be periodic with respect to the space variables. It is shown that if this condition is rejected, the qualitative character of the time behavior of the general solution remains the same. Moreover the general solution also includes as a particular case a homogeneous metric of the type VIII besides the homogeneous metric of the type IX.

1. INTRODUCTION

In[1] we investigated a general solution of the gravitational equations, close to the solution for a homogeneous Bianchi metric of type IX. The investigated solution was periodic in the spatial variables. In this paper we wish to call attention to the possibility of constructing also a solution that is not periodic in the spatial variables and has qualitatively the same oscillatory character in time as the solution in[1]. The obtained solution includes, as a particular case, besides the homogeneous metric of type IX, also the homogeneous metric of type VIII. We recall that homogeneous metrics can be represented in the form

$$-ds^2 = -dt^2 + (a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta)\,dx^\alpha dx^\beta, \tag{1}$$

where the three-vectors l_α, m_α, and n_α do not depend on the time, and the functions a, b, and c depend only on the time. In metrics of types IX and VIII, the three-vectors are subject to the conditions

$$\mathbf{l}\,\mathrm{rot}\,\mathbf{l} = \lambda,\quad \mathbf{m}\,\mathrm{rot}\,\mathbf{m} = \mu,\quad \mathbf{n}\,\mathrm{rot}\,\mathbf{n} = \nu,\quad \mathbf{l}[\mathbf{mn}] = 1. \tag{2}^*$$

Here λ, μ, ν = const, and all the non-diagonal scalar products of the form l curl m, etc., vanish. Condition $\mathbf{l}\cdot\mathbf{m}\times\mathbf{n} = 1$ (unlike the first three conditions) does not include any physical limitation and is a result of the special choice of coordinates. It was shown in[2,3] that a metric of type IX corresponds to the case when all three constants λ, μ, and ν have the same sign, and a metric of type VIII to the case when one of them is negative and the other two are positive. An investigation carried out in[2,3] shows that during the temporal evolution of these metrics (on approaching the singular point t = 0, where the determinant abc vanishes), an alternation of Kasner epochs and of long eras takes place. It is obvious that an analytic construction of the general solution should be carried out separately for each stage of the evolution. During the Kasner epoch, the metric is given by expression (1) with $(a^2, b^2, c^2) = (t^{2p_1}, t^{2p_2}, t^{2p_3})$ and with vectors l, m, and n that depend in arbitrary fashion on the spatial variables x, y, and z. We shall investigate further the form of the general solution for a long era. A long era is characterized by the fact that one of the functions a, b, or c decreases monotonically and subsequently, during the entire era, it becomes much smaller than the two others. Assume that this is the function c. If $c \ll a, b$, then in the coordinate system in which $l_3 = m_3 = 0$ and $n_3 = 0$ the components of the metric tensor g_{13}, g_{23}, and g_{33} turn out to be proportional to c^2, whereas the components g_{11}, g_{12}, and g_{22} are proportional to a^2 and b^2. Thus, the following inequalities are satisfied for the duration of the era

$$g_{33} \ll g_{ab},\quad g_{a3}{}^2 \ll g_{33} g_{aa}. \tag{3}$$

(The three Latin indices a, b, and c will henceforth assume the values 1 and 2.) Inasmuch as in the employed system the component g_{33} is equal to c^2 and depends only on the time, it is possible to introduce a new variable ξ by means of the transformation $dt = \sqrt{g_{33}}\,d\xi$, i.e., to change over to a coordinate system in which $-g_{00} = g_{33}$. It turns out (see[2,3]) that this is the most convenient system for the description of the solution within a single era, since the corresponding formulas assume the simplest form precisely in terms of the variable ξ. Therefore to construct the general solution in[1] we have chosen a coordinate system subject to the conditions

$$-g_{00} = g_{33},\quad g_{0\alpha} = 0. \tag{4}$$

It was shown that if one seeks a general solution of Einstein's equations $R_k^i = 0$ in the coordinate system (4), subject to the conditions (3), then the principal approximation to the solution is described by the metric

$$-ds^2 = g_{33}(dz^2 - d\xi^2) + g_{ab}\,dx^a dx^b, \tag{5}$$

in which the components g_{a3} are set equal to zero, while g_{33} and g_{ab} behave as if they depend only on the two variables z and ξ. In other words, all the terms containing derivatives with respect to the variables x^a in the equations $R_a^b = 0$, $R_3^3 = 0$, $R_0^0 = 0$, and $R_3^0 = 0$ turn out to be small in comparison with terms containing derivatives with respect to z and ξ only (this smallness is determined by the ratio $g_{33}/g_{ab} \ll 1$).

*[mn] ≡ m × n.

The equations $R_a^0 = 0$ and $R_a^3 = 0$ then determine the components g_{a3} from the solution of the preceding equations for g_{33} and g_{ab}, without imposing any new requirements whatever on the latter. The general solution obtained in this manner satisfies conditions (3), thus confirming the validity of the procedure described above.

It was also noted that the system (4) admits also of coordinate transformations that permit, without violating the conditions (3), the choice of the determinant $G = |g_{ab}|$ in the form

$$\sqrt{G} = f(x, y)\xi, \tag{6}$$

where f(x, y) is a certain specified function, the form of which is immaterial in what follows[1]. Of course, such a choice is possible only when the variable $\sqrt{G}$ has a time-like character.

If we now introduce the notation (the same as in[1])

$$g_{33} = e^{\psi}, \quad \dot{g}_{ab} = \varkappa_{ab}, \quad g_{ab}' = \lambda_{ab}, \tag{7}$$

Then Einstein's equations assume in the principal approximation the form

$$\dot{\varkappa}_a^b + \frac{1}{\xi}\varkappa_a^b - \lambda_a'^b = 0, \tag{8}$$

$$\dot{\psi} = -1/\xi + {}^1/_4\xi(\varkappa_a^b\varkappa_b^a + \lambda_a^b\lambda_b^a), \tag{9}$$

$$\psi' = {}^1/_2\xi\varkappa_a^b\lambda_b^a. \tag{10}$$

The raising and the lowering of the two-dimensional indices is carried out here with the aid of g_{ab}, and the dot and the prime denote differentiation with respect to ξ and z, respectively. The solution of Eqs. (8) should satisfy the condition (6).

2. INVESTIGATION OF THE EQUATIONS OF THE PRINCIPAL APPROXIMATION

To investigate Eqs. (8)—(10), it is convenient to introduce the following notation for the components g_{ab}:

$$g_{ab} = f\xi\begin{pmatrix} e^{\alpha}\operatorname{ch}\beta, & \operatorname{sh}\beta \\ \operatorname{sh}\beta, & e^{-\alpha}\operatorname{ch}\beta \end{pmatrix}, \tag{11}$$

which automatically take into account the condition (6). Then the system (8)—(10) assumes the form

$$\ddot{\alpha} + \frac{1}{\xi}\dot{\alpha} - \alpha'' = 2(\alpha'\beta' - \dot{\alpha}\dot{\beta})\operatorname{th}\beta, \tag{12}$$

$$\ddot{\beta} + \frac{1}{\xi}\dot{\beta} - \beta'' = \frac{1}{2}(\dot{\alpha}^2 - \alpha'^2)\operatorname{sh}2\beta, \tag{13}$$

$$\dot{\psi} = -\frac{1}{2\xi} + \frac{1}{2}\xi[\dot{\beta}^2 + \beta'^2 + (\dot{\alpha}^2 + \alpha'^2)\operatorname{ch}^2\beta], \tag{14}$$

$$\psi' = \xi[\dot{\beta}\beta' + \dot{\alpha}\alpha'\operatorname{ch}^2\beta]. \tag{15}$$

In[1] (where a somewhat different notation was used for the components g_{ab}), we confined ourselves to obtaining for Eqs. (12)—(15) a solution periodic in the variables z. This, as can be readily seen from (12) and (13), causes the functions α and β to tend to zero like $1/\sqrt{\xi}$ as $\xi \to \infty$. This result is obtained from the solution of Eqs. (12) and (13) in the linear approximation, if α and β are represented by Fourier series in z. Allowance for the nonlinear terms in the right sides of (12) and (13), the order of which is $1/\xi\sqrt{\xi}$, leads to an insignificant change of the phases of the oscillations of the linear approximation. It was shown that the solution constructed in this manner contains, as a particular case, the solution for a homogeneous metric of type IX.

In the present paper, as already mentioned, we point to the possibility of obtaining for Eqs. (12) and (13) a solution that is not periodic in z, but nevertheless possesses the same oscillating character at large ξ. In the region of large values of ξ, we seek the solution in the form of the expansions

$$\alpha = \rho z + \sigma\ln\xi + \theta + O(1/\sqrt{\xi}), \quad \beta = O(1/\sqrt{\xi}), \tag{16}$$

where ρ, σ, and θ are arbitrary functions of x and y.

Indeed, making the substitutions

$$\alpha = \rho z + \sigma\ln\xi + \theta + \frac{1}{\sqrt{\xi}}\alpha_1, \quad \beta = \frac{1}{\sqrt{\xi}}\beta_1, \tag{17}$$

we obtain from (12) and (13), assuming that $\alpha_1 \sim 1$ and $\beta_1 \sim 1$ (and the same for their derivatives)

$$\ddot{\alpha}_1 - \alpha_1'' = \frac{2\rho}{\sqrt{\xi}}\beta_1\beta_1' + O\left(\frac{1}{\xi}\right),$$
$$\ddot{\beta}_1 - \beta_1'' + \rho^2\beta_1 = -\frac{2\rho}{\sqrt{\xi}}\beta_1\alpha_1' + O\left(\frac{1}{\xi}\right). \tag{18}$$

We seek the solutions of (18) in the asymptotic region $\xi \to \infty$, in a class of functions that are bounded in ξ. In this case, owing to the presence of $1/\sqrt{\xi}$ in the right sides of (18), we can employ a method of successive approximations, neglecting in the first approximation the right sides of (18). The equation obtained in this case for α_1 is a wave equation and its general solution can be written in the form of a sum containing two arbitrary three-dimensional functions:

$$\alpha_1 = A(x, y, z + \xi) + B(x, y, z - \xi).$$

The second equation for β_1 is the Klein-Fock equation and its general solution also contains two arbitrary three-dimensional functions.

The subsequent analysis is best carried out by expanding the functions α_1 and β_1 in Fourier series[2] in a certain bounded region of z. In this case we have in first approximation

$$\alpha_1 = \sum_{n=-\infty}^{+\infty}[A_n e^{in\omega\xi} + B_n e^{-in\omega\xi}]e^{in\omega z}, \tag{19}$$

$$\beta_1 = \sum_{n=-\infty}^{+\infty}[C_n e^{i\omega_n\xi} + D_n e^{-i\omega_n\xi}]e^{in\omega z}, \tag{20}$$
$$\omega_n^2 = n^2\omega^2 + \rho^2,$$

which on the one hand satisfies[3] and conditions stipulated above that α_1 and β_1 be bounded, and on the other hand shows that it is legitimate to discard the terms of order of $1/\sqrt{\xi}$ in the right sides of (18). Indeed, sub-

[1]This function could be set equal to unity by using the aforementioned transformations. We retain it here only because its presence makes it easier to obtain the particular cases of the homogeneous metrics of type VIII and IX from the general solution.

[2]It may be that a solution could be sought in the form of Fourier integrals; this question has not been investigated fully. We therefore do not assert that expandability in Fourier series is a necessary requirement imposed on the coordinate dependence of the functions α_1 and β_2.

[3]It is assumed here that all the necessary convergence condtions are satisfied for the series in (19).

stituting in (18) the first-approximation solution (19) and varying the left-hand sides, we obtain equations for the determination of the corrections $\delta\alpha_1$ and $\delta\beta_1$:

$$(\delta\alpha_1)^{\cdot\cdot}-(\delta\alpha_1)''=\frac{2\rho}{\sqrt{\xi}}\beta_1\beta_1', \quad (\delta\beta_1)^{\cdot\cdot}-(\delta\beta_1)''+\rho^2\delta\beta_1=-\frac{2\rho}{\sqrt{\xi}}\beta_1\alpha_1'. \tag{21}$$

Expanding $\delta\alpha_1$ and $\delta\beta_1$ in a Fourier series, we then obtain equations for the determination of the Fourier coefficients $(\delta\alpha_1)_n$ and $(\delta\beta_1)_n$ as functions of the time. An essential factor is the absence from the right sides of these equations of terms with resonant frequencies (the frequency ωn for $(\delta\alpha_1)_n$ and $\sqrt{n^2\omega^2+\rho^2}$ for $(\delta\beta_1)_n$). Consequently, the integration of (21) yields $\delta\alpha_1 \sim 1/\sqrt{\xi}$ and $\delta\beta_1 \sim 1/\sqrt{\xi}$, i.e., the corrections actually turn out to be small.

The situation is somewhat different when it comes to the determination of the next terms of the expansions of α_1 and $\beta_1 \sim 1/\xi$, with allowance for the terms designated $O(1/\xi)$ in (18). A procedure analogous to that described above leads to equations for the coefficients $(\delta^2\alpha_1)_n$ and $(\delta^2\beta_1)_n$ with right-hand sides of the order of $1/\xi$, but containing resonant terms. We thus obtain corrections $\delta^2\alpha_1$ and $\delta^2\beta_1 \sim \ln\xi$. However, such an "instability" of the first approximation (19) is only illusory. This is connected with the variation of the amplitudes A_n, B_n, C_n, and D_n which actually are not constant but slowly-varying functions of ξ, of the form $\cos\ln\xi$ and $\sin\ln\xi$. A similar phenomenon takes place also in nonlinear oscillations[4]. The correct form of the amplitudes was determined by us in[1] for the case $\rho = 0$[4)].

Thus, with a sufficient degree of accuracy (neglecting the slow variation of the amplitudes), we can use the solution (17), (19). Substituting it in (14), we easily find the principal asymptotic terms of the function ψ. When $\rho \neq 0$, we obviously have

$$\psi = {}^1/_4\rho^2\xi^2[1+O(1/\xi)]. \tag{22}$$

On the other hand, if $\rho = 0$, then the principal term in ψ turns out to be linear in ξ, viz., $\psi = q\xi$, where q is an essentially positive function of x and y (see[1]). (The role of (15) reduces only to a determination of the arbitrary function $\psi_0(x, y, z)$ for the integration of Eq. (14)).

The foregoing analysis shows that when ξ decreases from a certain very large value $\xi_1 \gg 1$, the component of the metric tensor g_{33} decreases like $\exp({}^1/_4\rho^2\xi^2)$, whereas the behavior of the components g_{ab}, averaged over the oscillations, is described only by the power functions $\xi^{1+\sigma}$ and $\xi^{1-\sigma}$, as is seen from (11) and (17)[5)]. Thus, an era sets in in which the inequality $g_{33} \ll g_{ab}$, which was stipulated from the very beginning, holds.

The final stage of the era can be readily traced directly from Eqs. (8)—(10). A simple analysis made in[1] shows that when $\xi \to 0$ the solution has a Kasner-like asymptotic form

$$g_{ab} = l_a l_b \xi^{2p_1} + m_a m_b \xi^{2p_2},$$
$$l_1 m_2 - l_2 m_1 = f, \quad p_1 + p_2 = 1, \quad p_1 > 0, \quad p_2 > 0, \tag{23}$$
$$g_{33} \sim \xi^{p_1^2+p_2^2-1}.$$

Here l_a, m_a, and p_a are arbitrary three-dimensional functions (the conditions $l_1m_1 - l_2m_1 = f$ and $p_1 + p_2 = 1$ follow from (6)). Since $p_1, p_2 > 0$, the components g_{ab} decrease and the component g_{33} increases ($p_1^2 + p_2^2 - 1 < 0$). Consequently, sooner or later there sets in an instant when g_{33} becomes larger than g_{ab}, and the condition for the applicability of the approximation considered here no longer holds.

To verify the foregoing analysis, it is necessary to consider also the equation $R_a^0 = 0$ and $R_a^3 = 0$ and to check, after a solution is obtained for the components g_{a3}, that the second of the proposed inequalities, $g_{a3}^2 \ll g_{33}g_{aa}$, also is satisfied. We shall not do this in detail. An analysis perfectly analogous to the one in[1] shows that both when $\xi \gg 1$ and when $\xi \ll 1$ the components g_{a3} are proportional to g_{33}. Thus, when $g_{33} \ll g_{ab}$, the second condition is also satisfied.

Let us consider now the question of the degree of generality of the solution considered above. The physical leeway in the general solution is determined by four arbitrary functions of three spatial variables and three functions of two variables[5].

It must be ascertained first which coordinate transformations still remain admissible in our case; it is easy to show that if: 1) we retain the coordinate conditions (4), 2) we do not change the functional form (6) of the determinant $|g_{ab}|$, 3) we do not introduce into the component g_{a3}, by coordinate transformation, terms that are proportional to g_{ab} and violate the inequality $g_{a3}^2 \ll g_{33}g_{aa}$, and 4) we retain the diagonal asymptotic form of the matrix g_{ab} at large values of ξ

$$g_{ab} \sim f\begin{pmatrix} \xi^{1+\sigma}e^{\rho z+\theta}, & 0 \\ 0, & \xi^{1-\sigma}e^{-\rho z-\theta} \end{pmatrix}, \tag{25}$$

then we can still carry out a transformation of the form

$$z = \bar{z} + z_0(x, y). \tag{26}$$

Thus, the admissible transformations contain one arbitrary two-dimensional function and not a single three-dimensional one.

By choosing z_0 it is possible to cause $\theta(x, y)$ in solution (17) to vanish. We are then left with no transformations that contain two-dimensional functions. However, the solution still retains three arbitrary functions, which thus constitute a physical leeway. We have in mind the arbitrary function $\psi_0(x, y)$ of the integration of Eqs. (14) and (15), and two such functions contained in the component g_{a3} (for more details see[1]).

As to the leeway determined by the three-dimensional functions, it is contained in solution (17) of Eqs. (18) for α and β. This leeway is described by four three-dimensional functions.

A unique effect arises in the general case as a result of the presence of the term $\sigma\ln\xi$ in expressions (17) for the function α. When $\sigma \neq 0$, the oscillations of the metric coefficients occur against the background of

4) In [1] we also assumed that $\sigma = \theta = 0$. However, the quantity θ does not enter in the equations that determine α_1 and β_1 at all, and the term $\sigma\ln\xi$ in the function α affects only the order $1/\xi\sqrt{\xi}$ of (18). Thus, the formulas obtained in [1] for the amplitudes remain valid when $\rho = 0$.

5) To obtain the particular cases of homogeneous metrics of type VIII and IX from the general soultion, it is necessary to put $\sigma(x, y) = 0$. In the general case, there is a limitation on the region of admissible values of this function. From the conditions for the joining of the solution described above with the solution in the region of small values of ξ, which is given below, it follows that $\sigma < 1$.

a general decrease of these functions, which is nonlinear in ξ and has a different form for the different components of the metric tensor g_{ab} ($\xi^{1+\sigma}$ and $\xi^{1-\sigma}$). Since we did not encounter such a behavior of the metric in the homogeneous models, this question deserves a more thorough investigation.

3. CONCLUSION

As follows from the results of[1], the solution for the homogeneous metric of type IX is contained as a particular case in the general solution obtained above, describing one era. If we put in the solution (17) and (19), $\rho = \sigma = \theta = 0$, and change over to the notation used for g_{ab} in[1]

$$\begin{pmatrix} e^{\alpha}\operatorname{ch}\beta, & \operatorname{sh}\beta \\ \operatorname{sh}\beta, & e^{-\alpha}\operatorname{ch}\beta \end{pmatrix} = \begin{pmatrix} \operatorname{ch}\gamma + \frac{\chi}{\gamma}\operatorname{sh}\gamma, & \frac{\varphi}{\gamma}\operatorname{sh}\gamma \\ \frac{\varphi}{\gamma}\operatorname{sh}\gamma, & \operatorname{ch}\gamma - \frac{\chi}{\gamma}\operatorname{sh}\gamma \end{pmatrix}, \qquad (27)$$

$$\gamma^2 = \chi^2 + \varphi^2$$

(in the linear approximation we have simply $\alpha = \chi$ and $\beta = \varphi$), then we can use simply the results of[1].

We shall show now that when $\rho \neq 0$ the solution described above contains the particular case of a homogeneous metric of type VIII. The latter can be obtained from (A.11) of[2], by putting $\mu = \nu = 1$ and $\lambda = -1$ (the constant β is also assumed equal to unity). This yields

$$-ds^2 = -dt^2 + [(a^2\operatorname{sh}^2 z + b^2\operatorname{ch}^2 z)(1-y^2) + c^2y^2]dx^2$$
$$+ (a^2\operatorname{ch}^2 z + b^2\operatorname{sh}^2 z)\frac{dy^2}{1-y^2} - (a^2+b^2)\operatorname{sh}2z\,dx\,dy + c^2dz^2 - 2c^2y\,dx\,dz. \qquad (28)$$

Let us consider an era during which $c^2 \ll a^2, b^2$. We can then neglect in (28) the component $g_{13} \sim c^2$ and discard the term c^2y^2 in g_{11}. Then the transformation

$$dt = c\,d\xi, \quad x = \bar{x} + \bar{y}, \quad y = \operatorname{th}(\bar{x} - \bar{y}), \qquad (29)$$

yields in the principal approximation

$$-ds^2 = c^2(dz^2 - d\xi^2) + \frac{2ab}{\operatorname{ch}^2(x-y)}[e^{-2z}\operatorname{ch}\beta\,dx^2 + e^{2z}\operatorname{ch}\beta\,dy^2 + 2\operatorname{sh}\beta\,dx\,dy], \qquad (30)$$

where β denotes

$$\beta = \ln(b/a), \qquad (31)$$

and the bars over x and y have been omitted. Equations for c^2, ab, and β can be readily obtained from the exact equations (4.2) and (4.3) of[2], neglecting in them the small quantity c^2 and changing over to the variable ξ. This yields immediately $(ab)_{\xi\xi} = 0$, and without loss of generality we obtain $ab = a_0^2\xi$. Then the equations for β and ψ ($c^2 = e^{\psi}$) take the form

$$\beta_{\xi\xi} + \frac{1}{\xi}\beta_\xi + 2\operatorname{sh}2\beta = 0,$$
$$\psi_\xi = -\frac{1}{2\xi} + \frac{1}{2}\xi[\beta_\xi^2 + 4\operatorname{ch}^2\beta], \qquad (32)$$

and in the region $\xi \gg 1$ we have in first approximation

$$\beta = \frac{1}{\sqrt{\xi}}(C_0e^{2i\xi} + D_0e^{-2i\xi}), \quad \psi = \xi^2. \qquad (33)$$

It is easy to see that the metric (30) and Eqs. (32) are obtained from the general case (11)—(15) by putting

$$f(x,y) = \frac{2a_0^2}{\operatorname{ch}^2(x-y)}, \quad \alpha = -2z, \quad \beta = \beta(\xi). \qquad (34)$$

This means that in the solution (17), (19), and (22) we have $\sigma = \theta = 0$, $A_n = B_n = 0$, $\rho = -2$, and among the Fourier coefficients C_n and D_n only C_0 and D_0 differ from zero.

We note further the following circumstance, which justifies the employed Fourier expansion in z. The general solution obtained above is valid only in a certain limited interval of ξ, namely $\xi_1 - \xi_2 = \Delta\xi$, where ξ_1 and ξ_2 are the start and end of the era. We can therefore always separate in space a region S_1 bounded in z at the initial instant ξ_1, and a region S_2 at the instant ξ_2, such that the initial perturbations from the regions outside S_1 do not have time to influence the character of the solution in S_2 at the instant ξ_2. Thus, if we use a Fourier expansion in the region S_1, then it can be assumed that the solution obtained in this manner will actually be general for all the instants of time in the interval $\Delta\xi$ and in a certain bounded region of z, the dimensions of which are determined by the dimensions of S_2 at the final instant ξ_2 of the era.

Finally, we note the following cosmological aspect of the analysis presented here. We have seen that the general solution describing one era is close to the solutions for the homogeneous models of types IX and VIII, depending on whether the function $\rho(x, y)$ vanishes at our solution or not. In the general case the solution, of course, does not belong to any type, but the local properties of the space in the vicinity of a certain point (x_0, y_0, z_0) may be quite close to those that obtain in the homogeneous models. The type of model approximating the structure of space can change from point to point. For example, the function $\rho(x, y)$ can be equal to zero in some regions, and different from zero in others. Accordingly, in the former regions the evolution of space in time is close to that predicted by the homogeneous model of type IX, and in the latter by the homogeneous model of type VIII.

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